

The Generalised Liar Paradox: A Quantum Model and Interpretation*

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Abstract

The formalism of abstracted quantum mechanics is applied in a model of the generalized Liar Paradox. Here, the Liar Paradox, a consistently testable configuration of logical truth properties, is considered a dynamic conceptual entity in the cognitive sphere (Aerts, Broekaert, & Smets, 1999, 2000, Aerts, Broekaert, & Gabora 1999, 2000, 2002). Basically, the intrinsic contextuality of the truth-value of the Liar Paradox is appropriately covered by the abstracted quantum mechanical approach. The formal details of the model are explicated here for the generalized case. We prove the possibility of constructing a quantum model of the m -sentence generalizations of the Liar Paradox. This includes (i) the truth-falsehood state of the m -Liar Paradox can be represented by an embedded $2m$ -dimensional quantum vector in a $(2m)^m$ dimensional complex Hilbert space, with cognitive interactions corresponding to projections, (ii) the construction of a continuous ‘time’ dynamics is possible: typical truth and falsehood value oscillations are described by Schrödinger evolution, (iii) Kirchoff & von Neumann axioms are satisfied by introduction of ‘truth-value by inference’ projectors, (iv) time invariance of unmeasured state.

1 Introduction

Specific aspects of dynamics of general entities —not necessarily from the microphysical domain— can be successfully described by an abstracted formulation of quantum mechanics (Aerts, 1982, 1983ab, 1986, 1992, 1994, 1999). This approach is used here to seize some of the specific dynamical aspects of *conceptual entities* (Aerts, Broekaert, & Gabora, 1999, 2000, 2002).

The theoretical model which conceives a *conceptual entity* as a consistently testable configuration of properties in the sphere where personal and interpersonal cognitive interactions are taking place, is drawn from analogy with e.g. modeling of social entities in a social sphere, or quantum entities in the quantum sphere. Here conceptual entities are located in their proper ‘space’: the cognitive layer or sphere of reality. The nature of the conceptual entity and its interactions limits the analogy with physical or social modeling; essentially the transience of their identity and subjectivity in interaction, and their ontological status is different (Aerts, Broekaert, & Gabora, 1999, 2000).

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Specifically, in the application of the abstracted quantum formalism to cognition, the interaction between the *context* and a conceptual entity can be modeled. The context is the set of effective extraneous factors from physical surrounding and internal cognitive state as well. The latter is considered as the ever-fluctuating associative structure of the conceptual network of the mind. At present we envisage our ‘capacity of logical inference’ to figure as a *coercive* internal context. Which supposes that outcomes of the process of logical inference are endorsed as valid cognitive ‘input’ states for further reasoning.

The key factor of quantum mechanics that allows its application in the present modeling problem, is the contextual and indeterministic effect of the *measurement* process. Aspects of formal equivalence between abstracted quantum mechanics and the ‘concept in context’ cognitive model for the Liar Paradox will allow to cast the latter in a strict quantum-like model.

We suppose the cognitive entity of the Liar Paradox can be validly accessed using language. The most elementary form of the Liar Paradox may well be the natural linguistic expression “this proposition is false”. Classically its reasoning by logical inference leads to the well known logically contradictory evaluation, leaving indefinite its genuine logical state. Repetitive reasoning on the Liar Paradox sets in an oscillatory attribution of contradicting truth values.

The quantum-like model of Liar Paradox allows the non-deterministic contextual actualization of logical truth-values and the continuous deterministic evolution by reasoning at any subsequent instance of *time* as well.

We develop the formalism required for the Liar Paradox model in the next section.

2 The Quantum Model of the Liar Paradox

We let the Liar Paradox be a configuration of a number of sentences —propositions— referring to each other and claiming truth or falsehood of its target sentence in the configuration. The entity is stripped to its logical content using a formal shorthand notation. In this notation the simplest Liar Paradox is

$$1 \quad \neg \tag{1}$$

The first number is the sentence pointer. The second expression in row is the semantical content of the proposition, here the target sentence pointer number with a logical operator acting upon it (True \equiv 1 or False \equiv /).

Some 8-Liar Paradoxes become in this notation;

$$\begin{array}{ccc|ccc} 1 & \neg & & 5 & 6 & & 1 & \neg & & 5 & \neg \\ 2 & 3 & & 6 & 7 & & 2 & \neg & & 6 & \neg \\ 3 & 4 & & 7 & 8 & \text{or} & 3 & 8 & & 7 & \neg \\ 4 & 5 & & 8 & 1 & & 4 & 6 & & 8 & 2 \end{array} \tag{2}$$

In the following we will refer to (2,b) as an explicit example case. The formulation of the generalized liar Paradox is one single and ordered or unordered string of m concatenated sentences (‘daisy-chain’ configuration);

$$\begin{array}{l} 1 \quad O_1(2) \\ 2 \quad O_2(3) \\ \dots \\ m \quad O_m(1) \end{array} \quad \text{('ordered')} \tag{3}$$

Where O_i is at choice one of the logical operators $\mathbf{1}$ or $/$. In the basic configuration the sentence with pointer m semantically leads back to the initial sentence with pointer 1. Unordered configurations—considered as the result of a basis transformation—are:

$$1 \leq i, j \leq m, \quad i \quad O_i(j) \quad (\text{'unordered'}) \quad (4)$$

where the set $\{(i, j)\}$ is a permutation of the number basis of the set $\{(i, i + 1)\}$. We require all configurations to have sentence pointers ordered 1 to m , and consider different those which expose reversed reasoning ordering (pointer \rightarrow target \rightarrow pointer \rightarrow ...).

The number of such index configurations is $(m-1)!$. Each configuration has m relations to which is attributed a logical operator $\mathbf{1}$ or $/$. Paradoxical configurations require an uneven number k of $/$ -operators ($1 \leq k \leq m$). As k indistinguishable items can be allocated to m relations in $\frac{m!}{k!(k-m)!}$ manner, the total number of m sentence Liar Paradox configurations is $(m-1)! \sum_{k=1}^{\leq m} \frac{m!}{k!(k-m)!}$ (k uneven). The particular choice of configuration does not affect in any manner the general structure of the model, it merely changes the contingencies of the entity's dynamics.

In the next section we approach the construction of the model in three parts, i) the representation space and reasoning acts, ii) step evolution in particular configuration, iii) continuous evolution.

2.1 Modeling of the Representation Space and Reasoning Acts

Representation space

All sentences, components of the Liar Paradox, are equivalently described by sub-space vectors. The state of the Liar Paradox is represented by tensor products of state vectors of the sentence sub-spaces. To each sentence in the configuration two possible truth-falsehood values by *hypothesis*, and an a priori unknown number of truth-falsehood values by *inference* can be assigned. And each sentence state vector is attributed a sufficient number of dimensions such that reasoning dynamics occurs without degenerescence of states. That is, all substates produced by reasoning the Liar Paradox are unique and should occur only once during the completion of a reasoning cycle. We introduce hereto a Hilbert space with the minimum number of dimensions required to symmetrically embed the m -sentence configuration. The requirement of symmetrical representation reflects the equivalence of all sentences in the configuration. E.g. for the case $m = 2$, the symmetrical representation needs the Hilbert space $\mathbb{C}^4 \otimes \mathbb{C}^4$ (cf. the 'double Liar Paradox' in Aerts, Broekaert, & Smets, 2000). Let us suppose now that m sentences are in the Liar Paradox configuration, and let each sentence be represented by an n -dimensional subspace. The representation space is then the tensor coupled Hilbert space Σ .

$$\Sigma = \mathbb{C}_{(1)}^n \otimes \mathbb{C}_{(2)}^n \otimes \dots \otimes \mathbb{C}_{(m)}^n$$

Initial state

We consider the logically indefinite conceptual entity of the Liar Paradox as the initial situation of the reasoning process. The model then initially is in the *unmeasured* state with inexplicit truth value due to superposition of state with logically contradicting truth values. It is then in a state of time-invariance as each component is equally undetermined; the conceptual entity Liar Paradox is 'cognitively perceived but not logically evaluated'. This leads to the constraint of imposing an equiponderate initial state in the model.

For determination of the subspace dimension m in the appendix, we consider the expression for

the initial state Ψ_0 (for generality the equiponderate demand has not been explicated):

$$\Psi_0 = \sum_{i_1=1}^{i_1=n} \cdots \sum_{i_m=1}^{i_m=n} \alpha_{i_1 \dots i_m} \mathbf{e}_{i_1 \dots i_m} \quad (5)$$

where we systematically employ a double indices convention; the first index points at the entry level in a sentence state function, while the number of a sentence itself is indicated by the second subindex. E.g., $\alpha_{23} = 1$ indicates the state function of sentence 3 has ‘1’ in its 2-nd entry. The normalization condition for Ψ_0 is:

$$\sum_{i_1=1}^{i_1=n} \cdots \sum_{i_m=1}^{i_m=n} \alpha_{i_1 \dots i_m} \alpha_{i_1 \dots i_m}^* = 1$$

Representation of reasoning acts

The truth and falsehood ‘measurements’ —reasoning acts— on each sentence correspond to appropriately chosen projectors. These projectors put the prior state into a state representing truth or falsehood by hypothesis of the respective sentence. Each truth-falsehood by hypothesis projector represents a possible onset of the reasoning on the paradox, as the reasoning on it can start at any index. The subsequent reading, with logical inference, fix the truth-falsehood by inference state of the remaining sentences in the product. The next step of the dynamics is achieved by endorsing the inferred truth value into a hypothesized value. The sequential appearance of the *eigenstates* of the truth-falsehood by hypothesis measurements on the given sentences is thus realized. The dynamical quantum evolution reconstructs the inference sequence.

Without prior specification of the dimension, we define for each sentence with pointer i two projection operators. The truth by hypothesis projection operator on sentence i can in general be written as:

$$T_i = \sum_{j=1}^{j=n} \tau_{j_i} \mathbf{1}_1 \otimes \dots \mathbf{1}_{i-1} \otimes P_{j(i)} \otimes \mathbf{1}_{i+1} \dots \otimes \mathbf{1}_m \quad (6)$$

and the falsehood by hypothesis projection operator on sentence i :

$$F_i = \sum_{j=1}^{j=n} \phi_{j_i} \mathbf{1}_1 \otimes \dots \mathbf{1}_{i-1} \otimes P_{j(i)} \otimes \mathbf{1}_{i+1} \dots \otimes \mathbf{1}_m \quad (7)$$

The projectors operates strictly on the subspace with same sentence index. The basic projection operators P_j fulfill the usual requirements:

$$P_j = P_j^2 \quad \text{and} \quad P_j = P_j^\dagger$$

which leads to, $\forall i, j$;

$$\tau_{j_i}, \phi_{j_i} \in \{0, 1\} \quad (8)$$

Specific choices of coefficients (8) on the projectors will allow delineated interpretation per entry in the vector.

In the next section $n - 2$ ‘truth-falsehood by inference’ projectors —similar to (6, 7)— for each sentence are introduced in order to fulfill the complementarity of ‘false’ and ‘true’ operators according Kirchoff and von Neumann axioms.

We settle now the issue of the dimension of all subvectors in the m sentence configuration. When reasoning the Liar Paradox over one cycle, the possible degenerescence of occurring states is avoided by supplying sufficient dimensions n to each subspace. The initial state (17) is spanned over all n^m states, while strictly there are $2m$ relevant states for the reasoning process. For, outcomes of acts of logical inference are endorsed as valid ‘by hypothesis’ input for further cognitive acts. I.e., m outcomes for truth by hypothesis projections and m relevant outcomes for falsehood by hypothesis projections on Ψ_0 . The projection outcomes of the $2m$ truth-falsehood projectors on Ψ_0 set $2m$ constraints on the model. In the Appendix we prove that in order to satisfy $2m$ well chosen constraints on the system of m sentences, the subspace for each sentence needs n dimensions, with:

$$n = 2m \quad (9)$$

The model of the Liar Paradox entity is therefore constructed in a $(2m)^m$ dimensional Hilbert space.

2.2 Representation of Evolution: Stepwise Reasoning

The logically subsequent eigenstates of the reasoning acts—a truth-falsehood by hypothesis state in product relation with truth-falsehood by inference states— following any initial measurement, must be reproduced by dynamical evolution. Therefore at discrete moment—indexed j , $1 \leq j \leq 2m$ — of the time-ordering parameter, $t_j = j\frac{\pi}{2}$, at which an inferred logical value is endorsed into a hypothesized logical value, the state vector of the Liar Paradox should be of the form, modulo an irrelevant factorizable phase θ_j :

$$\Psi(t_j) = e^{i\theta_j} \mathbf{e}_{k_1(j)\dots k_i(j)\dots k_m(j)}$$

where $\{k_1(j)\dots k_i(j)\dots k_m(j)\}$ are the indices of the tensor product state at step n , out of $2m$, of reasoning on a specific Liar Paradox sentences configuration $\{O_i(j)\}$ with a truth-falsehood projectors convention (e.g. 8).

Logical reasoning acts put no conditions on the state functions at intermediary values of the time ordering parameter, but the quantum formalism allows the integration of the stepwise reasoning acts into a continuous evolution (next subsection).

The stepwise reasoning dynamics is constructed in the isomorphic single Hilbert space, instead of the tensor coupled Hilbert space representation. The transition is done by providing unequivocal translation of states. The basis vectors of the single Hilbert space with dimension n^m (eq. 9) have the index function:

$$\begin{aligned} \kappa(i_1, \dots, i_m) &= (2m)^{m-1}(i_m - 1) + (2m)^{m-2}(i_{m-1} - 1) \\ &+ \dots + 2m(i_2 - 1) + (i_1 - 1) + 1 \end{aligned} \quad (10)$$

as a function of the indices of the basis vectors in the tensor coupled representation. The inverse function κ is related to the decimal expression of the $2m$ -based digit sequence $i_1 i_2 \dots i_m$:

$$i_1 \dots i_m = (\kappa(i_1, \dots, i_m) - 1) |_{2m\text{-base}} + 1 \dots 1 \quad (11)$$

Where the expression $1 \dots 1$ has m digits. We choose the truth and falsehood by hypothesis operators as :

$$\begin{aligned} T_i &= \mathbf{1}_1 \otimes \dots \mathbf{1}_{i-1} \otimes T \otimes \mathbf{1}_{i+1} \dots \otimes \mathbf{1}_m \\ F_i &= \mathbf{1}_1 \otimes \dots \mathbf{1}_{i-1} \otimes F \otimes \mathbf{1}_{i+1} \dots \otimes \mathbf{1}_m \end{aligned} \quad (12)$$

with

$$T = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}_{2m \times 2m} \quad \text{and} \quad F = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}_{2m \times 2m} \quad (13)$$

Then, in each component vector of a sentence, the ‘truth by hypothesis’ state property corresponds entry with index $2m - 1$, and ‘falsehood by hypothesis’ state property corresponds to entry with index $2m$. The ‘truth and falsehood by inference’ operators are defined in direct relation to the assignment of the $2m - 2$ remaining entries.

For a given m sentence configuration having a *reasoning sequence* — for example in (2) $\{1_T, 3_F, 8_F, 2_F, 7_T, 4_F, 6_F, 5_T\}$ — i.e. a time-sequence of $2m$ eigenstate product vectors of the reasoning by inference states, the assignment procedure is:

- i) open $2m$ vectors of m tensorially coupled $2m$ dimensional sentence-vectors, ii) assign in a sentence-vector the value 1 respectively to entry $2m$ when ‘false’, and entry $2m - 1$ when ‘true’, iii) start by assuming sentence 1 is ‘true’, iii) by consecutive inference assign in each of the $2m$ tensor coupled states the proper truth or falsehood entries of the implied sentence-vectors, iv) for filling in the unknown entries unequivocally an *ad hoc* rule is supplied; assign the value 1, in a consecutive inference order and starting from a truth state (position $2m - 1$), in the next tensor coupled vector in the same sentence-vector to the position with index equal to previous index minus one, v) jump one tensor coupled state if it has the ‘false’ entry 1 at position $2m$.

The choice of truth-falsehood operators and the assignment procedure completely and unequivocally define the state vector. For example this gives for a 8-sentence Liar Paradox (2), in the tensor coupled space representation $\otimes_{i=1}^{i=8} \mathbb{C}^{16}_{(i)}$ the initial superposition state Ψ_0 :

$$\begin{aligned} \Psi_0 = & \frac{1}{\sqrt{16}} \{ \mathbf{e}_{15.10.8.12.7.13.4.9} \\ & + \mathbf{e}_{14.9.16.11.6.12.3.8} + \mathbf{e}_{13.8.7.10.5.11.2.16} + \mathbf{e}_{12.16.6.9.4.10.1.7} \\ & + \mathbf{e}_{11.7.5.8.3.9.15.6} + \mathbf{e}_{10.6.4.16.2.8.14.5} + \mathbf{e}_{9.5.3.7.1.16.13.4} \\ & + \mathbf{e}_{8.4.2.6.15.7.12.3} + \mathbf{e}_{16.3.1.5.14.6.11.2} + \mathbf{e}_{7.2.15.4.13.5.10.1} \\ & + \mathbf{e}_{6.1.14.3.12.4.9.15} + \mathbf{e}_{5.15.13.2.11.3.8.14} + \mathbf{e}_{4.14.12.1.10.2.16.13} \\ & + \mathbf{e}_{3.13.11.15.9.1.7.12} + \mathbf{e}_{2.12.10.14.8.15.6.11} + \mathbf{e}_{1.11.9.13.16.14.5.10} \} \end{aligned} \quad (15)$$

with e.g.:

$$\begin{aligned} \mathbf{e}_{15.10.8.12.7.13.4.9} &= \mathbf{e}_{15} \otimes \mathbf{e}_{10} \otimes \mathbf{e}_8 \otimes \mathbf{e}_{12} \otimes \mathbf{e}_7 \otimes \mathbf{e}_{13} \otimes \mathbf{e}_4 \otimes \mathbf{e}_9 \\ \mathbf{e}_{15} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)^t, \quad \dots \end{aligned} \quad (16)$$

In the reduced single Hilbert space of 16^8 dimensions the initial state using the indexfunction (eq. 10) is given by;

$$\begin{aligned} \Psi_0 = & \frac{1}{\sqrt{16}} \{ \mathbf{e}_{3917179961} + \mathbf{e}_{3640285992} + \mathbf{e}_{3345566240} + \mathbf{e}_{3210230023} \\ & + \mathbf{e}_{2789681382} + \mathbf{e}_{2503940053} + \mathbf{e}_{2217086916} + \mathbf{e}_{1930815155} \\ & + \mathbf{e}_{4060403106} + \mathbf{e}_{1642316945} + \mathbf{e}_{1355985807} + \mathbf{e}_{1321312894} \\ & + \mathbf{e}_{1034981885} + \mathbf{e}_{749633644} + \mathbf{e}_{463306331} + \mathbf{e}_{177012042} \} \end{aligned} \quad (17)$$

In the next subsection we will see the dynamical evolution spans a subspace of only $2m$ dimensions (the basis vectors of the initial state), the occupation of the space by the model is therefore rather scarce.

We conclude by recapitulating the consistent interpretation of each vector entry in the state functions in the general m -sentence case.

With respect to the ad hoc procedure (14), in column i , the state function of sentence i distinguishes $2m$ states of outcome typified by the index j according:

$1 \leq j \leq m - 1$, “Sentence i is true by inference according to its referent sentence” and “Sentence i is made hypothetically true after j inferences”

$m \leq j \leq 2(m - 1)$, “Sentence i is false by inference according to its referent sentence” and “Sentence i is made hypothetically false after $j + 1 - m$ inferences”

$j = 2m - 1$, “Sentence i is true by hypothesis”

$j = 2m$, “Sentence i is false by hypothesis”

Where the referent sentence of i is the sentence implying i , e.g. in Liar Paradox (2, b), ‘1’ is the referent sentence of ‘3’.

The respective projectors related to the detailed outcome states are simply the $2m \times 2m$ diagonal matrices with all elements zero, except unity at position (j, j) .

How does one ‘measure’ on the quantum model of a Liar Paradox? The reasoning on a Liar Paradox consists of two part-processes, ‘reading the sentence and inferring a sentence’s truth or falsehood’ according the intensional semantics of the subject sentence and ‘hypothesizing’, with eventual prior knowledge, truth or falsehood on that sentence. The reasoning process ‘compulsory’ continues by the repetition of this reading-inferring and hypothesizing act on the consecutive sentences. The initial reasoning starts by hypothesizing the truth value of a given sentence.

This means that in our description the Liar Paradox within the cognitive layer of reality is —before the measurement— not in a predictable true or false state. The ‘true state’ and the ‘false state’ of the sentence are specific states; eigenstates of the measurement projectors (13). In general, the state of the Liar Paradox is not one of these two eigenstates of a sentence. Due to the act of measurement, and in analogy with what happens during a quantum measurement, the state of the sentence changes (‘collapses’) into one of the two possible eigenstates, the ‘true by hypothesis state’ or the ‘false by hypothesis state’. This act of making a sentence true or false can be specifically described as ‘read the sentence, make the logical inference and hypothesize its truth or falsehood’. The compulsory consecutive reasoning is represented by the discrete unitary evolution operator which evolves a given state of sentence into its logically reasoned consecutive state.

We will expose this scheme for the 8-Liar Paradox (2,b), and see that an initial measurement followed by the sequence of logical inferences puts into work an oscillation dynamics that we can describe by a stepping evolution matrix, and eventually by a Schrödinger evolution over reasoning-time.

All discrete steps of reasoning on the generalised Liar Paradox of type (3) and (4) can be represented by a discrete $2m \times 2m$ evolution matrix U_D . The matrix U_D is conceived as a step matrix; i.e. with exactly one 1 on each row and column and all other elements identically zero. Such a step matrix is always equivalent to a basis transformation of the matrix with the elements of the lower off-diagonal and element $\{1, 2m\}$ equal to 1, or a similar one built on the higher off-diagonal

matrix.

$$U_D|_{\text{sub}} = \left(\begin{array}{cccccc} 0 & 0 & \dots & 0 & 1 & \\ 1 & 0 & \dots & 0 & 0 & \\ 0 & 1 & \dots & 0 & 0 & \\ \vdots & \vdots & \dots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & 0 & \end{array} \right) \Big|_{\text{base permutation}} \quad (18)$$

In the 8-Liar Paradox example (2) the discrete evolution submatrix, governing the reasoning evolution of sentences in both the tensor coupled and embedded description, is given by;

$$U_D|_{\text{sub}} = \left(\begin{array}{ccccc} 0 & \mathbf{1}_{6 \times 6} & 0 & \mathbf{0}_{6 \times 7} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \mathbf{0}_{7 \times 6} & \vdots & \mathbf{1}_{7 \times 7} & \vdots \\ 1 & \dots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \end{array} \right) \quad (19)$$

Notice the submatrix (index ‘sub’) discards all trivial dimensions from the description; only a $2m$ -dimensional subspace is actually employed in the evolution of m -sentences. This procedure can be applied in any finite dimension. While the m -sentence system has an exponentially increasing dimension of its description space, the relevant dynamics is still only taking place in a $2m$ -dimensional subspace, i.e. increasing linearly with the number of sentences.

2.3 Representation of Evolution: Continuous Reasoning

The reasoning on the m -sentence Liar Paradox is characterized by discrete moments of accumulated inferences-hypotheses till the completion of the full entity and observation of a contradictory truth-value for the initial sentence. The discreteness in the temporal process features explicitly in the logical reasoning; the evolution is characterized by the completion of consecutive inferences-hypotheses. The formalism of operational quantum mechanics allows a continuous time parameter of evolution. The introduction of continuous time in the model allows interpretation of intermediate states and qualitative duration of the reasoning on the Liar Paradox. Given the simplicity of the formal model no strict relation with psychological time is intended.

For the formal description at every instance of the time-ordering parameter, a procedure of diagonalization on the $2m \times 2m$ submatrix U_D is performed.

$$U_D|_{\text{diag}} = R U_D R^{-1} \quad (20)$$

where R is the $2m \times 2m$ diagonalization matrix.

The diagonalization procedure allows to solve the equation of the Schrödinger evolution operator with Stone’s Theorem —at ordering parameter equal to 1 unit of time— for the Hamiltonian:

$$H_{\text{sub}}|_{\text{diag}} = i \ln U_D|_{\text{diag}} \quad (21)$$

Inverting the procedure of diagonalization, the infinitesimal generator of the time-evolution —the submatrix hamiltonian— can be obtained.

$$H_{\text{sub}} = R^{-1} H_{\text{sub}}|_{\text{diag}} R \quad (22)$$

The unitary evolution submatrix can be constructed for any time τ (units of a single step in the reasoning). The continuity of the one-parameter group of unitary evolution operators allows intermediate moments of the time ordering parameter:

$$\forall \tau : \quad U_{\text{sub}}(\tau) = R^{-1} e^{i E_k \tau} g_{kk} R \quad (23)$$

where the E_k are the eigenvalues of the hamiltonian and g_{kk} is the $2m \times 2m$ diagonal unit matrix. The unitary evolution operator (eq. 23), the truth-falsehood projection operators (eq. 13), and the initial state Ψ_0 (eq. 17) allow the continuous time description of any m -sentence Liar Paradox of the type (3) and (4). The representation of the model in the tensor coupled space is straightforward, using the inverse function of the κ index (eq. 11).

In order to inspect qualitatively the reasoning evolution of the Liar Paradox, we make use of the truth and falsehood probabilities $P_{i,T}(\tau)$ and $P_{i,F}(\tau)$ respectively for each sentence i given the initial state —in short hand notation— of e.g. 1_T :

$$\begin{aligned} P_{i,T}(\tau) &= |\langle i_T | U(\tau) 1_T \rangle|^2 \\ P_{i,F}(\tau) &= |\langle i_F | U(\tau) 1_T \rangle|^2 \end{aligned} \quad (24)$$

In graphical representation it is easily seen how the probabilities evolve over time from a given truth value to their paradoxical opposite value.

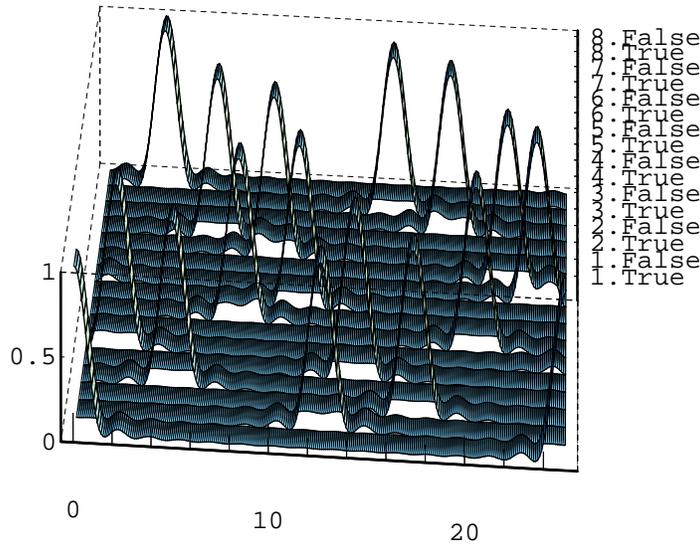


Figure 1: **Time evolution** of outcome probabilities for reasoning the 8-Liar Paradox (form. 2, b), with at $t = 0$, a ‘true’-measurement of sentence 1 on the initial state Ψ_0 . The ‘time’ t is an arbitrary continuous ordering parameter without physical interpretation. Logical contradiction is apparent after each interval $\Delta t = 8\frac{\pi}{2}$. The probability for a given outcome “sentence i is T/F at time t ” is obtained by taking the modulus squared of the scalar product of sentence of substate “ i is T/F ” and the initial state evolved till time t , i.e: $P(i.T/F, 1.T, t) = |\langle \psi_{i.T/F} | U(t) P_{1.T} \Psi_0 \rangle|^2$.

3 Conclusion

We have treated the Liar Paradox as a special case of a conceptual entity: a consistently testable configuration of truth properties expressed by sentences, and subject to our capacity of logical inference. The ‘contextuality’ of the reasoning process on the Liar Paradox — here provided by the logical conceptual network in the mind — allows the construction of an abstracted quantum

model. In the quantum model an initial hypothesis on a sentence engenders a time evolution of build up and collapse of logical states and eventually logically paradoxical content, without end. Evidently any real world reasoning on the Liar paradox does not expose this compulsory machine-like continuation of the process. Only the capacity of logical inference — here as a coercive internal context — has been accounted for. Ending the reasoning on the sentences needs the hypothesis of reestablishing the original superposition state of indefinite logical truth value.

Technically, we have found a procedure to solve in general the quantum mechanical modelling problem for the m sentence Liar Paradox. The formal model of truth behavior of the Liar Paradox needs to be constructed in a $(2m)^m$ dimensional Hilbert space. The exponential growth of the representation space is due to the demand of symmetric treatment of component sentences. The dynamical evolution of the Liar Paradox however spans only a subspace of $2m$ dimensions. The linear dimensional growth of the relevant space allows an adequate description of the Hamiltonian and unitary evolution operator. An indefinite state of the unreasoned Liar Paradox entity is obtained (eq. 17) and is time invariant when not reasoned on. The time-invariance of the initial state Ψ_0 (eq. 17) — $\Psi(0) = \Psi(\tau)$ — follows immediately from the fact that it is an eigenvector of the step matrix U_D (eq. 18).

What does the full quantum description of the Liar Paradox imply? The crucial feature making the Liar Paradox fit to be modeled by abstract quantum mechanics is its deterministic swaying to a coercive logical context, provided by the conceptual network of the mind.

The obtained model extends the static ‘entity + context’ configuration by providing a phenomenological dynamic. We have therefore been able in this case to introduce a time-propagator characterizing the concatenation of states of thought, albeit restricted to a non-evolving coercive logical context of logical inference. More general conceptual entities with variable internal context and environmental context will most certainly not fit a complete quantum model.

The appropriated autonomy of dynamics of the entity does not necessarily intend its ontological reality. This reading would be a literal interpretation of the physical analogy with the obtained complete quantum description. The cognitive person’s motivation of the entity by reasoning has been the reference for the dynamics of the conceptual entity. The temporal evolution of the entity is expected to originate intrinsically, when considered from the quantum mechanical analogy, while the construction mode of the evolution supposes the cognitive person’s motivation by reasoning. The indistinguishability of the ‘autonomous evolution of an ontological state’ and ‘coercive evolution by inner logical context’ in the model of the Liar Paradox can be interpreted as the mechanism that provides its *intentionality*. The conceptual entity Liar Paradox refers over time to the inner context of logical inference, which is indeed, in this particular case, sufficient ground for recognizing its intentionality.

Does the quantum model ‘solve’ some problems of self-reference in the Liar Paradox? When we attempt to understand the paradox of self-reference using classical logical categories, we are caught in a logical contradiction. In the present model this problem is avoided by the separation of incompatible logical values over time (Fig. 2). The model provides intrinsic contextuality to the Liar Paradox, distinguishing markedly its nature from conceptual entities with predetermined classical truth or falsehood values. The present model therefore suggests that the sphere of conceptual entities includes elements not subject to classical logical categories.

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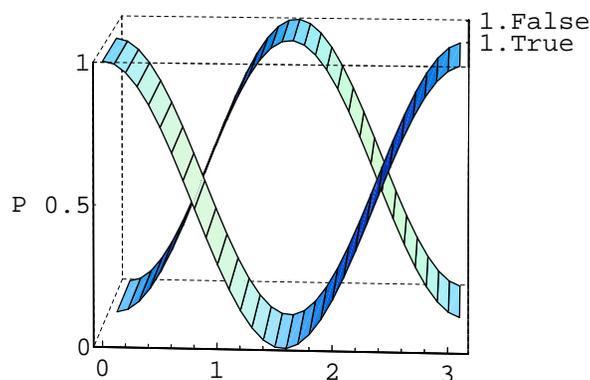


Figure 2: **Time evolution** of the 1-Liar Paradox (form 1). At $t = 0$ the entity is prepared in its True-state, the False-state is reached at $\Delta t = \frac{\pi}{2}$. Temporal separation of incompatible logical truth values circumvents logical paradox.

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Appendix: minimal dimension of sentence subspace

We prove the sentence vectors need $n = 2m$ dimensions in order not to have a degenerescence of states. In general we can write the action of the projectors (6, 7) on the initial state (5), $1 \leq i \leq m$:

$$T_i \Psi_0 = \sum_{j=1}^{j=n} \sum_{k_1=1}^{k_1=n} \dots \sum_{k_{i-1}=1}^{k_{i-1}=n} \sum_{k_{i+1}=1}^{k_{i+1}=n} \dots \sum_{k_m=1}^{k_m=n} \tau_{j_i} \alpha_{k_1 \dots j_{(i)} \dots k_m} \mathbf{e}_{k_1 \dots j_{(i)} \dots k_m}$$

Similarly for the false-projectors, $1 \leq i \leq m$;

$$F_i \Psi_0 = \sum_{j=1}^{j=n} \sum_{k_1=1}^{k_1=n} \dots \sum_{k_{i-1}=1}^{k_{i-1}=n} \sum_{k_{i+1}=1}^{k_{i+1}=n} \dots \sum_{k_m=1}^{k_m=n} \phi_{j_i} \alpha_{k_1 \dots j_{(i)} \dots k_m} \mathbf{e}_{k_1 \dots j_{(i)} \dots k_m}$$

This choice does not treat the meaning of the entries symmetrically over all sentence vectors in contrast to the choice (eqs. 13). For ease of proof the individual truth-falsehood projection operators are chosen such that simple indices occur on the outcome states. This can be done due to the equivalence with the case where specific base vector permutations give a unique matrix representation of the individual projection operator. We consider the constraints:

$$T_i \Psi_0 = t_i \mathbf{e}_{i \dots i \dots i} \quad (25)$$

$$F_i \Psi_0 = f_i \mathbf{e}_{m+i \dots m+i \dots m+i} \quad (26)$$

The coefficients t_i, f_i are restricted to: $0 < t_i \leq 1$ and $0 < f_i \leq 1$.

Taking into account the explicit expressions of the initial superposition state and the projection operators, these constraints lead to, $i \in 1, \dots, m$;

$$\begin{aligned} t_i \mathbf{e}_{i \dots i \dots i} &= \sum_{j=1}^{j=n} \sum_{k_1=1}^{k_1=n} \dots \sum_{k_{i-1}=1}^{k_{i-1}=n} \sum_{k_{i+1}=1}^{k_{i+1}=n} \dots \sum_{k_m=1}^{k_m=n} \tau_{j_i} \alpha_{k_1 \dots j_{(i)} \dots k_m} \mathbf{e}_{k_1 \dots j_{(i)} \dots k_m} \\ f_i \mathbf{e}_{m+i \dots m+i \dots m+i} &= \sum_{j=1}^{j=n} \sum_{k_1=1}^{k_1=n} \dots \sum_{k_{i-1}=1}^{k_{i-1}=n} \sum_{k_{i+1}=1}^{k_{i+1}=n} \dots \sum_{k_m=1}^{k_m=n} \phi_{j_i} \alpha_{k_1 \dots j_{(i)} \dots k_m} \mathbf{e}_{k_1 \dots j_{(i)} \dots k_m} \end{aligned}$$

And due to the orthogonality of the base vectors:

$$\begin{aligned} \forall i, j, k : i \leq m, j_{(i)} = i, k = i & \quad t_i = \tau_i \alpha_{i \dots i \dots i} \\ \forall i, j, k : i \leq m, \text{not } \{j_{(i)} = i, k = i\} & \quad 0 = \tau_{j_i} \alpha_{k_1 \dots j_{(i)} \dots k_m} \\ \forall i, j, k : i \leq m, j_{(i)} = m+i, k = m+i & \quad f_i = \phi_{m+i} \alpha_{m+i \dots m+i \dots m+i} \\ \forall i, j, k : i \leq m, \text{not } \{j_{(i)} = m+i, k = m+i\} & \quad 0 = \phi_{j_i} \alpha_{k_1 \dots j_{(i)} \dots k_m} \end{aligned}$$

Where the simplified notation has been used $\tau_i \equiv \tau_{ii}$ and $\phi_{m+i} \equiv \phi_{m+i, m+i}$. These equations have, taking into account condition (8), the unique solution;

$$\begin{aligned} \forall i, j, k : i \leq m, j_{(i)} = i, k = i & \quad \tau_i = 1 \\ & \quad \alpha_{i \dots i \dots i} = t_i \end{aligned}$$

$$\begin{aligned}
\forall i, j, k : i \leq m, j_{(i)} = m + i, k = m + i & \quad \phi_{m+i} = 1 \\
& \quad \alpha_{m+i\dots m+i\dots m+i} = f_i \\
\forall i, j, k : \forall i \leq m, \text{not } \{j_{(i)} = i, k = i\} & \quad \tau_{j_i} = 0 \\
& \quad \alpha_{k_1\dots j_{(i)}\dots k_m} = 0 \\
\forall i, j, k : \forall i \leq m, \text{not } \{j_{(i)} = m + i, k = m + i\} & \quad \phi_{j_i} = 0 \\
& \quad \alpha_{k_1\dots j_{(i)}\dots k_m} = 0
\end{aligned}$$

Having found a solution in a Hilbert space with $(2m)^m$ dimensions, we check whether a lesser dimension is adequate to represent the model. Strictly a space with $(2m)^m - 1$ dimensions should be checked for the consistency of the $2m$ constraints. Because individual sentences should not be distinct, we lower the dimension to the symmetric case of $(2m - 1)^m$ dimensions, i.e. assigning $n = 2m - 1$ dimensions to each sentence subspace.

We choose the first $2m - 1$ conditions identically as in the previous case, and complete the set with one more constraint where we are obliged to re-introduce at least m component base vectors indices, e.g.:

$$\begin{aligned}
\forall i \in \{1, \dots, m\} : & \quad T_i \Psi_0 = t_i \mathbf{e}_{i\dots i\dots i} \\
\forall i \in \{1, \dots, m - 1\} : & \quad F_i \Psi_0 = f_i \mathbf{e}_{m+i\dots m+i\dots m+i} \\
& \quad F_m \Psi_0 = f_m \mathbf{e}_{1\dots 12}
\end{aligned}$$

Actually the inevitable reintroduction of at least two indices of base vectors will lead finally to contradiction in the constraints.

The explicit expressions of the initial superposition state and the projection operators in the constraints lead to:

$$\forall i, j, k : i \leq m$$

$$j_{(i)} = i, k = i \quad t_i = \tau_i \alpha_{i\dots i\dots i} \quad (27)$$

$$\text{not } \{j_{(i)} = i, k = i\} \quad 0 = \tau_{j_i} \alpha_{k_1\dots j_{(i)}\dots k_m} \quad (28)$$

$$\forall i, j, k : i \leq m - 1$$

$$j_{(i)} = m + i, k = m + i \quad f_i = \phi_{i m+i} \alpha_{m+i\dots m+i\dots m+i} \quad (29)$$

$$\text{not } \{j_{(i)} = m + i, k = m + i\} \quad 0 = \phi_{j_i} \alpha_{k_1\dots j_{(i)}\dots k_m} \quad (30)$$

and

$$f_m = \phi_{2m} \alpha_{1\dots 12} \quad (31)$$

$$\forall j, k : \{k_1, \dots, k_{m-1}, j_{(m)}\} \neq \{1, \dots, 1, 2\} \quad 0 = \phi_{j_m} \alpha_{k_1\dots j_{(m)}} \quad (32)$$

From equation (31) and (8) we obtain

$$\phi_{2m} = 1 \quad \text{and} \quad \alpha_{1\dots 12} = f_m \quad (33)$$

next, consider equation (27) with $i = 1$: $\tau_1 \cdot \alpha_{1\dots 1} = t_1$. Which implies with equation (8):

$$\tau_1 = 1 \quad \text{and} \quad \alpha_{1\dots 1} = t_1 \quad (34)$$

Finally we put $i = j = k = 1$ and $k_m = 2$ in equation (28) to give: $\tau_1 \cdot \alpha_{1\dots 12} = 0$. Which implies:

$$\tau_1 = 0 \quad \text{or} \quad \alpha_{1\dots 12} = 0 \quad (35)$$

Conditions (33, b), (34, a) contradict (35). The $2m$ constraints are therefore too restrictive for the proposed dimension $(2m - 1)^m$ of the Hilbert space.

The model of the conceptual entity will therefore need to be constructed in a $(2m)^m$ dimensional Hilbert space.