THE ROLE OF THE MODULAR PAIRS IN THE CATEGORY OF COMPLETE ORTHOMODULAR LATTICE

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ABSTRACT. We study the modular pairs of a complete orthomodular lattice i.e. a CROC. We propose the concept of m-morphism as a mapping which preserves the lattice structure, the orthogonality and the property to be a modular pair. We give a characterization of the m-morphisms in the case of the complex Hilbert space to justify this concept.

INTRODUCTION

There are many formulations of quantum physics, the Hilbert-space formulation, the phase-space formulation, the C^*-algebra formulation, etc. All these formulations are equivalent and which one to choose for formulating and solving a particular problem depends on the nature of the problem at hand. It is thus important to be able to formulate the problem in terms independent of this choice. In pure mathematics we encounter the same situation. For example in algebraic geometry we like to define the objects independent of the coordinate field. This kind of difficulty was solved by the use of the concept of the category. So we will define the proper category corresponding to the objects describing a physical system. It is well known that these objects are the complete orthomodular lattices. The point is, however, to choose the best notion of morphism, such that we are able to describe in this category all the physical notions that we encounter in quantum physics and not more if possible. In this paper, we propose a notion of morphism called m-morphism to play this role and we discuss why such a notion is particularly well adapted to the physical problems.

THE ORTHOMODULAR LATTICES AND THE MODULAR PAIRS

Let us first recall the definition of the complete orthomodular lattices (see [1]), which are the

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objects of this category and discuss the properties of some particular pairs of elements of such lattices, the modular pairs, which play an important role in the notion of m-morphisms and in the classification of the complete orthomodular lattices.

DEFINITION 1. *A complete orthomodular lattice $\mathcal{L}$, i.e., a CROC, is a partially-ordered set such that:*

(a) *It is a complete lattice, i.e., for each family of elements $b_i \in \mathcal{L}$ there exists in $\mathcal{L}$ a greatest lower bound denoted by $\wedge_i b_i$ and a least upper bound denoted by $\vee_i b_i$.*

(b) *An involution is given which maps each element $b \in \mathcal{L}$ on another element of $\mathcal{L}$ denoted by b', in such a way that:*

$$b'' = b, \quad b \wedge b' = 0 \text{ and } b \vee b' = I, \quad a < b \Rightarrow b' < a', \quad \forall a, b \in \mathcal{L}.$$

0 *is the minimal element and* I *the maximal element of $\mathcal{L}$. In other words $\mathcal{L}$ is orthocomplemented.*

(c) *$\mathcal{L}$ is weakly modular:*

$$a < b \Rightarrow (b \wedge a') \vee a = b, \qquad \forall a, b \in \mathcal{L}.$$

$\mathcal{P}(\Gamma)$ the subsets of a set Γ ordered by the inclusion relation form a CROC for the involution which maps a subset on its complement. Such a CROC is distributive:

$$a \wedge (b \wedge c) = (a \wedge b) \vee (a \wedge c) \qquad \forall a, b, c \in \mathcal{P}(\Gamma).$$

$\mathcal{P}(\mathcal{H})$ the closed subspaces of a Hilbert space $\mathcal{H}$ ordered by the inclusion relation form a CROC for the involution which maps a closed subspace on its orthogonal complement. Such a CROC is not distributive except in dimension one (or zero) where the distributivity law is trivial.

In a partially-ordered set we call an atom an element which covers the minimal element 0 i.e. an element p such that $0 < x < p \Rightarrow x = 0$ or $x = p$. A CROC $\mathcal{L}$ is said to be atomic if for each $a \neq 0$ element of $\mathcal{L}$ there exists an atom $p < a$. $\mathcal{L}$ satisfies the covering law if for each $a \in \mathcal{L}$ and p an atom of $\mathcal{L}$, and such that $p \wedge a = 0$, $p \vee a$ covers a. A CROC which is atomic and satisfies the covering law is called a propositional system. $\mathcal{P}(\Gamma)$ as well as $\mathcal{P}(\mathcal{H})$ are propositional systems. An element of a propositional system is called a proposition and corresponds in principle to a well defined property of the physical system. In this interpretation $\mathcal{P}(\Gamma)$ describes a classical physical system and $\mathcal{P}(\mathcal{H})$, as is well known, a simple quantum physical system. To characterize the difference between propositional systems like $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\mathcal{H})$ it is useful to consider some particular pairs of propositions. Let us give the following definitions:

DEFINITION 2. *In a CROC $\mathcal{L}$ two elements a and b are said to be compatible, ($a \leftrightarrow b$) if and only if the sublattice generated in $\mathcal{L}$ by $\{a, b, a', b'\}$ is distributive.*

In $\mathcal{P}(\Gamma)$ all the pairs are compatible. In $\mathcal{P}(\mathcal{H})$ two closed subspaces are compatible if and only if the corresponding orthogonal projectors commute.

DEFINITION 3. *In a lattice* $\mathcal{L}$ *an ordered pair* (a, b) *of elements is said to be a modular pair,* $((a, b)M)$ *if and only if we have* [2] :

$$x \wedge (a \vee b) = (x \wedge a) \vee b \qquad \forall x \in \mathcal{L} \text{ such that } b < x.$$

THEOREM 4. *In a CROC if* $a \leftrightarrow b$ *then* $(a, b)M$.

Proof. Choose $x \in \mathcal{L}$ such that $b < x$, then $b \leftrightarrow x$ (see [1], Theorem (2.19)) and $x \wedge (a \vee b) = (x \wedge a) \vee b$ because, since we have $b \leftrightarrow x$ and $b \leftrightarrow a$ we can apply the distributivity law (see [1], Theorem (2.25)). □

The converse of Theorem 4 is, in general, not true. In particular, in $\mathcal{P}(\mathcal{H})$ all pairs are modular if $\mathcal{H}$ is finite-dimensional but not all pairs are compatible except in dimension one (or zero). When all pairs are modular the lattice is said to be modular. It was discovered by G. Birkhoff and J. von Neumann in the complex case [3] that $\mathcal{P}(\mathcal{H})$ is never modular when $\mathcal{H}$ is infinite-dimensional [4]. As we will see, it is possible with the concept of modular pairs to characterize the propositional systems. The weak modularity for example is trivially equivalent to the condition $(a'a)M$ $\forall a \in \mathcal{L}$. If we call a lattice $\mathcal{L}$ semimodular if and only if $(a, b)M \Rightarrow (b, a)M$ $\forall a, b \in \mathcal{L}$ [5] then

THEOREM 5. *An atomic CROC is a propositional system if and only if it is semimodular.*

Proof. Let $\mathcal{L}$ be a semimodular atomic CROC, p an atom and b an element of $\mathcal{L}$ such that $a \wedge p = 0$. We have to prove that $a \vee p$ covers a. For each $x \in \mathcal{L}$ it is trivial to verify that $(x, p')M$. By semimodularity we infer $(p', x)M$. Then if $a' \wedge p' < x < a'$ we find $a' \wedge (p' \vee x) = (a' \wedge p') \vee x = x$. Since I covers p', $p' \vee x$ is I or p'. If $p' \vee x = I$, then $a' = x$ and if $p' \vee x = p'$ we have $a' \wedge p' = x$. This proves that a' covers $a' \wedge p'$ which is equivalent to $a \vee p$ covers a. The converse is not so straightforward but it is a trivial consequence of Theorem 6 below. □

Before giving this theorem, we must, however, recall an important result (see [1], Theorem 3.14) which establishes a correspondence between propositional systems and projective geometries. If we call any atom of a propositional system a point and the subset of all the atoms less than any least upper bound of two distinct atoms a line we define in this manner a projective geometry. The propositions of the propositional system are canonically imbedded in the linear varieties of this projective geometry. A linear variety is a subset of points such that whenever it contains two distinct points it also contains the line defined by these two points. The set of linear varieties ordered by inclusion forms a complete lattice.

THEOREM 6. *In a propositional system* $\mathcal{L}$, *a pair* (a, b) *is a modular pair if and only if* $a \vee b = a + b$, *where* $a + b$ *is the linear variety generated by* a *and* b.

Proof. Suppose that $a, b \in \mathcal{L}$ such that $a \vee b = a + b$ which means that an atom is smaller than $a \vee b$ if and only if it is in $a + b$. We have to show that for $x \in \mathcal{L}$ such that $b < x$, every atom smaller than $x \wedge (a \vee b)$ is smaller than $(x \wedge a) \vee b$. Take q to be an atom smaller than $x \wedge (a \vee b)$ then q is on a line defined by an atom $p < a$ and an atom $r < b$ and since $b < x$, $r < x$. Since $q < x$ and $r < x$ every atom of this line is smaller than x and in particular $p < x$. Since $p < a$ we also have that $p < x \wedge a$. Hence q, being on a line defined by an atom $p < x \wedge a$ and an atom $r < b$, is smaller than $(x \wedge a) \vee b$. If q is smaller than a or b, the line is not defined, but then the result is trivial

For the converse let us prove that if $a \vee b \neq a + b$ then (a, b) is not a modular pair. Choose an atom $q < a \vee b$ which is not on a line defined by an atom smaller than a and an atom smaller than b. Then $(b \vee q) \wedge a = b \wedge a$ because $b \vee q = b + q$ (see [1], 3.16). If we choose $x = b \vee q$ we obtain

$$(x \wedge a) \vee b = (b \wedge a) \vee b = b \quad \text{and} \quad x \wedge (a \vee b) = x = b \vee q$$

which proves that the pair (a, b) is not a modular pair. □

This also proves that the relation $(a, b)M$ is symmetrical in $\mathcal{L}$, i.e., $\mathcal{L}$ is semimodular. Theorem 6 is a generalisation of a result due to G.W. Mackey (see [2], Theorem III.6).

Now we want to give some characterizations of modular pairs and for this we will first prove three lemmas which will enable us to restrict the study to some particular pairs.

LEMMA 7. *If a, b, c are elements of a lattice $\mathcal{L}$ such that $(a, b)M$ and $(a \vee b, c)M$ then we have also $(a_1, b \vee c)M$ for all a_1 such that $a < a_1 \vee b$.*

Proof. Let $x \in \mathcal{L}$ such that $b \vee c < x$, then from the hypothesis we can write:

$$x \wedge (a_1 \vee b \vee c) = (x \wedge (a \vee b)) \vee c = (x \wedge a) \vee b \vee c < (x \wedge a_1) \vee b \vee c$$

and the other inequality is trivial. □

Let $\mathcal{L}$ be a propositional system. We have seen that $x \vee p = x + p$ if p is an atom. Thus Lemma 7 implies that $(a, b \vee p)M$ when $(a, b)M$.

LEMMA 8. *If a, b, c are elements of a CROC $\mathcal{L}$ such that $(a, b)M$, $c \leftrightarrow a$ and $c \leftrightarrow b$ then we also have $(c \wedge a, c \wedge b)M$.*

Proof. Let $x \in \mathcal{L}$ such that $c \wedge b < x$. Since $c \leftrightarrow a$ and $c \leftrightarrow b$ we can apply the distributivity law and write

$$x \wedge ((c \wedge a) \vee (c \wedge b)) = x \wedge c \wedge (a \vee b).$$

By the same kind of argument we can prove

$$x \wedge c = c \wedge ((x \wedge c) \vee b).$$

Since by hypothesis $(a, b)M$, we therefore find

$$x \wedge ((c \wedge a) \vee (c \wedge b)) = c \wedge ((x \wedge c) \vee b) \wedge (a \vee b)$$

$$= c \wedge [(((x \wedge c) \vee b) \wedge a) \vee b] = (x \wedge c \wedge a) \vee (b \wedge c). \quad □$$

LEMMA 9. *If a and b are elements of a CROC $\mathcal{L}$, then the three following conditions are equivalent:*

$$(a, b)M, \quad ((a' \vee b') \wedge a, b)M \quad \textit{and} \quad (a, b \vee (a' \wedge b'))M.$$

Proof. Since $a' \vee b' \leftrightarrow a$ and $a' \vee b' \leftrightarrow b$

$$(a, b)M \Rightarrow ((a' \vee b') \wedge a,\ \ (a' \vee b') \wedge b)M$$

follows from Lemma 8.

Since $((a' \vee b') \wedge a) \vee ((a' \vee b') \wedge b) = (a' \vee b') \wedge (a \vee b)$ is compatible with $a \wedge b$, from Theorem 4 we have $((a' \vee b') \wedge (a \vee b),\ a \wedge b)M$. Therefore by Lemma 7

$$((a' \vee b') \wedge a,\ (a' \vee b') \wedge b)M \Rightarrow ((a' \vee b') \wedge a, b)M$$

because $((a' \vee b') \wedge b) \vee (a \wedge b) = b$.

Since $[(a' \vee b') \wedge a] \vee b = a \vee b$ by Lemma 7 we have

$$((a' \vee b') \wedge a, b)M \Rightarrow (a, b)M.$$

Since by Theorem 4 we have $(a \vee b,\ a' \wedge b')M$ then according to Lemma 7

$$(a, b)M \Rightarrow (a, b \vee (a' \wedge b'))M$$

and by Lemma 8 we find

$$(a, b \vee (a' \wedge b'))M \Rightarrow (a, b)M$$

since $a \wedge (a \vee b) = a$ and $(b \vee (a' \wedge b')) \wedge (a \vee b) = b$. □

In conclusion of the last lemma, for the study of the modularity we can restrict ourselves to the pairs (a, b) such that $a \wedge b = O$ and $a \vee b = I$.

The following theorem provides a criterion for a pair to be a modular pair.

THEOREM 10. *Let a and b be two elements of a CROC $\mathcal{L}$ such that $a \wedge b = O$ and $a \vee b = I$ and consider the projection $\emptyset x = (x \vee b) \wedge b'$ which applies the segment $[O, a]$ (the subset of all $x \in \mathcal{L}$ such that $x < a$) into the segment $[O, b']$. We have: $(a, b)M$ if and only if $\emptyset$ is surjective and $(b', a')M$ if and only if $\emptyset$ is injective.*

Proof. Suppose $(a, b)M$ and take $y \in [O, b']$. Define $x = (y \vee b) \wedge a$, then $\emptyset x = (((y \vee b) \wedge a) \vee b) \wedge b' = (y \vee b) \wedge b' = y$.

Let us prove the converse. Suppose $\emptyset$ is surjective and take $z \in \mathcal{L}$ such that $b < z$. We have to prove $z < (z \wedge a) \vee b$. Since $z \wedge b' \in [O, b']$, there exists a $y \in [O, a]$ such that $\emptyset y = z \wedge b'$. Since $b < z$ we have $z = (z \wedge b') \vee b = ((y \vee b) \wedge b') \vee b = y \vee b$ and because $y < z \wedge a$ we see that $z = y \vee b < (z \wedge a) \vee b$.

Suppose now $(b', a')M$ and take $x, y \in [O, a]$ such that $(x \vee b) \wedge b' = (y \vee b) \wedge b'$. This gives us also $x \vee b = y \vee b$ and $(x \vee b) \wedge a = (y \vee b) \wedge a$ which implies $(x' \wedge b') \vee a' = (y' \wedge b') \vee a'$. Since $(b', a')M$ and $x' > a'$ and $y' > a'$ we find $(x' \wedge b') \vee a' = x'$ and $(y' \wedge b') \vee a' = y'$ which shows us that $x = y$.

Conversely suppose that ϕ is injective. We have to prove that if $z > a'$, then $z = (z \wedge b') \vee a'$. Since $z' < (z' \vee b) \wedge a$ we have $\phi z' < \phi(z' \vee b) \wedge a$. We also have

$$\phi(z' \vee b) \wedge a = (((z' \vee b) \wedge a) \vee b) \wedge b' < (z' \vee b) \wedge b' = \phi z',$$

so $\phi z' = \phi(z' \vee b) \wedge a$ which implies that $z' = (z' \vee b) \wedge a$ or $z = (z \wedge b') \vee a'$. □

Let us take $a, b \in \mathcal{L}$ such that $a \wedge b = O$ and $a \vee b \neq I$. We can immediately apply Theorem 10 to the segment $[O, a \vee b]$ which is itself a propositional system. Thus we find the following generalisation: If $a \wedge b = O$ then $(a, b)M$ if and only if the mapping $\phi x = (x \vee b) \wedge b'$ is surjective from $[O, a]$ onto $[O, (a \vee b) \wedge b']$ and $(b', a')M$ if and only if ϕ is injective from $[O, a]$ into $[O, (a \vee b) \wedge b']$.

We will say that a CROC $\mathcal{L}$ is O-symmetric if and only if $(a, b)M \Rightarrow (b', a')M \;\; \forall a, b \in \mathcal{L}$. Since by Lemma 9 $(a, b)M$ is equivalent to $(a \wedge (a' \vee b'),\; b \vee (a' \wedge b'))M$ and $(b', a')M$ is equivalent to $(b' \wedge (a \vee b),\;\; a' \vee (a \wedge b))M$, $\mathcal{L}$ is O-symmetric if and only if $(a, b)M \Rightarrow (b', a')M \;\; \forall a, b \in \mathcal{L}$ such that $a \wedge b = O$ and $a \vee b = I$, or also according to the last theorem if and only if ϕ is injective whenever it is surjective or vice versa. This is the case for $\mathcal{P}(\mathcal{H})$, the lattice of the closed subspaces of a Hilbert space, according to the appendix since $\epsilon(a, b) = \epsilon(a', b')$.

Theorem 10 seems to be new (see [5], Theorem 38.7, p. 178) but we do not know if every propositional system is O-symmetric. However as a result due to Schreiner [6] the converse is true, more precisely.

THEOREM 11. *An O-symmetric orthocomplemented lattice $\mathcal{L}$ is semimodular.*

Proof. Let $a, b \in \mathcal{L}$ be such that $(a, b)M$. By Lemma 7 this implies $(c, b)M$ for all $c \in \mathcal{L}$ such that $a < c < a \vee b$. Hence $(b'c')M$ for all c' such that $a' \wedge b' < c' < a'$ since by assumption $\mathcal{L}$ is O-symmetric. This implies $a' \wedge (b' \vee c') = (a' \wedge b') \vee c'$ and by duality $c \wedge (b \vee a) = (c \wedge b) \vee a$ for all c such that $a < c < a \vee b$. Hence $(b, a)M$ since for all $x \in \mathcal{L}$ such that $a < x$,

$$x \wedge (b \vee a) = x \wedge (b \vee a) \wedge (b \vee a) = (x \wedge (b \vee a) \wedge b) \vee a = (x \wedge b) \vee a.$$ □

THE m-MORPHISM

DEFINITION 12. *A mapping μ from a CROC $\mathcal{L}_1$ into a CROC $\mathcal{L}_2$ is an m-morphism whenever:*

(a) $\mu(\vee_i b_i) = \vee_i \mu b_i$,

(b) $a < b' \Rightarrow \mu a < (\mu b)'$,

(c) $(a, b)M \Rightarrow (\mu a, \mu b)M$.

From the conditions (a) and (b) of Definition 12 it follows that an m-morphism is a c-morphism in the sense of [1], p. 29, and that:

$$\mu(\wedge_i b_i) = \wedge_i \mu b_i, \qquad (\mu b)' = \mu(b') \wedge \mu I_1, \qquad a \leftrightarrow b \Rightarrow \mu a \leftrightarrow \mu b.$$

Moreover ker μ the kernel of μ is a segment $[O_1, z_1]$ where z_1 is in the center of $\mathcal{L}_1$ (the subset of the propositions which are compatible with all propositions of $\mathcal{L}_1$).

LEMMA 13. *Let $\mathcal{L}$ be a* CROC *and* $[O, c]$ *a segment in $\mathcal{L}$ then the canonical injection of* $[O, c]$ *in $\mathcal{L}$ is an m-morphism.*

Proof. It is easy to verify (see [1], Theorem 2.31) that $[O, c]$ is a CROC for the involution (the orthocomplementation) given by $x \mapsto x^r = x' \wedge b$ and thus the canonical injection satisfies conditions (a) and (b) of Definition 12. Let (a, b) be a modular pair for $[O, c]$ i.e., $x \wedge (a \vee b) = (x \wedge a) \vee b$ for all $x \in \mathcal{L}$ such that $b < x < c$ and $a \vee b < c$ then $(a, b)M$ since for all $y \in \mathcal{L}$ such that $b < y$:

$$y \wedge (a \vee b) = y \wedge c \wedge (a \vee b) = (y \wedge c \wedge a) \vee b = (y \wedge a) \vee b. \qquad \square$$

THEOREM 14. *The* CROC*'s and the m-morphisms define a category which has the following properties:*

(a) *There exists a zero object.*

(b) *The direct product always exists.*

(c) *Any m-morphism μ factorizes:*

$$\begin{array}{ccc} \mathcal{L}_1 & \xrightarrow{\ \mu\ } & \mathcal{L}_2 \\ \pi \downarrow & & \uparrow j \\ \mathcal{L}_1/\ker \mu & \leftrightarrow & \operatorname{Im} \mu \end{array}$$

where j is the canonical injection and π is the projection $x_1 \mapsto x_1 \wedge z_1'$ for $\ker \mu = [O, z_1]$.

Moreover, if $\mu : \mathcal{L}_1 \to \mathcal{L}_2$ is injective and $\mathcal{L}_2$ semimodular (O*-symmetric*) *then $\mathcal{L}_1$ is semimodular* (O*-symmetric*).

Proof. This is a category since the composition of two m-morphisms is an m-morphism.

(a) The zero object is $\mathscr{P}(\phi)$ the CROC which contains only one element $O = I$.

(b) The direct product of a family $\mathcal{L}_\alpha$ of CROC's is nothing else than what is called in lattice theory, the direct union, i.e., the CROC $\vee_\alpha \mathcal{L}_\alpha$ obtained by taking the set of all families $\{x_\alpha\}$ where $x_\alpha \in \mathcal{L}_\alpha \quad \forall\alpha$, with the ordering $\{x_\alpha\} < \{y_\alpha\}$ defined by $x_\alpha < y_\alpha \quad \forall\alpha$ and the orthocomplementation $\{x_\alpha\} \mapsto \{x_\alpha\}' = \{x_\alpha'\}$. It is easily verified that $\{x_\alpha\} \leftrightarrow \{y_\alpha\}$ if and only if $x_\alpha \leftrightarrow y_\alpha \quad \forall\alpha$ and that $(\{x_\alpha\}, \{y_\alpha\})M$ if and only if $(x_\alpha, y_\alpha)M \quad \forall\alpha$. Furthermore, given a family of m-morphisms $\mu_\alpha : \mathcal{L} \to \mathcal{L}_\alpha$ then the mapping $x \to \{\mu_\alpha x\}$ from $\mathcal{L}$ into $\vee_\alpha \mathcal{L}_\alpha$ is an m-morphism μ such that:

$$\pi_{\alpha_0} \mu = \mu_{\alpha_0} \quad \forall\alpha_0$$

where π_{α_0} is the projection $\{x_\alpha\} \mapsto x_{\alpha_0}$.

(c) According to (b) this is trivial, since $\mathcal{L}$ in this case is the direct union of $[O, z_1]$ and $[O, z_1']$.

If μ is an injective m-morphism it is obvious from (c) that $(a, b)M \Leftrightarrow (\mu a, \mu b)M$. $\square$

The advantage of working in this category comes from the fact that an atomic sub-CROC of a propositional system is automatically a propositional system since by Theorem 14 it is semi-modular. On the other hand, the definition of an observable as c-morphism from a distributive CROC to a propositional system (see [1], p. 38) is not modified since by Theorem 4 such a c-morphism is always an m-morphism.

Let us consider the particular case of two propositional systems $\mathscr{P}(\mathcal{H}_1)$ and $\mathscr{P}(\mathcal{H}_2)$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are complex Hilbert spaces. Each isomorphism from $\mathscr{P}(\mathcal{H}_1)$ to $\mathscr{P}(\mathcal{H}_2)$ is induced according to Wigner theorem by a unitary or anti-unitary transformation (see [1], Theorem 3.28, for a generalization) of $\mathcal{H}_1$ on $\mathcal{H}_2$. For an m-morphism which is not an isomorphism we have the following characterization:

THEOREM 15. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces of dimensions $\geqslant 3$. Give a decomposition $\mathcal{H}_2 = \oplus_i V_i$ in orthogonal subspaces and for each i an isomorphism ϕ_i from $\mathscr{P}(\mathcal{H}_1)$ to $\mathscr{P}(V_i)$ then the mapping μ from $\mathscr{P}(\mathcal{H}_1)$ to $\mathscr{P}(\mathcal{H}_2)$ defined by $\mu a = \oplus_i \phi_i a$ is an m-morphism. Conversely each m-morphism from $\mathscr{P}(\mathcal{H}_1)$ to $\mathscr{P}(\mathcal{H}_2)$ which is not identically O_2 is of this form.*

Proof. It is easy to verify that such a mapping is a c-morphism i.e., satisfies the conditions (a) and (b) of Definition 12. Let us prove the condition (c). Take $a, b \in \mathscr{P}(\mathcal{H}_1)$ such that $(a, b)M$ then by Theorem 6:

$$\mu a \vee \mu b = \mu(a \vee b) = \mu(a + b) = \oplus_i \phi_i(a + b) = \oplus_i (\phi_i a + \phi_i b)$$

and from Lemma 9 we restrict ourselves to the case $a \wedge b = O_1$.

Take now $z \in \mathcal{H}_2$, hence $z = \Sigma_i z_i$ where $z_i \in V_i$. If $z \in \oplus_i (\phi_i a + \phi_i b)$ then $z_i \in \phi_i a + \phi_i b$ and $z_i = x_i + y_i$ with $x_i \in \phi_i a$ and $y_i \in \phi_i b$. Because $(a, b)M$, $\epsilon(a, b) \equiv \sup\{|\langle x, y\rangle|$, if $x \in a, y \in b$ and $\|x\| = \|y\| = 1\} < 1$ (see the Appendix).

Since every ϕ_i is induced by unitary or anti-unitary transformations $\epsilon(\phi_i a, \phi_i b) = \epsilon(a, b) \equiv \epsilon$ and since

$$\|x_i\|^2 \leqslant \frac{1}{1-\epsilon^2}\|z_i\|^2 \quad \text{and} \quad \|y_i\|^2 \leqslant \frac{1}{1-\epsilon^2}\|z_i\|^2 .$$

This proves $\Sigma_i x_i \in \oplus_i \phi_i a$, $\Sigma_i y_i \in \oplus_i \phi_i b$ and $z = \Sigma_i x_i + \Sigma_i y_i$.

Hence, $\oplus_i \phi_i(a + b) = \oplus_i \phi_i a + \oplus_i \phi_i b = \mu a + \mu b$ and by Theorem 6, $(\mu a, \mu b)M$. The converse is a deep result which is the main subject of [7]. □

From Theorem 16 we see that the notion of m-morphism is strong enough to allow us to translate any physical problem given in terms of CROC's into the language of operators in Hilbert space (see for example [8, 9]).

APPENDIX

If $\mathcal{H}$ is a complex Hilbert space and $a, b \in \mathscr{P}(\mathcal{H})$ such that $a \wedge b = O$ then $(a, b)M \Leftrightarrow \epsilon(a, b) < 1$ [10].

Proof. Suppose that $\epsilon(a, b) < 1$. From Theorem 6, we know that to prove that $(a, b)M$ it is sufficient to prove that $a + b$ is a closed subspace of $\mathcal{H}$. For this, take $z \in \overline{a + b}$, then $z = \lim z_n$ where $z_n = x_n + y_n$ with $x_n \in a$ and $y_n \in b$.

We also have:

$$\|z_n\|^2 = \|x_n + y_n\|^2 = \|x_n\|^2 + \|y_n\|^2 + 2\text{Re}\,\langle x_n, y_n\rangle$$
$$\leqslant \|x_n\|^2 + \|y_n\|^2 + 2\epsilon(a, b)\,\|x_n\|\,\|y_n\|$$

hence $0 \leqslant \|x_n\|^2 + \|y_n\|^2 + 2\epsilon(a, b)\,\|x_n\|\,\|y_n\| - \|z_n\|^2$. This inequality will only be fulfilled if we have

$$\|x_n\|^2 \leqslant \frac{\|z_n\|^2}{1 - \epsilon^2(a, b)} \quad \text{and} \quad \|y_n\|^2 \leqslant \frac{\|z_n\|^2}{1 - \epsilon^2(a, b)}$$

Thus, since $(z_n)_n$ is a Cauchy sequence, $(x_n)_n$ and $(y_n)_n$ will also be Cauchy sequences. Put $x = \lim z_n$ and $y = \lim y_n$, then $z = x + y$ with $x \in a$ and $y \in b$ and so $z \in a + b$.

Suppose now that $\epsilon(a, b) = 1$. We will prove that in this case $\overline{a + b} \neq a + b$, such that a and b cannot be a modular pair (Theorem 6). Indeed, if $\epsilon(a, b) = 1$, there exists a sequence of pairs of vectors (x_n, y_n), such that $x_n \in a$, $y_n \in b$ and $\lim \langle x_n, y_n\rangle = 1$. We construct this sequence in such a way that (x_n) and (y_n) are orthonormal sequences, such that $\langle x_n, y_m\rangle = 0$ if $m \neq n$ and $\langle x_n, y_n\rangle \geqslant 1 - 2^{-n-1}$. If $z_n = x_n - y_n$ we have: $\|z_n\|^2 \leqslant 2^{-n}$ and $\langle z_n, z_m\rangle = 0$ if $n \neq m$. From this we see that $\Sigma_n z_n$ converges, put $z = \Sigma_n z_n$. We can see now that $z \notin a + b$. Indeed if $z = x + y$ where $x \in a$ and $y \in b$ then the components of x in the orthonormal base completed starting with x_n and $t_n = (1 - \langle x_n, y_n\rangle^2)^{-1/2}(y_n - \langle x_n, y_n\rangle x_n)$ are exactly x_n. This is a contradiction because $\Sigma_n x_n$ diverges in norm. □

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