Description of Many Separated Physical Entities Without the Paradoxes Encountered in Quantum Mechanics

Dirk Aerts

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We show that it is impossible in quantum mechanics to describe two separated physical systems. This is due to the mathematical structure of quantum mechanics. It is possible to give a description of two separated systems in a theory which is a generalization of quantum mechanics and of classical mechanics, in the sense that this theory contains both theories as special cases. We identify the axioms of quantum mechanics that make it impossible to describe separated systems. One of these axioms is equivalent to the superposition principle. We show how these findings throw a different light on the paradox of Einstein, Podolsky, and Rosen.

1. INTRODUCTION

We shall show that the quantum mechanical description of two separated physical systems is wrong. More precisely, it is impossible in quantum mechanics to give a description of separated physical systems. We will give a description of separated systems in a more general theory. We shall say that this theory is more general because it is possible to define five axioms, such that when these five axioms are fulfilled, the theory becomes equivalent to quantum mechanics (eventually with Abelian superselection rules). Of these five axioms, three do not cause any trouble for the description of separated systems, but the last two axioms both make it impossible to describe separated systems. One of these two axioms is equivalent do to the fact that the set of states of the system has a vector space structure and hence is also

1 Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium.
equivalent with the superposition principle, the other axiom is more or less equivalent with the possibility of representing observables by operators. The strange phenomena that are predicted if one does describe separated systems by quantum mechanics were remarked on already a long time ago. They were called paradoxes. Some people like Einstein and Schrödinger claimed that since these strange phenomena do not correspond at all to our daily experience there had to be something wrong with quantum mechanics. Einstein, Podolsky, and Rosen end their famous paper(1) in the following way: "We left open the question whether or not a complete theory exists. We believe however that such a theory is possible." Schrödinger writes in a paper about the same situation(2): "This paradox must be regarded as indicating a serious deficiency of quantum mechanics." Other people, one could say: under the guidance of Bohr, claimed that quantum mechanics is right with the overwhelming argument that, now that we know quantum mechanics, arguments that have their origin in common sense cannot be a critique on quantum mechanics.

If one reflects a little bit then it becomes clear that this strange debate among the most brilliant physicists of their time comes from the fact that it is indeed impossible to translate even the simplest intuitive idea into the theory of quantum mechanics. This comes from the fact that quantum mechanics is a very complicated technical theory where the relation of the basic object, namely the complex wave function representing the state of the system, with the real system is not clear at all. Already in the thirties it was argued that it would be better to detach quantum theory from this too specific Hilbert space formalism. This gave rise to a mathematical generalization of the theory, now commonly called the algebraic approach to quantum mechanics. Another mathematical generalization was initiated by Birkhoff and von Neumann in 1936. (3) Birkhoff and von Neumann remarked that the structure of propositions of a system described by quantum mechanics is not Boolean algebra as it is for a system described by classical mechanics. The mathematical generalization that studies the structure of the proposition calculus of a physical system is usually called quantum logic. These approaches, although the mathematical formalism is more general and introduces only the basic structure needed for the interpretation of the theory, suffer from the same shortcoming as the orthodox formalism. There is no clear interpretation for the symbols that are used. Piron introduced the concept of "question." A question is a test that can be carried out on the physical system and that gives one of the answers "yes" or "no." On the basis of this physically very transparent concept Piron shows that the set of propositions of an arbitrary physical system is a complete lattice. (4) He also defines a set of axioms on this lattice, such that when these axioms are satisfied, the theory becomes a theory of quantum mechanics (with super-
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selection rules). A detailed analysis of the interpretation of the concept of question can be found in Ref. 5. The more general theory of which we are speaking is a theory constructed with this basic concept of question. We will however not only use the concept of question to justify the structure of the set of propositions, we will hold the concept of question as the basic work object in the theory. In this way we have a concrete theory that really describes the set of properties of a real physical system and not a theory that has as basic structure an abstract lattice. As we shall see, such a concrete theory is much more powerfull, since we can define very easily interesting new concepts by merely using our intuition as a guide (e.g., the concept of separated questions, the concept of classical questions, the concept of primitive questions).

In Section 2 we introduce the concept of entity. In Section 3 we give an outline of this theory. In Section 4 we show in what way this theory can be discovered in classical mechanics and in quantum mechanics. In Section 5 we give a set of axioms that reduces the theory to a theory equivalent with quantum mechanics. As remarked, this is also done by Piron in Ref. 4. The mathematical structure of the lattice of propositions that Piron gets when all of his axioms are satisfied, is a complete orthocomplemented weakly modular atomic lattice satisfying the covering law. It is this structure that makes it impossible to describe two separated systems. More precisely, it is the weak modularity and the covering law that cause the trouble. The completeness and the atomicity of the lattice cause no trouble. The orthocomplementation of the lattice also does not cause any trouble as a mathematical structure. However, we had to give another interpretation of this orthocomplementation. Taken all this into account we formulate 5 axioms that gives us a good structure for the set of properties, such that we can use Piron's representation theorem to show that the theory becomes equivalent to quantum mechanics when the five axioms are satisfied. In Section 6 we introduce the concepts of separated questions and separated entities. We built the property lattice of an entity consisting of two separated entities. In Section 7 we finely analyze the validity of the axioms for the property lattice of an entity consisting of two separated entities. We see that axiom 1, 2, and 3 are valid, but axiom 4 and axiom 5 are not valid if one of the two entities is not classical. As a consequence we can decide that quantum mechanics cannot describe separated entities. We see that this fact is at the basis of the well-known paradoxes of Einstein–Podolsky–Rosen that manifest themselves at several places in quantum mechanics. Hence these situations do not really represent a paradox, but are due to a fundamental structural inability of quantum mechanics to deal with the situation of separated entities. A more complete analysis of the Einstein–Podolsky–Rosen paradox and the Bell inequalities, and also of the question
whether this shortcoming is also present in the known generalizations of quantum mechanics can be found in Ref. 6.

We have to remark that from a logical point of view there are two possible conclusions. Either quantum mechanics is considered to be wrong and then it should be replaced by a theory that makes it possible to describe separated entities (for example the theory proposed in this article) or otherwise we believe that quantum mechanics can describe all entities that appear in nature, but then we have to conclude that separated entities do not exist.

Our opinion is that this choice is not a metaphysical one, because we define separated entities by means of separated questions. The definition of separated questions is experimentally verifiable, and it is very easy to find examples of separated questions. It is these separated question that cannot be described by quantum mechanics. One can argue further and say that perhaps separated questions do exist but no separated entities. We can remark then that although in this paper we only treat the case of two separated entities, it is really the separated questions that cannot be described by quantum mechanics (see Ref. 9). In Section 6.1 we also show that the concept of separated entities is not an idealization, once we agree that the concept of entity is a good idealization.

2. THE CONCEPT OF ENTITY

Physics describes the phenomena that take place. The universe is what we call the collection of all these phenomena. One way to understand this universe is to concentrate on certain phenomena that "strike the eye," certain phenomena that we can experience without being forced to experience also all the other phenomena that are taking place. We give names to these phenomena. For example: rain, tree, electron, etc. Since we can experience such a phenomenon apart from all other phenomena, it makes sense to attribute properties to the phenomenon and to study these properties.

We shall study the set of properties that we want to attribute in this way to a certain phenomenon. The "thing" described by this set of properties, and which is an idealization of the phenomenon in the sense explained above, we shall call an "entity."

By the set of properties of a entity, we do not mean the set of all possible properties that we could attribute to the phenomenon defining the entity. No, we restrict ourselves to those properties that we know at this moment and we hope that these properties will remain interesting properties in the future. It seems to us that we cannot do more. Hence the set of properties of an entity is a well-defined set which is complete in the sense
that it contains really all that we know about the phenomenon. Among the
properties of an entity, some are “actual,” the entity has them “in acto,” and
others are “potential,” the entity has the possibility of obtaining them. The
evolution of the entity is the changing of actual properties into potential
properties and potential properties into actual properties. In physics we say
that the state of the entity changes. Hence the state of the entity is the
collection of all actual properties.

3. THE DESCRIPTION OF AN ENTITY

We shall define carefully the notion of actual property and of potential
property. This was done by Piron in Refs. 4 and 5. An arbitrary statement
about a phenomenon does not in general define the property of the
phenomenon. No, it must be possible to construct a test for this statement.

3.1. Testing of Properties and the Concept of Truth

Such a test consists of an experiment that can be performed on the
phenomenon. If the experiment gives us the expected outcome, we will say
that the answer of the test is “yes.” If the experiment does not give us the
expected outcome, we will say that the answer of the test is “no.”

A proposal for such a test will be called a “question.” Hence to define a
question one has to define:

- the measuring apparatus used to perform the test,
- the manual of operation of the apparatus,
- a rule that allows us to interpret the result in terms of “yes” and
  “no.”

When will we agree on the fact that the entity has actually the property that
we want to test? Let us look at an example: We consider the entity which is
a piece of wood. We have in mind the property of “burning well.” A test for
this property consists of taking the piece of wood and setting it on fire. If we
perform this test on a piece of dry wood, the piece of wood will in general be
destroyed by the test.

So the property of “burning well” is a property that the piece of wood has before we made the test. And even before we had decided to make the
test.

Of course it is after having done a number of tests with a certain type of
pieces of wood and having always discovered the answer to be yes, we
decide that the one new piece of wood of this type, whereon we never
performed the test, has actually the property of burning well. We will say that the question corresponding to this test is “true” in this case. Let us repeat:

A question \( a \) of an entity \( S \) is said to be “true” (and the corresponding property is said to be “actual”) iff when we should decide to perform the test proposed by \( a \), the expected answer “yes” would come out with certainty.

To exhibit an entity with an actual property we proceed for example as follows: we prepare a collection of identical entities in a well-defined way. If we make the test on each element of the collection and we see by statistics that the probability of obtaining the answer “yes” is 1, then we claim that the one new entity prepared in the same way has actually this property. We want to describe however first of all the collection of actual and potential properties of an entity, and not fix the way in which a property can be defined operationally.

3.2. Inverse Questions

If \( a \) is a question of the entity \( S \), we can consider the question that consists of proposing the same test as the one proposed by \( a \), but changing the role of yes and no. We will denote this new question by \( a^{-} \) and we will call it the inverse question of \( a \). Hence \( a^{-} \) is true iff, when we should decide to perform the test corresponding to \( a \), we would find with certainty the answer “no.” If \( a \) is a property of an entity \( S \) tested by a question \( a \), then the statement “\( a \) is not actual” is not necessarily tested by the question \( a^{-} \). Hence, \( a^{-} \) is not the negation of \( a \). It is possible that neither \( a \) nor \( a^{-} \) are true. In general the statement “\( a \) is not actual” is not even a property of the entity \( S \), because it is in general not possible to test this statement.

3.3. Testing Several Properties at Once

In general the results of a test of one property are profoundly influenced by the testing of another property. In most cases it makes no sense to perform two tests on the entity. But still it is true that an entity can have several properties that are actual at once.

There is indeed a way to construct a question that tests the actuality of several properties at once. Let us analyze this first on an example. We take again a piece of wood as an entity. We consider the following two properties of the piece of wood

\( c \): “The piece of wood burns well”

\( d \): “The piece of wood floats on water”

A question \( y \) testing \( c \) consists of setting the wood on fire and giving the answer “yes” if it burns. A question \( \delta \) testing \( d \) consists of making the wood
float on water and giving the answer “yes” if it floats. If we perform first the
test δ, and make the wood float on water, we have brought the wood into a
state of being wet wood and as a result the wood will not burn anymore. On
the other hand if we perform the test γ and burn the wood, it will not float
anymore on water. So if it is necessary to perform both test γ and δ in order
to see whether the wood has both properties c and d, then we will find that
this is never the case. However we all agree on the fact that for most pieces
of wood both properties c and d are actual. If we analyze very carefully we
see that the fact that a piece of wood has both properties c and d means that
if we choose one of the two tests γ or δ we are certain to obtain the answer
“yes” no matter what is our choice. This leads us to the following definition
of a new question.

Given two questions γ and δ we define a new question γ · δ, that
consists of choosing one of the two questions at random and performing the
test corresponding to this chosen question, and attributing the answer
obtained in this way. We will call this question the “product” of γ and δ, and
γ · δ is true iff γ is true and δ is true. Hence γ · δ tests whether the wood
burns well and floats on water.

In general, if we have a family of properties a_i and questions a_i testing
a_i, we will define a question that tests the actuality of all the properties a_i as
follows:

We choose as we want, at random, or not one of the a_i, and accord to
π_i a_i the answer obtained by performing the test of this chosen question.

We will denote this question by π_i a_i and call it the product of the a_i.
Clearly π_i a_i is true iff a_i is true for every i.

3.4. A Generating Set of Questions

Let us denote by Q the set of questions for the entity S. We want to
remark again that Q is not the set of all possible questions performable on
the phenomenon corresponding to the entity S. No, Q is a well-defined set of
questions about the phenomenon, and defines in this way the entity S. We
will consider the set Q to be closed for the “product” operation and for the
“inverse” operation. Hence if a_i ∈ Q, then π_i a_i ∈ Q and if a ∈ Q then
a^- ∈ Q. If a_i are arbitrary questions then π_i a_i is known exactly if we know
all the a_i. Indeed the measuring apparatus and the manuals that we need are
just a collection of the measuring apparatus, and the manuals that we need for
the a_i. We see also that (π_i a_i)^- = π_i a_i^- . Hence it is sufficient to specify some
subset of Q, that is closed for the “inverse” operation, in order to generate by
products the whole of Q.

A subset with this property we will call a generating set of questions.
Hence if $G$ is a generating set of questions, then if $\alpha \in G$ we have $\alpha^\sim \in G$ and
\[ Q = \langle \pi_i \alpha_i \mid \alpha_i \in G \rangle \]

3.5. A Physical Law on the Questions of an Entity

If we have the situation that whenever a question $\alpha$ is true then also the question $\beta$ is true for an entity, we shall denote this as follows:
\[ \alpha < \beta \]
and we shall say "$\alpha$ is stronger than $\beta$.

It is easy to find numerous examples of this physical law. In general this law has the following properties:

1. $\alpha < \alpha$
2. if $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$

where $\alpha$, $\beta$, and $\gamma$ are questions of the entity. The physical law $\alpha < \beta$ is a preorder relation on the set of questions.

3.6. Properties of an Entity

If $\alpha$ and $\beta$ are questions of an entity $S$ such that
\[ \alpha < \beta \quad \text{and} \quad \beta < \alpha \]
then we will say that $\alpha$ is "equivalent" to $\beta$ and we will denote $\alpha \approx \beta$. If $\alpha \approx \beta$ then $\alpha$ and $\beta$ test the same property of the entity. That is why we shall identify the properties of the entity with the classes of equivalence of questions. Then a property $\alpha$ is "actual" iff we have a question $\alpha$ testing $\alpha$ that is true. If $\alpha$ is actual then every question testing $\alpha$ is true. The collection of properties of the entity we will denote by $\mathcal{S}$. If $G$ is a generating set of questions, then we will call the collection of properties tested by the questions of $G$ a generating set of properties.

The collection of questions that are never true we will denote by $O$. For an arbitrary question $\alpha$, we have $\alpha \cdot \alpha^\sim \in O$. It is easy to see that $O \subseteq \mathcal{S}$.

We introduce the concept of "trivial question." A trivial question is a question that is always true. An example of a trivial question is the following: we do anything that we want with the entity, and we give always the answer "yes."

If $\tau_1$ is a trivial question and $\tau_2$ is a trivial question, then $\tau_1 \approx \tau_2$. Hence all these trivial questions define a property that we will denote by $I$. If $\tau$ is a trivial question, then $\tau^\sim \in O$.

The preorder relation $<$ on the set of questions induces a relation on the set of properties, if $a, b \in \mathcal{S}$
\[ a < b \iff a \prec \beta \text{ for } a \in a \text{ and } b \in b \]
This relation satisfies the following properties:

1. \( a < a \) \( \forall a, b, c \in \mathcal{L} \)
2. \( a < b \) and \( b < a \) then \( a = b \)
3. \( a < b \) and \( b < c \) then \( a < c \)

This shows that \( < \) is a partial order relation on \( \mathcal{L} \). It is clear that \( O < a < I \) for \( a \in \mathcal{L} \). If, \( a, b \in \mathcal{L} \), then \( a < b \) means that if the property \( a \) is actual for \( S \) then also \( b \) is actual for \( S \). Let us consider an arbitrary family of properties \( a_i \). For every property we consider a question \( a_i \in a_i \). Let us denote the property tested by \( \pi_i a_i \) by \( \bigwedge_i a_i \). Then we have:

4. \( \bigwedge_i a_i < a_j \) for every \( j \)
5. \( b < a_j \) for every \( j \Rightarrow b < \bigwedge_i a_i \) for \( b \in \mathcal{L} \)

An element of a partially ordered set that satisfies (4) and (5) is called an infimum (or greatest lower bound) of the family \( a_i \) for the partial order relation. Let us define now for an arbitrary family \( a_i \) of properties

6. \( \bigvee_i a_i = \bigwedge_{a_i < b, i, b \in \mathcal{L}} b \)

This definition makes sense, since for every \( b \in \mathcal{L} \) we have \( b < I \). It is easy to verify that:

7. \( a_j < \bigvee_i a_i \ \forall j \)
8. \( a_j < b \ \forall j \Rightarrow \bigvee_i a_i < b \ \forall b \in \mathcal{L} \)

An element of a partial ordered set that satisfies (7) and (8) is called a supremum (or least upper bound) of the family \( a_i \). Hence for every family \( a_i \) of properties of \( S \), there exists a property \( \bigwedge_i a_i \) which is an infimum and a property \( \bigvee_i a_i \) which is a supremum. A partially ordered set where this is the case is called a “complete lattice.” Hence \( \mathcal{L} \) is a complete lattice. We will therefore call \( \mathcal{L} \) the “property lattice” of the entity \( S \). If \( a, b \in \mathcal{L} \) then \( a \wedge b \) is actual iff \( a \) is actual and \( b \) is actual. This shows that the infimum has the meaning of “and.” If \( a \) is actual or \( b \) is actual, then \( a \vee b \) is actual, but in general the contrary is not satisfied. We shall analyze later on what is the meaning of this.

3.7 The Set of States of an Entity

The state of an entity is the set \( e \) of all actual properties. It is interesting to remark that the state is totally determined by the infimum of this collection of \( e \)’s. Indeed if

\[
p = \bigwedge_{e \in e} a
\]
then

\[ \varepsilon = \{ a \mid p < a, a \in \mathcal{P} \} \]

In the following we will always represent the state \( \varepsilon \) of the entity by this property \( p \). We will denote by \( \Sigma \) the set of all states of \( S \). \( \Sigma \) is a partially ordered set with the partial order inherited from \( \mathcal{P} \). We can see now that

1. If \( a \in \mathcal{P} \) then \( a \) is actual iff the entity is in a state \( p \) such that \( p < a \).
2. If \( a, b \in \mathcal{P} \), then \( a < b \) iff whenever \( p \in \Sigma \), \( p < a \) then \( p < b \).
3. If \( a \in \mathcal{P} \), then \( a = \bigvee \_p \)

\[ p < a, \quad p \in \Sigma \]

From (3) follows that every property is the supremum of all the states that make this property actual. Therefore we will say that \( \Sigma \) is a "full set of states" for \( \mathcal{P} \).

3.8. An Orthogonality Relation

If \( p \) and \( q \) are two states of \( S \), we will say that \( p \) is orthogonal to \( q \), iff there exists a question \( \gamma \) such that \( \gamma \) is true if \( S \) is in the state \( p \), and \( \gamma \) is true if \( S \) is in the state \( q \). We will denote then \( p \perp q \). If \( p, q, r, s \in \Sigma \) then

1. \( p \perp q \Rightarrow q \perp p \)
2. \( p \perp q \) and \( r < p \) and \( s < q \) then \( r \perp s \)
3. \( p \perp q \Rightarrow p \land q = O \)

From the remark at the end of Section 3.2 we see that states can be nonorthogonal, nor equal. We shall say that two properties \( a, b \in \mathcal{P} \) are orthogonal iff for every \( p, q \in \Sigma \) such that \( p < a \) and \( q < b \) we have \( p \perp q \).

We shall also denote \( a \perp b \). The relation \( \perp \) on \( \mathcal{P} \), we shall call an orthogonality relation. This orthogonality relation has the following properties:

4. \( a \perp b \Rightarrow b \perp a \quad a, b, c, d \in \mathcal{P} \)
5. \( a \perp b \) and \( c < a, d < b \) then \( c \perp d \)
6. \( a \perp b \Rightarrow a \land b = O \).

4. CLASSICAL MECHANICS AND QUANTUM MECHANICS

We remarked that the formalism that we put forward is more general than classical mechanics and more general than quantum mechanics. By
more general we mean, that it is possible to define a set of axioms such that when these axioms are satisfied our formalism reduces to a formalism equivalent to quantum mechanics or to classical mechanics. We want to mention that no claim of truth is implied in the term axiom as it is used here. The axioms must merely be seen as physical hypothesis. In this case it is of course of great importance to understand the nature of these physical hypothesis. We will indeed try to introduce the axioms in such a way that it becomes clear what are the physical hypothesis underlying the axioms. Let us first demonstrate in which way we find our formalism in classical mechanics and in quantum mechanics.

4.1. Classical Mechanics

In classical mechanics an entity \( S \) is described in a state space \( \Gamma \). Observables are functions on the state space to some outcome space. A yes-no observable is a function \( f \) from \( \Gamma \) to the outcome set \( \{ \text{yes}, \text{no} \} \). These yes-no observables represent the test of the entity. Remember that a yes-no observable is totally determined if we know the subset \( f^{-1}(\{\text{yes}\}) \) of \( \Gamma \). Hence, yes-no observables can be represented by subsets of the state space. Every statement of the entity can also be represented by a subset of the state space, namely the subset of all those states of the entity that make the statement true. And with every subset that corresponds to such a statement. Let us denote by \( \mathcal{P}(\Gamma) \) the lattice of all subsets of \( \Gamma \), then \( \mathcal{P}(\Gamma) \) describes the statements of the entity. Of course every statement does not have to correspond with a property of the entity. The statement has to be testable. In classical mechanics we make the following hypothesis

1. There exist a set \( K(\Gamma) \) of subsets of \( \Gamma \) such that for every \( A \in K(\Gamma) \) it is possible to construct a question \( \pi_i \phi(A_i) \) such that \( \pi_i \phi(A_i) \) is true iff the entity is in state \( \{P\} \). This hypothesis asks that for every \( P \in \Gamma \) there exists \( A_i \in K(\Gamma) \) such that \( \{P\} = \bigcap_i A_i \).

2. For every state \( \{P\} \) it is possible to construct a question \( \pi_i \phi(A_i) \) such that \( \pi_i \phi(A_i) \) is true iff the entity is in state \( \{P\} \). This hypothesis asks that for every \( P \in \Gamma \) there exists \( A_i \in K(\Gamma) \) such that \( \{P\} = \bigcap_i A_i \).

The property lattice that we can build on the hand of the generating set of questions \( \{\phi(A)/A \in K(\Gamma)\} \), we will denote by \( \mathcal{L}(\Gamma) \). If \( A \in \mathcal{P}(\Gamma) \) such that \( A = \bigcap_i A_i \) where \( A_i \in K(\Gamma) \), then \( \pi_i \phi(A_i) \) is true iff \( P \in A \) where \( \{P\} \) is the state of the entity. Hence \( \pi_i \phi(A_i) \) tests the statement corresponding to \( A \). The property corresponding to \( \pi_i \phi(A_i) \) we will denote by \( v(A) \). In this way we define a map

\[
\nu : \mathcal{P}(\Gamma) \rightarrow \mathcal{L}(\Gamma) \\
A \rightarrow \nu(A)
\]
which we will call the "interpretation map" of the lattice of statements \( \mathcal{P}(\mathcal{F}) \). The states of the entity are the properties \( v(\{P\}) \) where \( P \in \mathcal{F} \) and the orthogonality relation becomes trivial

\[
v(\{P\}) \perp v(\{Q\}) \text{ iff } P \neq Q
\]

4.2. Quantum Mechanics

An entity \( S \) as a quantum mechanical system is described in a complex Hilbert space \( \mathcal{H} \). The states of \( S \) are represented by the rays \( \hat{x} \) of the Hilbert space. The observable are represented by self-adjoint operators on the Hilbert space. A yes-no observable is an observable with two alternatives "yes" and "no." Such an observable is represented by a projection operator \( P \). Again such an observable is totally determined by the closed subspace \( \mathcal{P}(\mathcal{H}) \) of the Hilbert space. Hence the role that in classical mechanics is played by the subsets of the phase space is in quantum mechanics played by the closed subspaces of the Hilbert space. The abstract lattice that describes the entity will be \( \mathcal{P}(\mathcal{H}) \) which is the set of all closed subspaces of the Hilbert space \( \mathcal{H} \). Let us construct now the property lattice corresponding to this abstract lattice. Therefore we make the following hypothesis:

1. There exists a set \( K(\mathcal{H}) \) of closed subspaces such that for every \( A \in K(\mathcal{H}) \) it is possible to construct a question \( \alpha \) such that \( \alpha \) is true iff \( x \in A \), and \( \tilde{\alpha} \) is true if \( x \in A^c \) where \( \tilde{x} \) represents the state of the entity. We will denote \( \alpha \) by \( \phi(A) \).

2. For every state \( \hat{x} \) it is possible to construct a question \( \pi_i \phi(A_i) \) such that \( \pi_i \phi(A_i) \) is true iff the entity is in the state \( \tilde{x} \). Hence we ask that for every \( x \in \mathcal{H} \) there exists \( A_i \in K(\mathcal{H}) \) such that \( \tilde{x} = \bigcap_i A_i \).

The property lattice that we build on the hand of this primitive set of questions \( \{\phi(A) / A \in K(\mathcal{H})\} \) we will denote by \( \mathcal{L}(\mathcal{H}) \). If \( A \in \mathcal{P}(\mathcal{H}) \) such that \( A = \bigcap_i A_i \) where \( A_i \in K(\mathcal{H}) \) then the question \( \pi_i \phi(A_i) \) is true iff \( x \in A \) where \( \tilde{x} \) represents the state of the entity. The property tested by \( \pi_i \phi(A_i) \) we will denote by \( v(A) \). In this way we define a map

\[
v : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})
\]

\[
A \rightarrow v(A)
\]

which is the interpretation map on the abstract lattice \( \mathcal{P}(\mathcal{H}) \). The states of the entity are the elements \( v(\hat{x}) \) where \( x \in \mathcal{H} \). And if \( v(\hat{x}) \perp v(\hat{y}) \) then \( \langle x, y \rangle = 0 \). Hence the orthogonality relation is not trivial in this case. This is the main difference between classical and quantum mechanics.
5. THE AXIOMS

5.1. The Primitive Questions

The formalism explained in Section 3 is a generalization of both classical mechanics and quantum mechanics. Let us try to see what are the axioms that can be defined such that this formalism reduces to quantum mechanics or to classical mechanics. By the first axiom we want to express the fact that the experimenter does not construct his experiments at random. No, he has a certain knowledge of how to construct good experiments. How can we express this fact? Let us first see what strange properties a general question as we defined it in Section 3 can have: Suppose that \( a \) is an arbitrary property of an entity \( S \). If \( \alpha \) and \( \beta \) are questions testing \( a \), then \( \alpha \approx \beta \). In general however there is no relation between \( \alpha' \) and \( \beta' \). This is the reason that it is in general not true that the entity has a weakest orthogonal property to another property. To see this fact even better, we remark that it is always possible to find a question testing \( a \) such that the inverse of this question is never true. Indeed if \( \tau \) is a trivial question, then \( a \cdot \tau \in a \) and \( a^\tau \cdot \tau \in \varnothing \). Suppose now that the entity is in a state such that \( a \) is not actual. In this case, for an arbitrary question \( \alpha \) testing \( a \), “no” is a possible answer. If \( \tau \) are trivial questions we can consider the question \( \gamma_n = \alpha \cdot \pi^\tau \cdot \tau \), then \( \gamma_n \) is a question testing \( a \). If \( \alpha \) choose during the performance of the question \( \gamma_n \), at random between the \( n + 1 \) questions of the product, we have a probability for the answer “yes” which is greater than \( n/n + 1 \). Hence by taking \( n \) to be very big, we can make this probability very near to 1, and in this way hide the most important property of the question \( a \), namely that it has, “no” as a possible outcome.

This is a procedure we can apply for every property of the entity. Of course, we all agree that although \( \gamma_n \) is a question that tests the property \( a \), it is a question that tests \( a \) in a very bad way. A good experimentalist will never perform a test of this kind consciously.

We want to introduce the concept of “primitive question.” A primitive question should not be a question as \( \gamma_n \). What should it be? Let us look again at an example.

Suppose we consider the entity which is a piece of wood and the question \( \nu \) that tests the property \( c \) of burning well. We already remarked that the question \( \nu \) consists of setting the piece of wood on fire and getting the answer “yes” if it burns. Of course this \( \nu \) is also a product question. Indeed we choose for example among different ways to set the piece of wood on fire. We can set the wood on fire by using a match, but we can also set the wood on fire by using a bazooka. These two ways correspond to nonequivalent questions, and as long as we did not specify whether we shall set
the wood on fire by using a match or by using a bazooka, the question that we propose contains the product of these two possibilities.

Hence a primitive question must be a question that eliminates as much as possible all nonequivalent components of the question, such that it becomes a product of equivalent questions. The experimentalist does this by stipulating in detail the conditions under which the experiment must be done. We can remark that \( \gamma _{a} \) is never true. Hence if \( a \) is an arbitrary question, then \( a \) will be more primitive if there are more states of the entity where \( a \) is true. This inspires us to the following definition.

**Definition 1.** If \( a \) is a question testing the property \( a \) such that \( a \) tests the property \( h \), then \( a \) is a primitive question if and only if whenever the entity is in a state orthogonal to \( a \), then \( a \) is true and whenever the entity is in a state orthogonal to \( h \), then \( a \) is true.

If \( a \) is a primitive question then also \( a \) is a primitive question.

**Theorem 2.** If \( a_{i} \) are questions of an entity \( S \), then \( \pi a_{i} \) is a primitive question if and only if for every \( i, j \) we have \( a_{i} \cong a_{j} \) and \( a_{i} \) are primitive questions.

**Proof.** \( a_{i} \in a_{j} \), \( \pi a_{i} \in a \), \( a_{i} \in b \), \( \pi a_{i} \in b \). If \( a_{i} \) is true, the entity is in a state \( p \perp b \). But then \( p \perp b \). As a consequence \( \pi a_{i} \) is true. This shows that \( a_{i} \cong a_{j} \) for every \( i, j \). Suppose the entity is in a state \( p \perp a_{j} \), then \( p \perp a \). Hence \( \pi a_{i} \) is true. As a consequence \( a_{j} \) is true. This shows that \( a_{j} \) is a primitive question.

The question \( \gamma \) that tests whether the piece of wood burns well, properly defined is a primitive question. Also the question \( \delta \) that tests whether the piece of wood floats on water is a primitive question. The question \( \gamma \cdot \delta \) is not a primitive question.

**Theorem 3.** If \( a \) and \( b \) are primitive questions such that \( a \cong b \), then \( a \cong b \).

**Proof.** \( a \in a \) and \( b \in a \). If \( a \) is true, then the entity is in a state orthogonal to \( a \). As a consequence \( b \) is true.

**Definition 4.** A property \( a \) of an entity will be called a primitive property, if it can be tested by a primitive question.

Let us denote the set of primitive questions by \( P \), and the set of primitive properties by \( \mathcal{F} \). Then \( \mathcal{F} \) is a partially ordered set, but it is in general not a lattice. We can see that \( \mathcal{F} \) is an orthocomplemented partially ordered set.
Theorem 5. If $\mathcal{E}$ is the set of primitive properties of an entity $S$ and $a \in \mathcal{E}$ and $a \models a$ is a primitive question, we will denote by $a'$ the property tested by $a$. The map $': \mathcal{E} \rightarrow \mathcal{E}$ is an orthocomplementation, namely

1. $a < b \Rightarrow b' < a'$ for $a, b \in \mathcal{E}$
2. $a'' = a$
3. $a \wedge a' = O$

Also the following properties are satisfied

4. $c < a' \Leftrightarrow c \perp a$ for $c \in \mathcal{L}$
5. $a \vee a' = I$, if $a_i \in \mathcal{E}$ then $\bigwedge_i a_i \in \mathcal{E}$ iff $\bigvee_i a'_i \in \mathcal{E}$ and in this case $(\bigwedge_i a_i)' = \bigvee_i a'_i$

Proof. If $a, b \in \mathcal{E}$ such that $a < b$ and $b'$ is actual, then the entity is in a state $p \perp b$. Hence $p \perp a$. But then $a'$ is actual. If $a \in \mathcal{E}$ and $a \in a'$ is a primitive question, then $a \models a$. This shows that $(a')' = a$. Clearly $a \wedge a' = O$. If $c \in \mathcal{L}$ such that $c < a'$ and $p$ and $q$ are states such that $p < c$ and $q < a$, then there exists $a \in a$ such that $a$ is true if the entity is in the state $q$ and $a$ is true if the entity is in the state $p$. Hence $p \perp q$. If on the other hand $c \perp a$ and $c$ is actual, then the entity is in a state $p \perp a$. From this follows that $a'$ is actual.

5.2. The First Two Axioms

We are now in a position to formulate the first axiom.

Axiom 1. If $S$ is an entity, then the primitive questions of $S$ form a generating set of questions for the property lattice.

With this first axiom we put forward the fact that a physical theory is built on the hand of primitive questions. Quantum mechanics satisfies this first axiom since all the questions $\phi(A)$ where $A \in K(\mathcal{H})$ and $\mathcal{H}$ is the Hilbert space describing the entity are primitive questions. Also classical mechanics satisfies this first axiom. There is another axiom satisfied by quantum mechanics and by classical mechanics. Suppose we have an entity described in a Hilbert space $\mathcal{H}$ with a property lattice $\mathcal{L}(\mathcal{H})$. If $\nu(A) \in \mathcal{L}(\mathcal{H})$ we can ask ourself what is the meaning of $\nu(A^\perp)$.

$\nu(A^\perp)$ actual $\Leftrightarrow$ the state $\nu(x)$ of $S$ is such that $x \in A^\perp$

$\Leftrightarrow$ the state $\nu(x)$ of $S$ is orthogonal to $\nu(A)$

So $\nu(A^\perp)$ is the property that is actual iff the state of the entity is orthogonal to $\nu(A)$. For an arbitrary entity $S$ with a property lattice $\mathcal{L}$ it is in general
not true that for every property \( a \) there exists another property that is actual
iff the entity is in a state orthogonal to \( a \). This will in general be the case
only for the primitive properties. This enables us to formulate the second
axiom.

**Axiom 2.** If \( S \) is an entity and \( p \) is a state of \( S \) then there exists a
question that is true iff \( S \) is in a state orthogonal to \( p \).

**Theorem 6.** Suppose that \( \mathcal{X} \) is the property lattice of an entity and
axiom 1 and axiom 2 are satisfied by then

1. for \( a \in \mathcal{X} \) there exists a unique property \( b_a \) such that \( b_a \) is actual iff
   the entity is in a state orthogonal to \( a \).
2. if \( a \) is a primitive property then \( b_a = a' \).

**Proof.** For every state \( p \), there exists such a property \( b_p \). For \( a \in \mathcal{X} \)
define then \( b_a = \bigwedge_{p \sqsubseteq a} b_p \).

We will in the following denote this unique property \( b_a \) by \( a' \), and we
will call it the orthocomplement of \( a \). It is in fact the weakest property
orthogonal to \( a \).

**Theorem 7.** If \( \mathcal{X} \) is the property lattice of the entity \( S \) and axiom 1
and axiom 2 are satisfied by then for \( a, b, a_i \in \mathcal{X} \) we have

1. \( a \leq b \Rightarrow b' \leq a' \)
2. \( a \wedge a' = O \)
3. \( (a')' = a \)
4. \( b \leq a' \Leftrightarrow b \perp a \)
5. \( a \vee a' = 1 \), \( (\forall \; a_i) = \bigwedge \; a_i \), \( (\forall \; a_i) = \bigvee \; a_i \)

which shows that the map \( \cdot : \mathcal{X} \rightarrow \mathcal{X} \) is an orthocomplementation and \( \mathcal{X} \)
is an orthocomplemented lattice.

**Proof.** (1) and (2) are easy to check. Clearly \( a \leq a'' \). If \( a \in \mathcal{X} \) then
\( a = \bigwedge_i a_i \) where \( a_i \) are primitive properties. Suppose \( a'' \) is actual.
The entity is then in a state \( p \perp a' \). If \( p \perp a' \) then \( p \perp a' \forall i \). So \( p \leq a'' \forall i \), but then
\( p < a_i \forall i \) and as a consequence \( p < a \). This shows that \( a'' < a \).

To summarize we can say that the first axiom postulates the existence
of a generating set of properties such that for every property \( a \) of this
generating set there exists a weakest orthogonal property \( a' \) such that there
exists \( a \in a \) and \( a' \in a' \). The second axiom postulates the existence of such
a weakest orthogonal property \( a' \) for every property \( a \) of the entity. Remark
however that the second axiom does not ask for the existence of a question \( a \in a \) such that \( \tilde{a} \in \tilde{a}' \). This would indeed provoke difficulties since such a question \( a \in a \) and \( \tilde{a} \in \tilde{a}' \) can evidently not be a product question.

5.3. The Example of the Spin

Let us construct the lattice of the spin properties of a spin 1/2 particle as an example. The questions that we will consider are the following: \( a_{\theta, \phi} \): we take a Stern–Gerlach apparatus and put it in the direction \((\theta, \phi)\) and record the answer "yes" if the spin 1/2 particle is deflected up. If the spin 1/2 particle is deflected down, we record the answer "no." We take \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \). We know experimentaly that if a particle is deflected up by a Stern–Gerlach apparatus in the direction \((\theta, \phi)\), then the question \( a_{\theta, \phi} \) is true for the particle immediately after the passage. But the experiment also shows that if for example \( a_{\theta, \phi} \) is true and we make the test for the question \( a_{\theta', \phi'} \) where \( \theta' = \theta + \pi/2 \) then both possible answers have the same probability. It is also possible to see that \( \tilde{a}_{\tilde{\theta}, \tilde{\phi}} \approx a_{\pi - \theta, \pi + \phi} \). Further we have \( a_{\theta, \phi} \land a_{\theta', \phi'} \in O \) if \( \theta \neq \theta' \) or \( \phi \neq \phi' \). Let us call \( a_{\theta, \phi} \) the property: "the particle is polarized in the \((\theta, \phi)\) direction." Then \( a_{\theta, \phi} \in a_{\theta, \phi} \) and \( a_{\pi - \theta, \pi + \phi} \in a_{\theta, \phi} \). We also have:

\[
a_{\theta, \phi} \land a_{\theta', \phi'} = O \quad \text{if} \quad \theta \neq \theta' \quad \text{or} \quad \phi \neq \phi'
\]

\[
a_{\theta, \phi} + a_{\pi - \theta, \pi + \phi}
\]

We can also see that in the quantum mechanical model of such a spin 1/2 particle \( a_{\theta, \phi} \) is supposed to be a primitive question and axiom 1 and axiom 2 are satisfied. Then

\[
a'_{\theta, \phi} = a_{\pi - \theta, \pi + \phi}
\]

and

\[
a_{\theta, \phi} \lor a_{\theta', \phi'} = (a'_{\theta, \phi} \land a'_{\theta', \phi'})' = (a_{\pi - \theta, \pi + \phi} \land a_{\pi - \theta', \pi + \phi})' = O' = I
\]

Hence the property lattice that we get is the following

\[
\mathcal{L} = \mathcal{F} = \{O, a_{\theta, \phi}, I\} \quad \text{and} \quad \Sigma = \{a_{\theta, \phi}\}
\]

5.4. The Third Axiom, Atomicity of the States

Quantum mechanics is a more specific theory than our theory even when the two axioms are satisfied. We cannot help that the formulation of these last three axioms is rather technical. This is not due to a failure of our formalism but just reflects the fact that quantum mechanics is a very
technical theory. We will then also see that two of these three axioms are not satisfied in nature. If \( \mathcal{L} \) is a complete lattice and \( p \in \mathcal{L} \) we will say that \( p \) is an atom of \( \mathcal{L} \) iff for \( a \in \mathcal{L} \) and \( a < p \) we have \( a = O \) or \( a = p \). So an atom is the strongest element different from \( O \). It is clear that in quantum mechanics the states of an entity are represented by atoms of the property lattice \( \mathcal{P}(\mathcal{H}) \). This is however a property that cannot be deduced from axiom 1 and axiom 2. If we want to find a theory as quantum mechanics we must introduce it as an axiom.

**Axiom 3.** The states of an entity \( S \) are represented by atoms of the property lattice of the entity.

As a consequence of this axiom the property lattice of an entity becomes an atomic lattice. Remark that in classical mechanics axiom 3 is also satisfied.

### 5.4. The Fourth Axiom, Weak Modularity of the Property Lattice

Suppose that \( S \) is an entity described by quantum mechanics in a Hilbert space \( \mathcal{H} \). We know that if \( A \in \mathcal{P}(\mathcal{H}) \), then \( A + A^\perp = \mathcal{H} \). This is just the projection theorem of Hilbert spaces. More generally if \( A, B \in \mathcal{P}(\mathcal{H}) \) and \( A \subseteq B \), then \( B = A + (A^\perp \cap B) \) where \( A \perp A^\perp \cap B \) and \( A^\perp \cap B \in \mathcal{P}(\mathcal{H}) \). Hence for two closed subspaces \( A, B \) of the Hilbert space such that \( A \subseteq B \) we can find a closed subspace \( C \) such that \( C \perp A \) and \( B = A + C \). In lattice theory, a lattice \( \mathcal{L} \) with an orthogonality relation is said to be “weakly modular” iff for every \( a, b \in \mathcal{L} \) such that \( a < b \) we can find a \( c \in \mathcal{L} \) such that \( c \perp a \) and \( a \lor c = b \). As we just showed, the property lattice \( \mathcal{P}(\mathcal{H}) \) of an entity described by quantum mechanics is weakly modular. Also the property lattice \( \mathcal{L}(I) \) of an entity described by classical mechanics is weakly modular. Let us introduce this as an axiom.

**Axiom 4.** If \( \mathcal{L} \) is the property lattice of an entity and \( a, b \in \mathcal{L} \) such that \( a < b \), then it is possible to find a property \( c \in \mathcal{L} \) such that \( c \perp a \) and \( a \lor c = b \).

This axiom is already a technical axiom and it is not very easy to see what it means physically. In the following we will show that this axiom is not satisfied in the property lattice of two separated entities.

### 5.5. The Fifth Axiom, the Covering Law

Suppose again that \( S \) is an entity described by quantum mechanics in a Hilbert space \( \mathcal{H} \). Take \( A \in \mathcal{P}(\mathcal{H}) \) and \( x \in \mathcal{H}, x \in A \). Then \( A + x \in \mathcal{P}(\mathcal{H}) \),
and we have the following: If we take \( X \in \mathcal{P}(\mathcal{L}) \) such that \( A \subseteq X \subseteq A + \bar{x} \), then \( X = A \) or \( X = A + \bar{x} \). In lattice theory one says that \( A + \bar{x} \) “covers” \( A \). So if \( \mathcal{L} \) is a lattice and \( a, b \in \mathcal{L} \) we will say that “\( b \) covers \( a \)” iff for \( c \in \mathcal{L} \) such that \( a < c < b \) we have \( c = a \) or \( c = b \). With this definition an atom is in fact a property covering \( O \). Also in classical mechanics when an entity is described by a state space \( \Gamma \) we know that if \( A \in P(\Gamma) \) and \( P \in \Gamma \), \( P \not\in A \) then \( A \cup \{P\} \) covers \( A \). Let us write this in the form of an axiom.

**Axiom 5.** If \( \mathcal{L} \) is the property lattice of an entity \( S \) and \( a \in \mathcal{L} \) and \( p \) is a state of \( S \) such that \( a \wedge p = O \), then \( a \vee p \) covers \( a \).

This axiom is also called the covering law. It is again of a very technical nature. Also this axiom will not be satisfied in the property lattice of two separated entities.

### 5.6. A Theory Satisfying the Five Axioms is a Quantum Theory with Superselection Rules

Axiom 1, 2, 3, 4, and 5 are satisfied in quantum mechanics and also in classical mechanics. In a certain sense also the inverse is true. These five axioms force our formalism to be a formalism equivalent to quantum mechanics with superselection rules. If there are no superselection rules we get ordinary quantum mechanics in one Hilbert space. If all states are separated by a superselection rule we get classical mechanics. So even with these five axioms satisfied the formalism is more general than quantum mechanics in the sense that it introduces in a natural way superselection rules. This fact will be analyzed deeper in Ref. 7. Indeed in Ref. 7 we show that by introducing the concept of classical question, it is possible to study the classical part of an arbitrary entity and the nonclassical part of an arbitrary entity. To be able to explain this more easily we have to introduce the concept of direct union of lattices. Suppose \( \mathcal{L}_i \) is a family of complete lattices with full sets \( \Sigma_i \) and orthogonality relations \( \perp \). We shall define the direct union of the \( \mathcal{L}_i \) denoted by \( \bigoplus_i \mathcal{L}_i \). An element \( b \in \bigoplus_i \mathcal{L}_i \) will be written \( \bigoplus_i b_i \) where \( b_i \in \mathcal{L}_i \). We define a partial order relation and an orthogonality relation as follows:

\[
\bigoplus_i b_i < \bigoplus_i c_i \iff b_i < c_i \forall i
\]

\[
\bigoplus_i b_i \perp \bigoplus_i c_i \iff b_i \perp_i c_i \forall i
\]

and a set

\[
\Sigma = \{ \bigoplus_{j \neq i} O_i \bigotimes p_j \text{ where } p_j \in \Sigma_j \}
\]
then it is possible to show that $\bigodot_i \mathcal{L}_i$ is a complete lattice with a full set $\mathcal{S}$ and an orthogonality relation $\perp$. We introduce now the concept of classical question. A classical question is a question that has a certain outcome. A classical property is a property such that there exists a product of classical questions testing this property. We denote then the set of all classical properties by $\mathcal{C}$ and show that it is a sublattice of $\mathcal{L}$. We introduce the concept of classical mixture, as being the collection of all classical properties that are actual. We call $\Omega$ the collection of all the classical mixtures of the entity, it is the states space of the entity if we study only the classical part of the entity. This study can then be done on the hand of a theory as classical mechanics. But we can show more. If $\omega$ is a classical mixture of the entity we can consider the collection $\mathcal{L}_\omega$ of properties that are stronger than $\omega$. These are the properties that we encounter if we decide that this classical study is to rough. Then we can show that in general $\mathcal{L}$ is isomorphic to $\bigodot_\omega \mathcal{L}_\omega$.

This shows that we can really study separately the classical properties of an entity and the nonclassical properties.

Hence the degree of classicality of an entity is not defined by the number of atoms it contains, but by the nature of the questions that we can perform on the entity. The whole of this study is shown in Refs. 6 and 7 without the need of the axioms to be satisfied. The proof of such a decomposition existed already for the case of a weakly modular complete orthocomplemented atomic lattice and was given by Piron.4 Also the interpretation of this decomposition as representing superselection rules of the entity was already used extensively by Piron.4 One could say that it indicated in a certain sense the first shortcoming of orthodox quantum mechanics.

Piron shows that a weakly modular orthocomplemented complete atomic lattice $\mathcal{L}$ satisfying the covering law is isomorphic to the direct union $\bigodot_i \mathcal{L}_i$ of irreducible weakly modular orthocomplemented complete atomic lattices $\mathcal{L}_i$ satisfying the covering law. It is for such an irreducible weakly modular orthocomplemented complete atomic lattice $\mathcal{L}$ satisfying the covering law that Piron proves a representation theorem.

**Theorem 8.** If $\mathcal{L}$ is an irreducible complete orthocomplemented weakly modular atomic lattice satisfying the covering law and having at least four orthogonal atoms, then $\mathcal{L}$ is isomorphic to $P(V)$ where $V$, $D$, $\phi$, $*$ is a “generalized Hilbert space” which means that:

1. $V$ is a vectorspace over a division ring $D$
2. $\phi$ is a definite Hermition form on $V$
3. $*: D \to D$ is an involutive anti-automorphism.
\[ P(V) \text{ is defined as follows: If } M \subseteq V, \text{ then } M^1 = \{ x \mid \phi(x, y) = 0, y \in M, x \in V \} \text{ and } P(V) = \{ M^{-1} \mid M \subseteq V \}. \text{ So if we call } M^{-1} \text{ the closure of } M, \text{ then } P(V) \text{ is the set of all closed subspaces of } V. \text{ We have the additional property } P(V) \text{ is the set of all closed subspaces of } V. \text{ We have the additional property } M \subseteq P(V), M + M^1 = V. \]

For a proof of this theorem see Ref. 4 ch. 3. To see that a generalized Hilbert space deserves this name we have the following theorem:

**Theorem 9.** Every generalized Hilbert space over the complex field, with an involution which is the complex conjugation is a complex Hilbert space.

**Proof.** See Refs. 4 and 8.

This is to show that the five axioms that we propose are indeed the axioms of quantum mechanics, because they force the property lattice to be a complete orthocomplemented weakly modular atomic lattice satisfying the covering law. As we remarked already, it is the weak modularity introduced by axiom 4 and the covering law introduced by axiom 5 that are not satisfied in the property lattice of two separated entities. We shall have to analyze what it means for the structure of the theory if we drop these axioms. We shall see that this forces us to drop the superposition principle in quantum mechanics. The set of states of two separated systems cannot be described by a vectorspace structure.

### 6. SEPARATED QUESTIONS AND SEPARATED ENTITIES

#### 6.1. Separated Entities

We want to give a description of an entity \( S \) that consists of two separated entities \( S_1 \) and \( S_2 \). To be able to do this we have to analyze very carefully what we mean by the word "separated." Before we do this let us see what this means intuitively for us. To be able to speak of two separated entities \( S_1 \) and \( S_2 \), it must first of all be possible to perform experiments on \( S_1 \) and \( S_2 \) separately. More properly speaking for every two properties \( a_1 \) of \( S_1 \) and \( a_2 \) of \( S_2 \) it must be possible to construct an experiment that tests both properties. If this is not the case we cannot even say that \( S \) consists of two entities. We want to have more. This experiment has to be of such a kind that the answer that we get for the testing of \( a_1 \) is not "influenced" by the testing of \( a_2 \) and the answer that we get for the testing of \( a_2 \) is not "influenced" by the testing of \( a_1 \). Surely we have to explain what we mean by "influenced." We defined an entity as an idealization of the phenomenon in the sense that we consider only these properties of the phenomenon that are not influenced by the rest of the universe. It is the same meaning of
influence that we use when we talk about separated entities. Hence we could say that an entity is separated from the rest of the universe. In the same way as we imagine that an entity is in fact an idealization of the real phenomenon we can imagine that two phenomena are in fact never separated. But this is not the point, even when two phenomena are not separated the entities corresponding to these phenomena shall very often be separated. In fact every time we consider an entity that is a part of the outer world of another entity, these two entities will be separated. And again, if we should find in the future new interesting properties of the phenomena this can change the entities in such a way that they are not separated anymore. It becomes clear now that the idealization that we make lies in the concept of entity and not in the concept of separated. Once we agree that to do physics we have to introduce this concept of entity it is clear that most of the entities that we consider will be separated.

6.2. Separated Questions

If $\alpha$ and $\beta$ are two questions of the entity $S$, then in general it makes no sense to perform both questions together on the entity since the performance of one of the questions can change the state of the entity in such a way that it becomes irrelevant to perform the other question. We shall say that we can perform both questions $\alpha$ and $\beta$ together iff there exists an experiment $E(\alpha, \beta)$ having four outcomes that we shall label by $\{yes, yes\}$, $\{yes, no\}$, $\{no, yes\}$, and $\{no, no\}$, and such that

$\alpha$ is true iff we are certain to get one of the outcomes $\{yes, yes\}$ or $\{yes, no\}$ for the experiment $E$.

$\alpha^-$ is true iff we are certain to get one of the outcomes $\{no, yes\}$ or $\{no, no\}$ for $E$.

$\beta$ is true iff we are certain to get one of the outcomes $\{yes, yes\}$ or $\{no, yes\}$ for $E$.

$\beta^-$ is true iff we are certain to get one of the outcomes $\{yes, no\}$ or $\{no, no\}$ for $E$.

Remember that $E$ does not define a unique question since it has four possible outcomes. It is indeed the experiment $E$ that we perform in the laboratory if we want to test two questions together. From the experiment $E$ we can construct new questions. We will define the question

$\alpha \Delta \beta$ which consists of performing the experiment $E$ and attributing the answer "yes" if we get the outcome $\{yes, yes\}$. If we get one of the outcomes $\{yes, no\}$, $\{no, yes\}$, or $\{no, no\}$ we attribute the answer "no."
We also define the question
\[ \alpha \nabla \beta \] which consists of performing the experiment \( E \) and attributing the answer "yes" if we get one of the outcomes \{yes, yes\}, \{yes, no\}, or \{no, yes\}. If we get \{no, no\} we attribute the answer "no."

We also define the question
\[ \alpha \ominus \beta \] which consists of performing the experiment \( E \) and attributing the answer "yes" if we get one of the outcomes \{yes, yes\} or \{no, no\}. If we get \{yes, no\} or \{no, yes\} we attribute the answer "no."

What is now the meaning of these new questions. We can prove the following theorem.

**Theorem 10.** If \( \alpha \) and \( \beta \) are two questions that can be performed together then

\[ \alpha \triangle \beta \text{ true } \iff \alpha \text{ true and } \beta \text{ true} \]

As a consequence if \( \alpha \) and \( \beta \) are questions of \( S \) that can be performed together and \( a \) and \( b \) are the corresponding properties of \( S \) we have \( \alpha \triangle \beta \approx a \cdot \beta \) and hence \( \alpha \triangle \beta \subseteq a \land b \). \( \alpha \triangle \beta \) is a question that tests whether \( \alpha \) is true and \( \beta \) is true. Intuitively we would perhaps think that \( \alpha \nabla \beta \) is the question that tests whether \( \alpha \) is true or \( \beta \) is true. Let us see that this is in general not the case. Suppose that we have the situation that the entity \( S \) is in such a state that whenever the test of \( \alpha \) gives us the answer "no," the test of \( \beta \) gives us the answer "yes" and whenever a test of \( \beta \) gives us an answer "no," a test of \( \alpha \) gives us an answer "yes." This is what happens for example if we have an entity \( S \) consisting of two spin 1/2 particles in the singlet spin state, and we perform a test \( \alpha_{\theta, \phi} \) of the spin of one of the particles in direction \((\theta, \phi)\) in one region of space and a test \( \beta_{\theta, \phi} \) of the other particle in the same direction in an opposite region of space. The experiment \( E(\alpha_{\theta, \phi}, \beta_{\theta, \phi}) \) consists of performing \( \alpha \) and \( \beta \) together, and clearly for \( E \) we will always have one of the outcomes \{yes, no\} or \{no, yes\}. Hence \( \alpha \nabla \beta \) is true even when \( \alpha \) is not true and \( \beta \) is not true. To be able to have such a situation the performance of one of the questions has to "influence" the answer that we have for the other question. This is the reason that one of the outcomes for the experiment \( E \), namely in this case the outcome \{no, no\} will never occur when the entity is in such a state. We come now to the point where we can define what we mean by two questions \( \alpha \) and \( \beta \) that do not influence each other. We will call such questions *separated questions*. 
Definition 11. We will say that two questions $\alpha$ and $\beta$ of an entity $S$ that can be performed together are separated iff, when for an arbitrary state of the entity there is a certain chance to obtain one answer for $\alpha$ and another one for $\beta$, then there is for this state of the entity a certain chance to obtain this combination for $E(\alpha, \beta)$.

So for example if the entity is in a state such that yes is a possible answer for $\alpha$ and no is a possible answer for $\beta$ then $\{\text{yes, no}\}$ is a possible answer for $E(\alpha, \beta)$.

Definition 12. We will say that two properties $a, b$ of $S$ are separated iff there exist questions $\alpha \in a$ and $\beta \in b$ that are separated.

Theorem 13. If $\alpha$ and $\beta$ are two questions that are separated, and $a$ and $b$ are the properties tested by $\alpha$ and $\beta$ then

$\alpha \triangledown \beta$ true $\iff$ $\alpha$ true or $\beta$ true

$\alpha \ominus \beta$ true $\iff$ ($\alpha$ true and $\beta$ true) or ($\alpha$ false and $\beta$ false)

we also have

$\bar{(a \triangle \beta)} = \bar{a} \triangledown \bar{\beta}$ and $\bar{(a \triangledown \beta)} = \bar{a} \triangle \bar{\beta}$

$\bar{(a \ominus \beta)} = \bar{a} \ominus \bar{\beta} = a \ominus \beta$.

Proof. If $\alpha$ is true or $\beta$ is true, then $\alpha \triangledown \beta$ is true. Suppose now that $\alpha$ is not true and $\beta$ is not true. Then there is a certain chance to have the answer “no” for $\alpha$ and there is a certain chance to have the answer “no” for $\beta$. Since $\alpha$ and $\beta$ are separated, there is a certain chance to have the outcome $\{\text{no, no}\}$ for $E(\alpha, \beta)$. This shows that $\alpha \triangledown \beta$ is not true. From this follows that

$\alpha \triangledown \beta$ true $\Rightarrow$ $\alpha$ true or $\beta$ true

$\alpha \ominus \beta$ true $\iff$ we are certain that a test of $E(\alpha, \beta)$ gives us one

of the outcomes $\{\text{yes, yes}\}$ or $\{\text{no, no}\}$

$\iff (\alpha \triangledown \bar{\beta}) \cdot (\bar{\alpha} \triangledown \beta)$ true

$\iff (\alpha$ true or $\beta$ false) and ($\alpha$ false or $\beta$ true)

$\iff \alpha \triangle \beta$ true or $\bar{\alpha} \triangle \bar{\beta}$ true.

6.3. The Superposition Principle and Superselection Rules

Suppose that we have an entity $S$ and two properties $a, b$ of $S$. We know that

$a \wedge b$ is actual iff $a$ is actual and $b$ is actual
Hence \( a \land b \) is the property \( a \) "and" \( b \). We also defined

\[
\bigvee_{a < x, b < x, x \in \mathcal{X}} x
\]

We can see that \( a \land b \) is a property that is defined by \( a \) and \( b \) while \( \bigvee_{a < x, b < x, x \in \mathcal{X}} x \) is a property that depends on all the other properties of the entity. If we add new properties to \( \mathcal{X} \) then \( a \lor b \) will change in general. We know that \( a \lor b \) actual does not in general imply that \( a \) is actual or \( b \) is actual. Even when \( a \) and \( b \) are properties that can be tested together this is not necessarily the case as we saw on the hand of an example in Section 6.2. We also see this fact very well if the entity \( S \) is described by quantum mechanics in a Hilbert space \( \mathcal{H} \). If \( A, B \in \mathcal{P}(\mathcal{H}) \) then

\[
\nu(A) \lor \nu(B) = \nu(\text{lin} (A, B))
\]

The vectors that we get by making superpositions of vectors in \( A \) and vectors in \( B \) will represent the states that make \( \nu(A) \lor \nu(B) \) actual, but not \( \nu(A) \) nor \( \nu(B) \). This shows that the superposition principle in quantum mechanics is profoundly connected with this occurrence of states that make \( \nu(A) \lor \nu(B) \) actual but not \( \nu(A) \) nor \( \nu(B) \). The fact that some superposition states are never observed made one introduce the concept of superselection rule. Two states \( p \) and \( q \) are separated by a superselection rule if superpositions of these states are not observed. Let us introduce this concept in our formalism.

**Definition 14.** If we have an entity \( S \), and \( a \) and \( b \) are two properties of the entity \( S \) we will say that \( a \) and \( b \) are separated by a superselection rule (s.s.r.) iff for every state \( p \) of \( S \) such that \( p < a \lor b \) we have \( p < a \) or \( p < b \).

Let us see that in classical mechanics every two properties are separated by a superselection rule. Indeed if \( A, B \in \mathcal{P}(\mathcal{F}) \) and \( \nu(P) < \nu(A) \lor \nu(B) \), then \( \nu(P) < \nu(A) \) or \( \nu(P) < \nu(B) \). As we remarked already in general the property \( a \lor b \) depends on a lot of properties of the entity. There is however a case where \( a \lor b \) only depends on \( a \) and on \( b \). This is when we know a way to test whether \( a \) is actual or \( b \) is actual. We can show that this is just the case when \( a \) and \( b \) are separated by a superselection rule.

**Theorem 15.** If \( a \) and \( b \) are two properties of an entity \( S \) then \( a \) and \( b \) are separated by a superselection rule iff there exists a question \( \gamma \) that tests whether \( a \) is actual or \( b \) is actual. In this case \( \gamma \in a \lor b \).

**Proof.** If \( a, b \) are s.s.r. we just have to take \( \gamma \in a \lor b \). Suppose that there exists a \( \gamma \). Let us call \( c \) the property tested by \( \gamma \). If the entity is in a state \( p < a \lor b \). Since \( a < c \) and \( b < c \) we have \( a \lor b < c \). Hence \( p < c \). But then \( a \) is actual or \( b \) is actual. As a consequence \( p < a \) or \( p < b \).
From this theorem follows that the existence of superpositions is due to the lack of particular tests that can be done on the entity and this again shows that a property is not just a statement. The existence of a superselection rule between \( a \) and \( b \) is due to the existence of a question that can test whether \( a \) is actual or \( b \) is actual.

The superposition principle as it is formulated in quantum mechanics gives a wrong impression. Indeed one is tempted to think that the possibility of building new states by making superpositions of states is something quantum mechanics has more than classical mechanics. While it is just the other way around, it is because for entities described by quantum mechanics it is in general impossible for two properties \( a \) and \( b \) to construct a measuring apparatus that would enable us to define a question \( \gamma \) that tests whether \( a \) is actual or \( b \) is actual, that one has the superposition principle in quantum mechanics. It is because of this wrong impression about the superposition principle that people wonder why commonly we almost never observe a superposition of two states of a macroscopical system. The answer is that for a macroscopical system we can in general construct an experiment that leads to a question of the type \( \gamma \) in theorem 15 (an example of a construction procedure for such a question \( \gamma \) for a macroscopical system is given in Refs. 6 and 7).

**Theorem 16.** If \( a \) and \( b \) are two separated properties of an entity \( S \) then \( a \) and \( b \) are separated by a superselection rule.

*Proof.* If \( \alpha \) and \( \beta \) are separated questions testing \( a \) and \( b \), then \( \alpha \nabla \beta \) tests whether \( a \) is actual or \( b \) is actual.  

### 6.4. The Description of Separated Entities

We want to build the property lattice of an entity \( S \) consisting of two separated entities \( S_1 \) and \( S_2 \). In the following \( P_1, Q_1, E_1, L_1, \Sigma_1, \perp_1, P_2, Q_2, E_2, L_2, \Sigma_2, \perp_2, P, Q, E, L, \Sigma, \perp \) will represent the set of primitive questions, the set of questions, the set of primitive properties, the property lattice, the set of states, the orthogonality relation of \( S_1 \), \( S_2 \), and \( S \). Let us consider an entity \( S_1 \). If \( \alpha_1 \) is a question of \( S_1 \), then \( \alpha_1 \) is defined without making any reference to the outer world of \( S_1 \). This means that the answer that we get for the question \( \alpha_1 \) does not depends on what happens with the outer world of \( S_1 \). This however does not mean that the performance of the question \( \alpha_1 \) does not change the state of the outer world. No, in general the performance of the question \( \alpha_1 \) will profoundly change the state of the outer world of \( S_1 \). If \( S_2 \) is an entity that is separated from \( S_1 \) and so is a part of the outer world of \( S_1 \), then the question \( \alpha_1 \) can be performed on the entity \( S \).
which is the entity consisting of $S_1$ and $S_2$. And it makes sense not only to perform $a_1$ on $S_1$, but it is the same question whether we perform it on $S_1$ or on $S$. This shows that $Q_1 \subset Q$. As we remarked already, the performance of the question $a_1$ can however change considerably the state of $S_2$, it can even destroy $S_2$. This has a consequence that if we take an arbitrary question $a_2$ of $S_2$, it is not evident that we shall be able to find an experiment $E(a_1, a_2)$. It is only in the case that there exists a question $a'_1$ equivalent to $a_1$ such that $a'_1$ is equivalent to $a_2$ and such that $a'_1$ does not change the state of $S_2$ that we can construct $E(a'_1, a_2)$. But this is an experiment that is equivalent to $E(a_1, a_2)$.

**Definition 17.** If we have an entity $S$ consisting of two entities $S_1$ and $S_2$, then $S_1$ and $S_2$ are said to be separated iff every question of $S_1$ is separated from every question of $S_2$.

As we remarked already

$$Q_1 \subset Q \quad \text{and} \quad Q_2 \subset Q$$

If $a_1$ is a question of $S_1$ and $a_2$ a question of $S_2$ such that $a_1$ is separated from $a_2$, we can construct the following questions of $S$

$$|a_1, a_2, a_1 \sim, a_2 \sim, a_1 \oplus a_2, a_1 \sim \oplus a_2, a_1 \oplus a_2, a_1 \triangledown a_2, a_1 \triangledown a_2 \sim, a_1 \triangledown a_2 \sim, a_1 \triangledown a_2 \sim, a_1 \triangledown a_2 \sim, a_1 \triangle a_2, a_1 \cap a_2, a_1 \cap a_2, a_1 \cap a_2, a_1 \cap a_2, a_1 \cap a_2 |$$

These questions are constructed by considering the experiment $E(a_1, a_2)$. We want $S$ to contain nothing else than $S_1$ and $S_2$, and therefore we can considered this set to be a generating set for the questions of $S$. Hence,

$$G = \{ a_1, a_2, \beta_1 \triangle \beta_2, \gamma_1 \triangledown \gamma_2, \delta_1 \ominus \delta_2 \mid a_1, \beta_1, \gamma_1, \delta_1 \in Q_1 \}
$$

$$\text{and} \ a_2, \beta_2, \gamma_2, \delta_2 \in Q_2 \}$$

is a generating set for $S$. So $Q = \{ \pi_i a_i \mid a_i \in G \}$ is the set of questions of $S$. Hence an arbitrary question of $S$ is of the following form

$$a_1 \cdot a_2 \cdot \pi_i a'_1 \triangle a'_2 \cdot \pi_i a'_1 \triangledown a'_2 \cdot \pi_k a'_1 \ominus a'_2$$

where $a_1, a'_1, a_k \in Q_1$ and $a_2, a'_2, a'_k \in Q_2$. We will denote $Q$ by $Q_1 \oplus Q_2$ and call it the separated product of $Q_1$ and $Q_2$. The property lattice of $S$ we will denote by $L_1 \oplus L_2$. We have:

$$L_1 \oplus L_2 = \\{ a_1 \wedge a_2 \wedge \pi_i (a'_1 \wedge a'_2) \wedge \pi_k (a'_1 \wedge a'_2) \}$$

where $a_1, a'_1, a'_k, b'_k \in L_1$ and $a_2, a'_2, b'_k \in L_2$ and there exists $a'_k, a'_2$ such that $a'_k \in a'_1, a'_2 \in a'_2$ and $a'_1 \sim \in b'_1, a'_2 \sim \in b'_2$. 

6.5. The States of the Entity Consisting of Two Separated Entities

To determine the structure of the states of $S$ we first prove the following lemma.

**Lemma 18.** Suppose $\alpha \in Q_1 \otimes Q_2$. If $S$ is in a state such that $\alpha$ is true, then there exist $\alpha_1 \in Q_1$ and $\alpha_2 \in Q_2$ such that $\alpha_1 \triangle \alpha_2$ is true. Moreover, $\alpha_1 \triangle \alpha_2 < \alpha$.

*Proof.* If $\alpha \in Q_1 \otimes Q_2$ then

$$a = a_1 \cdot a_2 \cdot \pi_1 a_1^1 \triangle a_2^1 \cdot \pi_2 a_1^2 \vee a_1^1 \pi_1 a_2^k \oplus a_2^k$$

If $\alpha$ is true then $\alpha_1 \cdot \pi_1 a_1^1$ is true and $\alpha_2 \cdot \pi_2 a_2^1$ is true, and $a_1^1 \vee a_2^1$ is true for every $j$ and $(a_1^k$ and $a_2^k)$ or $(a_1^k$ and $a_2^k)$ is true for every $k$. Call $j_1$ those $j$ such that $\alpha_1^1$ is true and call $j_2$ those $j$ such that $\alpha_2^1$ is true. Call $k_1$ those $k$ such that $\alpha_1^k$ and $\alpha_2^k$ is true, and $k_2$ those $k$ such that $\alpha_1^k$ and $\alpha_2^k$ is true. If we put

$$\beta_1 = a_1 \cdot \pi_1 a_1^1 \cdot \pi_1, a_1^1 \pi_1 a_2^1 \cdot \pi_1 a_1^1 \pi_1, a_1^1 \pi_1 a_1^1 \pi_1$$

and

$$\beta_2 = a_2 \cdot \pi_1 a_2^1 \cdot \pi_1, a_2^1 \pi_1 a_2^1 \cdot \pi_1 a_2^1 \pi_1, a_2^1 \pi_1 a_2^1 \pi_1$$

then $\beta_1 \triangle \beta_2$ is true. Remark that since for every $j$ we have $\alpha_1^1$ or $\alpha_2^1$ true we have $\{|j_1| \cup |j_2| = |j|\}$. In an analogous way we have $\{|k_1| \cup |k_2| = |k|\}$. Suppose now that $\beta_1 \triangle \beta_2$ is true, then $\alpha_1 \cdot \pi_1 a_1^1$ is true and $\alpha_2 \cdot \pi_2 a_2^1$ is true, and for every $j$ we have $\alpha_1^1$ is true or $\alpha_2^1$ is true so in any case $a_1^1 \vee a_2^1$ is true. In an analogous way $\alpha_1^k \oplus a_2^k$ is true for every $k$. This shows that $\alpha$ is true. Hence $\beta_1 \triangle \beta_1 < \alpha$. ■

**Theorem 19.** If $\Sigma_1$ is the set of states of $S_1$ and $\Sigma_2$ is the set of states of $S_2$ then $\{p_1 \land p_2 \mid p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2\}$ is the set of states of $S$.

*Proof.* If $\varepsilon$ is the set of all properties of $S$ that are actual, then the state of $S$ is represented by $p = \land_{\alpha \in \varepsilon} a$. From the above lemma we know that there exists $a_1 \in \Sigma_1$ and $a_2 \in \Sigma_2$ such that $a_1 \wedge a_2$ is actual. Suppose now that $p_1$ is the state of $S_1$ and $p_2$ the state of $S_2$ than

$$p_1 \land p_2 < a_1 \land a_2 < p < p_1 \land p_2$$

This shows that $p = p_1 \land p_2$. If on the other hand the entities $S_1$ and $S_2$ are in states $p_1$ and $p_2$, then if $q_1 \land q_2$ is the state of $S$ we have

$$q_1 \land q_2 < p_1 \land p_2$$
Now \( q_1 \) is actual and \( q_2 \) is actual. So \( p_1 < q_1 \) and \( p_2 < q_2 \). But then \( p_1 \lor p_2 < q_1 \land q_2 \) which shows that

\[
q_1 \land q_2 = p_1 \land p_2
\]

This theorem shows that every state of \( S \) is determined by a state of \( S_1 \) and a state of \( S_2 \), just as we would expect. We will denote the set of states of \( S \) by \( \Sigma_1 \otimes \Sigma_2 \) and we will call it the separated product of \( \Sigma_1 \) and \( \Sigma_2 \).

**Theorem 20.** If \( a_1, b_1 \in \Sigma_1 \) and \( a_2, b_2 \in \Sigma_2 \) we have

1. \( a_1 \land a_2 = O \iff a_1 = O_1 \) or \( a_2 = O_2 \)
   \( a_1 \lor a_2 = I \iff a_1 = I_1 \) or \( a_2 = I_2 \)
2. if \( a_1 \land a_2 \neq O \) and \( b_1 \lor b_2 \neq I \)
   \( a_1 < b_1 \) and \( a_2 < b_2 \iff a_1 \land a_2 < b_1 \land b_2 \)
   \( \iff a_1 \lor a_2 < b_1 \lor b_2 \)
3. \( a_1 < b_1 \) or \( a_2 < b_2 \iff a_1 \land a_2 < b_1 \lor b_2 \)

### 6.6. The Orthogonality Relation

**Theorem 21.** If \( p_1 \land p_2, q_1 \land q_2 \in \Sigma_1 \otimes \Sigma_2 \) then:

\[
p_1 \land p_2 \perp q_1 \land q_2 \iff p_1 \perp q_1 \text{ or } p_2 \perp q_2
\]

**Proof.** Suppose that \( p_1 \perp q_1 \). Then these exists a \( a_1 \in Q_1 \) such that \( p_1 < a_1 \) and \( q_1 < a_1^- \). For \( p_2, q_2 \in \Sigma_2 \) we have

\[
p_1 \land p_2 < a_1 \quad \text{and} \quad q_1 \land q_2 < a_1^-
\]

which shows that \( p_1 \land p_2 \perp q_1 \land q_2 \). Suppose now that \( p_1 \land p_2 \perp q_1 \land q_2 \). Then there exists \( \alpha \in Q \) such that

\[
p_1 \land p_2 < \alpha \quad \text{and} \quad q_1 \land q_2 < \alpha^-
\]

If \( \alpha = a_1 \cdot \alpha_2 \cdot \pi_i a_1^i \triangle a_2^i \cdot \pi_j a_1^j \lor a_2^j \cdot \pi k a_1^k \otimes a_2^k \) then

\[
p_1 < a_1 \quad \text{and} \quad p_2 < a_2
\]
\[
p_1 < a_1^i \quad \text{and} \quad p_2 < a_2^i \lor i
\]
\[
p_1 < a_1^j \quad \text{or} \quad p_2 < a_2^j \lor j
\]
\[
(p_1 < a_1^k \text{ and } p_2 < a_2^k) \quad \text{or} \quad (p_1 < a_1^- \text{ and } p_2 < a_2^-) \forall k
\]
\[ a^- = a^- \cdot \pi_i a_i^\sim \bigvee a_i^\sim \cdot \pi_j a_j^\sim \bigtriangleup a_j^\sim \cdot \pi_k a_k^\sim \ominus a_k^\sim \] so

\[ q_1 < a_1^\sim \text{ and } q_2 < a_2^\sim \]
\[ q_1 < a_1^\sim \text{ or } q_2 < a_2^\sim \forall i \]
\[ q_1 < a_1^\sim \text{ and } q_2 < a_2^\sim \forall j \]
\[ (q_1 < a_1^k^- \text{ and } q_2 < a_2^k^-) \text{ or } (q_1 < a_1^k \text{ and } q_2 < a_2^k^-) \forall k \]

Call \( j_1 \) those \( j \) such that \( p_1 < a_1^j \) and \( j_2 \) those \( j \) such that \( p_2 < a_2^j \), call \( i_1 \) those \( i \) such that \( q_1 < a_1^i \) and \( i_2 \) those \( i \) such that \( q_2 < a_2^i \). For an arbitrary \( k \) we have

\[ \{(p_1 < a_1^k \text{ and } p_2 < a_2^k) \text{ or } (p_1 < a_1^k^- \text{ and } p_2 < a_2^k^-)\} \]

and

\[ \{(q_1 < a_1^k^- \text{ and } q_2 < a_2^k) \text{ or } (q_1 < a_1^k \text{ and } q_2 < a_2^k^-)\} \]

Hence:

\[ (p_1 < a_1^k \text{ and } p_2 < a_2^k \text{ and } q_1 < a_1^k^- \text{ and } q_2 < a_2^k) \] (1)

or

\[ (p_1 < a_1^k \text{ and } p_2 < a_2^k \text{ and } q_1 < a_1^k \text{ and } q_2 < a_2^k^-) \] (2)

or

\[ (p_1 < a_1^k^- \text{ and } p_2 < a_2^k^- \text{ and } q_1 < a_1^k^- \text{ and } q_2 < a_2^k) \] (3)

or

\[ (p_1 < a_1^k^- \text{ and } p_2 < a_2^k^- \text{ and } q_1 < a_1^k \text{ and } q_2 < a_2^k^-) \] (4)

Call \( k_1 \) those \( k \)'s such that (1) is true, \( k_2 \) those \( k \)'s such that (2) is true, \( k_3 \) those \( k \)'s such that (3) is true and \( k_4 \) those \( k \)'s such that (4) is true. Let us define

\[ \beta_1 = a_1 \cdot \pi_i a_i^1 \cdot \pi_j a_j^1 \cdot \pi_k a_k^1 \cdot \pi_k a_k^k \]
\[ \beta_2 = a_2 \cdot \pi_i a_i^2 \cdot \pi_j a_j^2 \cdot \pi_k a_k^2 \cdot \pi_k a_k^k \]

then it is easy to check that one of the two questions always exist if \( a \in O \). Then we see that

\[ p_1 < \beta_1 \text{ and } q_1 < \beta_1^- \]
or

\[ p_2 < \beta_2 \quad \text{and} \quad q_2 < \beta_2^* \]

This shows that \( p_2 \perp q_1 \) or \( p_2 \perp q_2 \).

On the side of this theorem the orthogonality relation in \( \mathcal{L}_1 \odot \mathcal{L}_2 \) is totally characterized once we know the orthogonality relations in \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

6.7. The Primitive Questions

**Theorem 22.** If \( \alpha \) is a primitive question of \( S \), such that neither \( \alpha \) nor \( \alpha^* \) is a trivial question, then \( \alpha \) is one of the following forms

- \( \alpha_1 \) where \( \alpha_1 \) is a primitive question of \( S_1 \)
- \( \alpha_2 \) where \( \alpha_2 \) is a primitive question of \( S_2 \)
- \( \beta_1 \triangle \beta_2 \) where \( \beta_1 \) is a primitive question of \( S_1 \) and \( \beta_2 \) is a primitive question of \( S_2 \)
- \( \gamma_1 \triangledown \gamma_2 \) where \( \gamma_1 \) is a primitive question of \( S_1 \) and \( \gamma_2 \) is a primitive question of \( S_2 \)
- \( \delta_1 \uplus \delta_2 \) where \( \delta_1 \) is a primitive question of \( S_1 \) and \( \delta_2 \) is a primitive question of \( S_2 \)

**Proof.** \( \alpha \) is a question of \( S \), so in general

\[ \alpha = \alpha_1 \cdot \alpha_2 \cdot \pi_i a_i^l \triangle \alpha_2^l \cdot \pi_j a_j^l \triangledown a_2^l \cdot \pi_k a_k^l \uplus a_2^l \]

where \( a_1, a_1^l, a_1^l, a_1^k \) are questions of \( S_1 \) and \( a_2, a_2^l, a_2^l, a_2^k \) are questions of \( S_2 \). Since \( \alpha \) is a primitive question, \( \alpha \) and \( \alpha^* \) must both be a product of equivalent questions. Now \( \alpha_1 \approx \alpha_2^l \) and \( \alpha_1^l \approx \alpha_2 \) is only possible if \( \alpha_1 \) or \( \alpha_2 \) is a trivial question, and \( a_2 \) or \( a_2^l \) is a trivial question. But then \( \alpha \) or \( \alpha^* \) is a trivial question. The same reasoning can be made for the other components of \( \alpha \), and this enables us to show that \( \alpha \) must be one of the forms \( \alpha_1, \alpha_2, \beta_1 \triangle \beta_2, \gamma_1 \triangledown \gamma_2, \delta_1 \uplus \delta_2 \). It is then possible to show that \( \alpha_1, \beta_1, \gamma_1, \delta_1 \) must be primitive questions of \( S_1 \) and \( \alpha_2, \beta_2, \gamma_2, \delta_2 \) must be primitive questions of \( S_2 \).

From this theorem follows that the set of primitive questions of \( S \) is the following

\[ P = \{ \alpha_1, \alpha_2, \beta_1 \triangle \beta_2, \gamma_1 \triangledown \gamma_2, \delta_1 \uplus \delta_2 \} \]

\[ \alpha_1, \beta_1, \gamma_1, \delta_1 \in P_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in P_2 \]
Proof: Suppose that $S_1$ and $S_2$ satisfy axiom 2. Consider a state $p_1 \land p_2$ of $S$. There exists a question $a_1$ of $S_1$ and a question $a_2$ of $S_2$ such that $a_1$ is true iff $S_1$ is in a state $q_1 \perp p_1$, and $a_2$ is true iff $S_2$ is in a state $q_2 \perp p_2$

$$a_1 \lor a_2 \text{ true } \iff (S_1 \text{ is in a state } q_1 \perp p_1) \text{ or } (S_2 \text{ is in a state } q_2 \perp p_2)$$

$$\iff S \text{ is in state } q_1 \land q_2 \perp p_1 \land p_2$$

This shows that $S$ satisfies axiom 2. Suppose now on the other hand that $S$ satisfies axiom 2. Consider a state $p_1$ of $S_1$. Then $p_1$ is a property of $S$. As a consequence there exists a question $a$ of $S$ such that $a$ is true iff $S$ is in a state orthogonal to $p_1$. Now $q_1 \land q_2 \perp p_1 \iff q_1 \perp p_1$. So $a$ is true iff $S_1$ is in a state $q_1 \perp p_1$. It is easy to show that in this case $a$ is equivalent to a question of $S_1$.

7.3. The Validity of Axiom 3

Theorem 26. Axiom 3 is satisfied for $S$ iff axiom 3 is satisfied for $S_1$ and for $S_2$.

Proof: Suppose that $S$ satisfies axiom 3. Take $p_1 \in \Sigma_1$ and $O < a_1 < p_1$ for $a_1 \in \mathcal{L}_1$. Then $O < a_1 \land p_2 < p_1 \land p_2$ for $p_2 \in \Sigma_2$. As a consequence $a_1 \land p_2 = O$ or $a_1 \land p_2 = p_1 \land p_1$. From this follows that $a_1 = O$ or $a_1 = p_1$. This shows that $S_1$ satisfies axiom 3. In an analogous way we show that $S_2$ satisfies axiom 3. Suppose now that $S_1$ and $S_2$ satisfy axiom 3. Take $p_1 \land p_2 \in \Sigma_1 \otimes \Sigma_2$. If $a \in \mathcal{L}_1 \otimes \mathcal{L}_2$ and $O < a < p_1 \land p_2$, then $a = (a \land p_1) \land (a \land p_2)$. Then $a \land p_1 = O_1$ or $a \land p_1 = p_1$ and $a \land p_2 = O_2$ or $a \land p_2 = p_2$. If $a \land p_1 = O_1$ or $a \land p_2 = O_2$ then $a = 0$. Otherwise $a = p_1 \land p_2$.

7.4. Why Quantum Mechanics Cannot Describe Separated Entities

We will show in this section that the property lattice of the entity consisting of two separated entities does not satisfy axiom 4 and does not satisfy axiom 5 if both entities are nonclassical entities. To be able to show this we have to characterize classical entities (i.e., entities described by classical mechanics). As we remarked in section 5, this is done in great detail in Refs. 6 and 7. We will just retain one of the characterizations of Refs. 6 and 7 that is sufficiently intuitive and that we will use to prove the main theorem of this paper.
Theorem 27. An entity $S$ is a classical entity whenever two states that are different are also orthogonal.

Proof. See Refs. 6 and 7. 

Hence a classical entity is characterized by the fact that its orthogonal relation becomes trivial. We shall now show that axiom 4 and axiom 5 make it impossible to have states that are separated by a superselection rule and are not orthogonal. It is just such states that do exist for an entity consisting of separated entities.

Theorem 28. Suppose that $S$ is an entity and axiom 1, axiom 2, and axiom 3 are satisfied for $S$. If axiom 4 is satisfied or axiom 5 is satisfied then two arbitrary different states $p$ and $q$ of $S$ that are separated by a superselection rule are orthogonal.

Proof. Suppose that $p$ and $q$ are two states of $S$ that are separated by a superselection rule. Then

$$(p \lor q) \land q' = (p \land q') \lor (q \land q') = p \land q'$$

Now $O < p \land q' < p$ which shows that $p \perp q$ or $p \land q' = O$. Suppose now that $p \land q' = O$ and axiom 4 are satisfied. Then $(p \lor q) \land q' = O$. From this follows:

$$[(p \lor q) \land q'] \lor q = q$$

or

$$(p \lor q) \land (q' \lor q) = q$$

or

$$p \lor q = q \quad \text{which implies that} \quad p = q$$

Suppose now that $p \land q' = O$ and axiom 5 is satisfied, then $q \lor [p' \land q'] = I$. Hence $I$ covers $p' \land q'$. But $p' \land q' < p' \land I$. So $p' \land q' = p'$, but then $p' = q'$ which implies that $p = q$. So in both cases we find $p = q$ or $p \perp q$.

Consequence. Suppose that axiom 1, axiom 2, axiom 3 are satisfied for an entity $S$. Then axiom 4 can only be satisfied if for any two different states of the entity that are not orthogonal, it is impossible to find a question $y$ that test whether the entity is in the state $p$ or whether the entity is in the state $q$. The same conclusion can be drawn concerning axiom 5. It is just such a defended situation that we can easily create for two separated entities. Let us show this.
Theorem 29. Suppose that $S$ is the entity consisting of two separated
entities $S_1$ and $S_2$, and suppose that axiom 1, 2, and 3 are satisfied. If $p_1, q_1$
are two different states of $S_1$ and $p_2, q_2$ are two different states of $S_2$, then
$p_1 \land p_2$ and $q_1 \land q_2$ are separated by a superselection rule.

Proof. Suppose that $\alpha_1$ tests $p_1$, $\alpha_2$ tests $p_2$, $\beta_1$ tests $q_1$, and $\beta_2$ tests $q_2$.
Consider then the question

\[(\alpha_1 \lor \beta_2) \cdot (\beta_1 \lor \alpha_2)\]

\[(\alpha_1 \lor \beta_2) \cdot (\beta_1 \lor \alpha_2) \text{ true } \iff (\alpha_1 \text{ true and } \beta_2 \text{ true}) \, \text{ or } \, (\alpha_1 \text{ true and } \alpha_2 \text{ true}) \, \text{ true}
\]

\[\iff (\alpha_1 \text{ true and } \beta_2 \text{ true}) \, \text{ or } \, (\alpha_1 \text{ true and } \alpha_2 \text{ true}) \, \text{ or } \, (\beta_1 \text{ true and } \beta_2 \text{ true}) \, \text{ or } \, (\alpha_2 \text{ true and } \beta_2 \text{ true}) \, \text{ true}
\]

\[\iff (\alpha_1 \land \alpha_2 \text{ true or } \beta_1 \land \beta_2 \text{ true}) \, \text{ true}
\]

\[\iff \text{ S is in state } p_2 \land p_2 \text{ or S is in state } q_1 \land q_2 \]

Now we easily see that if $p_1 \not\in p_1 \land q_1$ and $p_2 \not\in p_2 \land q_2$, then $p_1 \land p_1 \not\in q_1 \land q_2$.
This is a situation which cannot occur when axiom 4 or axiom 5 are satisfied
(Theorem 28). So here we see very clearly what goes wrong if axiom 4 or
axiom 5 are satisfied. However, we can show more. Let us write down the
main theorem of this paper.

Theorem 30. Suppose that $S$ is the entity consisting of two separated
entities $S_1$ and $S_2$, and axiom 1, axiom 2, axiom 3 are satisfied. If axiom 4 or
axiom 5 is satisfied for $S$, then one of the two entities $S_1$ or $S_2$ is a classical
entity.

Proof. Suppose that $S_2$ is not a classical entity. From Theorem 27
follows that in this case there exists two states $p_2$ and $q_2$ of $S_2$ such that
$p_2 \neq q_2$ and $p_2 \not\in q_2$. Suppose that $p_1$ and $q_1$ are arbitrary states of $S_1$ such that
$p_1 \neq q_1$. From Theorem 29 follows that $p_1 \land p_2$ and $q_1 \land q_2$ are
separated by a superselection rule. If now axiom 4 or axiom 5 (one of the
two axioms suffices) is satisfied then from Theorem 28 follows that
$p_1 \land p_2 \perp q_1 \land q_2$. From Theorem 21 follows then that $p_1 \perp q_1$.
Hence we show that two arbitrary different states of $S_1$ are orthogonal. Applying again
Theorem 27 we conclude that $S_1$ is a classical entity.

Conclusion. From this theorem we can conclude that whenever two entities $S_1$ and $S_2$ have at least one nonclassical property (as is the case for
entities described by quantum mechanics), the property lattice of the entity $S$
consisting of the two separated entities $S_1$ and $S_2$ cannot satisfy axiom 4 and cannot satisfy axiom 5. As a consequence such an entity $S$ can certainly not be described by quantum mechanics, since quantum mechanics satisfies axiom 4 and axiom 5. This is the reason why one encounters paradoxical situation if one does describes two separated entities by quantum mechanics. The essential thing that makes it impossible for a theory satisfying axiom 4 or axiom 5 to describe separated entities is that fact that such a theory cannot describe nonorthogonal states that are separated by a superselection rule. Indeed from Theorem 29 follows that the entity consisting of separated entities has nonorthogonal states that are separated by a superselection rule. It is axiom 4 and axiom 5 that cause the trouble. Let us analyze what these axioms mean. It is axiom 5 that makes it possible to show that the set of states has a vectorspace structure. This vectorspace structure is essential to be able to formulate the superposition principle for the states. Hence we must conclude that the superposition principle is not something which must be taken as an axiom for a theory. This is not so strange if we consider the analysis made of the superposition principle in Section 5.5. It is axiom 4 that makes the vectorspace form the structure that we need in an essential way if we want to represent observables in a satisfactory way by operators. Indeed in all the generalizations of quantum mechanics where axiom 5, and often also axiom 3 has been dropped, axiom 4 is still retained. Sometimes axiom 4 is explicitly postulated (e.g., quantum logical approach), this out of analogy with ordinary quantum mechanics, and of course also because it is more easy to calculate in a weakly modular lattice, than in a lattice which is not weakly modular, because then a lot of nice theorems are not true anymore. Sometimes the generalization in question automatically satisfies axiom 4. This is the case for most of the algebraic approaches (e.g., von Neumann algebra approach). We want to remark also, that although Theorem 30 which is in a certain sense our main theorem, is formulated and proved in the concrete property lattice formalism that we put forward, it can very easily be reformulated and reproved in another formalism (e.g., quantum logic). We make this remark to draw attention to the fact that the conclusion of this theorem, namely that we cannot describe two nonclassical separated systems if the lattice of properties is weakly modular, remains valid for another approach. There is really a structural incompatibility between the weak modularity and the concept of separated physical systems, that is not due to the specific interpretation we put forward. This is the reason why we think that the conclusion of Theorem 5 should have some repercussion on quantum logic and on other generalizations of quantum mechanics. One could think that the fact that quantum mechanics cannot describe separated entities is not so bad, since when two entities $S_1$ and $S_2$ are separated, we can describe them separate, both by quantum mechanics. This is indeed what
people do. The incapacity of quantum mechanics to deal with separated entities has also a consequence, that it is impossible to give a meaning in quantum mechanics to the concept of separation of entities. Quantum mechanics predicts that correlation effects due to the fact that the entities are not separated must always be present, even when the entities have been separated. Hence it is impossible to give a description of the act of separation of two entities, and for the same reason it is impossible to give a description of the measuring procedure in quantum mechanics. If one does try to describe these situation in quantum mechanics, paradoxical situations occur. Let us analyze shortly one of these paradoxes. A more detailed analysis can be found in Refs. 6 and 9.

7.5. The Einstein Podolsky Rosen Paradox

In their paper(1) Einstein, Podolsky, and Rosen examine the following two alternatives:

1. the quantum mechanical description of reality given by the wave function is not complete.

2. when operators corresponding to two physical quantities do not commute, the two quantities cannot have simultaneous reality.

Obviously these two sentences cannot both be wrong. Indeed, if two quantities corresponding to noncommuting operators have simultaneous reality, this should be described by the wave function of the entity under consideration if (1) is wrong. Since this is not the case in quantum mechanics, we conclude that one of the two sentences has to be right. So we must have one of the three cases

A (1) false and (2) true

B (1) true and (2) false

C (1) true and (2) true

Once EPR come to this conclusion, they consider the situation of two separated entities $S_1$ and $S_2$. By applying quantum mechanics to describe these two separated entities, they show that (2) is false. As a consequence they conclude that (1) has to be true. I want to draw the attention to the fact that EPR did not “prove” the falseness of (2). Indeed, in the example of the two separated entities they use quantum mechanics to describe these two separated entities and to arrive at the falseness of (2). To be allowed to do so, they must make the hypothesis:

1. quantum mechanics describes in a correct and complete way two separated entities.
Then we indeed have

\[(3) \text{ true } \Rightarrow (2) \text{ false } \Rightarrow (1) \text{ true}\]

from which we can conclude that (3) is false or (1) is true. As we showed in
the foregoing (3) is indeed false. Then it becomes also clear that the proof of
EPR is a proof ex absurdum. If (3) is true, then (1) is true. And it is obvious
that the results that one finds during a proof ex absurdum are not
necessarily true. This is the reason why they did not prove the falseness of
(2). It seems to be that EPR were aware of this fact, but it is clear that later
on people reinterpreted the paper of EPR in the following way: “Quantum
mechanics is not a complete theory since when operators corresponding to
two physical quantities do not commute, the two quantities cannot have
simultaneous reality” or, in other words, “quantum mechanics is not a
complete theory because (2) is true in quantum mechanics.” This is a wrong
deduction of the EPR paper. However it is this deduction that leads directly
to the following thought: To avoid the EPR paradox we have to build a
theory that takes into account the fact that even when operators do not
commute the corresponding physical quantities can have simultaneous
reality. This wrong deduction of the EPR paper which takes again for
granted that EPR proved the falseness of (2) made it seem possible to find a
solution of the paradox by introducing classical hidden variables. Now that
we know that (2) is just a sentence used in the EPR paper to make their
proof ex absurdum, we can ask ourselves which of the three alternatives, A,
B, and C is true. In Refs. 6 and 9 we again show, this time by pointing out
the missing elements of reality, that quantum mechanics is not a complete
theory. In Ref. 7 we show an experimental example that (2) is true. Hence
of the three alternatives, it is C that is true, quantum mechanics is not a
complete theory and there exist properties that have no simultaneous reality.
To resolve the incompleteness of quantum mechanics it is not at all
necessary to construct a theory with classical hidden variables. In fact in
Ref. 7 we show the contrary. Every theory can be seen as a classical theory
with nonclassical (eventually quantum) hidden variables. The incompleteness
problem of quantum mechanics disappears if one replaces quantum
mechanics by the theory we put forward. Indeed this theory is complete from
the start. Let us show this.

8. ELEMENTS OF REALITY AND COMPLETENESS OF THE
THEORY

Let us recall the definition of an element of reality given by Einstein,
Podolsky, and Rosen: “If without in any way disturbing a system, we can
predict with certainty the value of a physical quantity, then there exists an
element of reality corresponding to this physical quantity.” If we know that
the proposal of a test has a result that is certain, then we know that one of
the questions $a$ or $\alpha$ corresponding to this test is true. So we see that the
“true questions” that we defined are just the elements in our theory that
correspond with the elements of reality of the entity. Now our theory
examines a set of questions of a phenomenon. This set of questions defines
an entity. The condition of completeness put forward by EPR is the
following: “A theory is complete is every element of reality has a coun-
terpart in the theory.” Certainly EPR did not mean that a theory should
describe all the possible elements of reality of the phenomenon. A theory
never describes exactly the phenomenon, but always an entity corresponding
to this phenomenon. Therefore, we should like to put this criterium of
completeness in a slightly different way. We would say that: “A theory is
complete if it can describe every possible element of reality of the
phenomenon, without leading to contradictions.” In Refs. 6 and 9 we show
that this is not the case for quantum mechanics. Per construction this is the
case for the theory that we put forward. If we add elements of reality to the
entity, we have to add the corresponding questions to the theory. We will
never find any contradictions, since the structure of the theory does not
change by adding questions or by taking questions away.

CONCLUSION

Classical mechanics is a theory that has the following prejudice. If we
can predict with certainty the value of a physical quantity, for a certain state
of the physical system, then this is the case for every state of the physical
system. Or in other words, if there is an element of reality present for a
certain state of the system, then this element of reality is present for every
state of the system (see the analysis in Refs. 5, 6, and 7). We can see
experimentally that this is not true (see Refs. 6 and 7). Quantum mechanics
takes this fact into account by introducing observables that are not
compatible. The mathematical structure of quantum mechanics is however
too specific, and in applying this mathematical structure one unconsciously
introduces other prejudices? Namely the fact that it is not possible to have
nonorthogonal states that are separated by a superselection rule. When we
want to describe the separation of two physical systems, then automatically
states of this kind exist. This is the reason that quantum mechanics and as
far as I know also its generalizations cannot describe this happening. The
theory that we used does not have these shortcomings. It can describe
classical situations and quantum situations, and also a mixtures of both. For
the question of systems that are getting separated, we can expect that the quantum correlations, that are now always present, shall die out if the systems get more separated, and it must be possible to describe this. We also want to remark that the theory shall perhaps make it possible to describe the measuring process, since it seems that a measuring process is in a certain sense a unification and then again a separation of the measuring apparatus, and the physical system.

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