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Quantum Probability, the Classical Limit and Non-Locality.

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ABSTRACT.

We investigate quantum mechanics using an approach where the quantum probabilities arise as a consequence of the presence of fluctuations on the experimental apparatuses. We show that the full quantum structure can be obtained in this way and define the classical limit as the physical situation that arises when the fluctuations on the experimental apparatuses disappear. In the limit case we come to a classical structure but in between we find structures that are neither quantum nor classical. In this sense, our approach not only gives an explanation for the non-classical structure of quantum theory, but also makes it possible to define, and study the structure describing the intermediate new situations. By investigating in which way the non-local quantum behaviour disappears during the limiting process we can explain the 'apparent' locality of the classical macroscopical world.

1. Introduction.

Many aspects of quantum mechanics remain puzzling, still today, more than sixty years after its birth. We want to concentrate on three of these paradoxical aspects, and explain in which way the approach that we have elaborated in our Brussels group, can provide an answer to them.

1.1 The Origin of the Quantum Probabilities.

The probabilistic behaviour of quantum entities has been of great concern to the community of physicists working on the foundations of the theory. Indeterminism in itself is not a problem, after all also statistical mechanics is a non-deterministic theory. The structure of the quantum probabilities however seems to make it impossible to construct an underlying, even hypothetical, deterministic theory. This indicates that quantum indeterminism does not find its origin in a lack of knowledge about a deeper objective reality, as it is the case for the probabilistic structure

of classical statistical mechanics. As a consequence, the quantum probabilities are often considered as 'intrinsically present in nature', and whatever this means, if it is true, it makes the existence of an 'objective' reality problematic. Whether quantum indeterminism originates in a lack of knowledge about a deeper, still unknown, reality of the entity under study, has been investigated in great detail. A possible 'deterministic' generalization of a probabilistic theory in this sense has been called a 'hidden variable' theory. Already during the first years of quantum theory, John Von Neumann formulated a 'no-go-theorem' [1] about hidden-variable theories for quantum mechanics. Von Neumann's original no-go-theorem was criticised by several authors [2], and then refined by others [3], leading finally to the following conclusion : when one forces a 'mathematical' hidden variable theory, recovering all the predictions of quantum mechanics, it always contains an aspect that makes it very un-physical, namely, the hidden variables defining the micro-state depend explicitly on the experiment that one considers. In this sense, this kind of hidden variables, 'contextual' as they have been called, cannot be given a physical meaning when attached to the physical entity under study, and therefore they were considered to be of no physical interest. With these results many of the mathematically oriented physicists were convinced that 'physical' hidden variables theories were impossible for quantum mechanics. John Von Neumann wrote down his no-go-theorem in 1932, and three years later Einstein, Podolsky and Rosen presented their famous paper [4], where they show that quantum mechanics is an incomplete theory. EPR proof that, if quantum mechanics can be correctly applied to a situation of separated physical entities, it is possible to define elements of reality of the states of the entities, not contained in the quantum description. This means that there do exists hidden variables in the sense of Von Neumann. David Bohm presented another version of the EPR reasoning, and constructed explicitly a hidden variable theory for quantum mechanics. Also John Bell was convinced by the result of the EPR paper, the incompleteness of quantum mechanics, and the presence of hidden variables to complete it. He derived his famous inequalities, that should be violated if nature follows quantum mechanics in its description of entities detectable in widely separated regions of space. This was the situation at the end of the sixties : Von Neumann's theorem and its refinements giving rise to the belief in the impossibility of hidden variable theories, and the EPR reasoning and its elaborations, indicating the existence of hidden variables completing quantum mechanics. In our Brussels group, we have developed an approach [7,8,9,10,11,12,13], where the quantum probabilities are derived as a consequence of the presence of fluctuations in the experimental apparatuses. The quantum state

is considered to be a pure state, hence there are no hidden variables in the sense of Von Neumann, describing a deeper reality of the entity under study, but the experiments are not pure. Different experiments, considered to be equivalent on the macroscopical level, and hence presented by the same mathematical concept in the theory, are not identical on a deeper level. This lack of knowledge about the deeper reality of the experimental apparatuses and their interaction with the entity, is in our approach the origin of the presence of the quantum probabilities. We can easily understand why our approach allows us to construct 'classical' models giving rise to quantum probability structures, or with other words, why our examples escape the no-go-theorems about hidden variable theories. Indeed, on a purely mathematical level, we can describe the fluctuations in the experimental apparatuses by means of a variable, and interpret this as a hidden variable. The mathematical model then reproduces, as an hidden variable theory, the quantum probabilities. But, since the variable belongs to the experimental apparatus, and not to the state of the entity, this hidden variable theory is highly contextual, and hence escapes the no-go-theorems.

1.2 The Classical Limit

Although there are connections on different levels between quantum mechanics and classical mechanics, the fundamental relation between both theories is still obscure. Many general axiomatic approaches lead to structures that incorporate quantum as well as classical theories, but also here the fundamental nature of the relation is not understood. Let us indicate the situation connected with the study of the relation between quantum and classical by the problem of the 'classical limit'. In our approach there is a straightforward way to investigate the relation between quantum mechanics and classical mechanics. Indeed, if the quantum probabilities find their origin in the presence of fluctuations on the experimental apparatuses, we consider the quantum situation as the one where these fluctuations are maximal, the classical situation as the one where they are zero, and in between we have a new type of situation, that we call 'intermediate'. These 'intermediate' situations give rise to a structure, for the collection of properties and for the probability model, that is neither quantum nor classical. In our approach we have parametrized the amount of fluctuations by means of a number $\epsilon \in [0, 1]$, where $\epsilon = 0$ corresponds to the classical situation of no fluctuations, leading to classical structures (Boolean lattice of properties and Kolmogorovian probability model), $\epsilon = 1$ to the quantum situation of maximal fluctuations, leading to quantum structures (quantum lattice of properties and quantum probability model), and $0 < \epsilon < 1$ to intermediate

situations, giving rise to a property-lattice and a probability model that are neither Boolean or Kolmogorovian nor quantum. In [12,13] we study explicitly a physical example, that we have called the ϵ-example, and that gives rise to a model that is isomorphic with the quantum mechanical description of the spin of a spin $\frac{1}{2}$ quantum entity for the case $\epsilon = 1$. For $\epsilon = 0$ the ϵ-example describes a classical mechanics situation, and for values of ϵ between 0 and 1 we find intermediate situations, with structures that are neither quantum nor classical, as is shown explicitly in [12,13]. The limit $\epsilon \to 0$ is the classical limit in our approach. We have generalized this approach to the realistic situation of a quantum entity described in an infinite dimensional Hilbert space $L^2(\Re^3)$, and again we define a continuous limit process, $\epsilon \to 0$, corresponding to a continuous decrease of fluctuations on the measurement apparatuses. Starting with a presentation of the two dimensional case in section 2, we investigate the n-dimensional case in section 3, and the general, realistic and infinite dimensional case, in section 4.

1.3 Non-Locality.

The theoretical prediction and experimental verification of non-local quantum behaviour, in the form of non-local correlations (the experiments to verify Bell's inequalities), and as a manifestly present reality (Rauch's experiments with the neutron-interferometer), is a more recent, but as difficult to understand aspect of quantum theory. Many physicist have pointed out many problems caused by the presence of non-locality, only to mention : 'causality', and 'classical separability' of distant entities. The investigations of Bell's inequalities have lead to experiments on quantum entities that one tries to keep in a quantum mechanical superposition state (non-product state), while they are detected in largely separated regions of space. These experiments have shown that the quantum non-separation of the two entities, giving rise to a type of correlations that have been called quantum correlations, remains intact, even under these extreme experimental situations [14,15]. Experiments of the group of Helmut Rauch with ultra-cold neutrons and a neutron interferometer have confronted us in a more direct way with this non-local behaviour of quantum entities [16]. Since the macroscopical world, described by classical theories, does not contain non-local effects, we can investigate in which way the non-locality disappears in the classical limit corresponding to disappearing fluctuations on the experimental apparatuses in our approach. The answer is the following : non-local states remain present, even in the limit for disappearing fluctuations, but their number is very small (the measure of the set of non-local states is zero in the classical limit for one entity), and they are unstable, in the

sense that a small disturbance will change a non-local state into a local one. That is the reason why they are not detected in the macroscopical world, and we analyse this situation in section 5.

2. The Two Dimensional Case.

2.1 Introduction of the ϵ-Example.

The physical entity S that we want to consider is a particle characterized by a parameter q (in [7] this parameter was a negative charge, in [8] it was a mass). The particle is located on a sphere of radius r, and can move on this sphere, and the different locations of the particle on the sphere represent the states of the particle. Let us consider the centre of the sphere as the origin of a three dimensional vectorspace, then every point on the surface of the sphere can be indicated by a vector v of length r, and that is the way in which we shall represent these points. A particle in point v will be said to be in state p_v (see fig. 1). We shall denote the set of possible states of the entity by Σ, and hence $\Sigma = \{p_v \mid v \text{ is a point of the sphere}\}$.

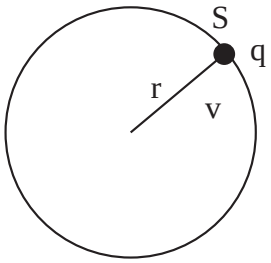

fig 1 : A representation of the state of the entity.

For an arbitrary point u of the surface of the sphere we define the experiment e_u as follows : we choose two particles characterized by parameters q_1 and q_2 (these are positive charges in [7] and masses in [8]) such that $q_1 + q_2 = Q$. The parameters q_1 and q_2 are chosen randomly in the interval $[0, Q]$ according to a probability measure that we will explicitly describe immediately. This random choice of the parameters q_1 and q_2 represents the 'lack of knowledge' about the experimental situation. Once the two parameters q_1 and q_2 are chosen we put the two particles diametrically on the sphere, such that q_1 is in the point u and q_2 in the point $-u$. This is the set-up of an experiment e_u. Let us describe what happens next. We can see easily that the three particles are located in a plane, particle q in a point v, and particles q_1 and q_2 in diametrically opposed points u and $-u$ (see fig 2).

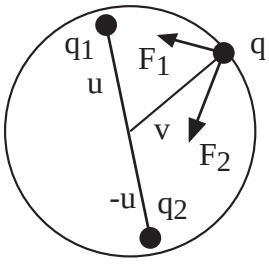

fig 2 : A representation of the experiment.

We suppose that there are forces of attraction between the particles, a force F_1 between q and q_1 and a force F_2 between q and q_2, that are directed along the connecting lines between the particles and proportional to the product of the parameters characterizing the particles divided by the square of the distance between the particles (Coulomb forces if q, q_1 and q_2 represent

charges, as in [7], and gravitational forces if q, q_1, and q_2 represent masses, as in [8]). Introducing a constant C we have :

$$\begin{aligned} F_1 &= C \cdot \frac{q_1 q}{4r^2 sin^2(\frac{\theta(u,v)}{2})} = \frac{C}{2} \cdot \frac{q_1 q}{r^2 - u \cdot v} \\ F_2 &= C \cdot \frac{q_2 q}{4r^2 cos^2(\frac{\theta(u,v)}{2})} = \frac{C}{2} \cdot \frac{q_1 q}{r^2 + u \cdot v} \end{aligned} \tag{1}$$

We suppose that the particle q moves under the influence of the two forces F_1 and F_2, and finally, since the motion happens in a viscous medium, arrives at rest at one of the two points u or $-u$, depending on the magnitude of the forces of attraction between the three particles. If $|F_1| > |F_2|$, q will end up at u, and the outcome $o_{u,1}$ occurs for the experiment e_u, and this experiment has transformed the state p_v into the state p_u. If $|F_1| < |F_2|$, q will end up at $-u$, and the outcome $o_{u,2}$ occurs for the experiment e_u, and this experiment has transformed the state p_v into the state p_{-u}. Let us now describe explicitly in which way the parameters q_1 and q_2 fluctuate in the interval $[0, Q]$ for an experiment e_u. We choose q_1 to fluctuate uniformly in an interval $[q_{1,min}, q_{1,max}] \subset [0, Q]$. The constraint $q_1 + q_2 = Q$ implies that then q_2 fluctuates uniformly in an interval $[q_{2,min}, q_{2,max}]$ with $q_{2,min} = Q - q_{1,max}$ and $q_{2,max} = Q - q_{1,min}$. Hence an experiment e_u is defined by two real parameters $q_{1,min}$, $q_{1,max}$ in $[0, Q]$ and by a point u on the sphere. For reasons of simplicity, we reparametrize, introducing two new real parameters ϵ and d :

$$d = \frac{\frac{Q}{2} - \frac{q_{1,min} + q_{1,max}}{2}}{\frac{Q}{2}} \tag{2}$$

characterizes the dissymmetry of the distribution, and we have $d \in [-1, 1]$

$$\epsilon = \frac{q_{1,max} - q_{1,min}}{Q} \tag{3}$$

characterizes the width of the distribution, and $\epsilon \in [0, 1]$. From this follows that we can define the fluctuation intervals by means of the parameters d and ϵ :

$$[q_{1,min}, q_{1,max}] = [\frac{Q}{2}(1 - d - \epsilon), \frac{Q}{2}(1 - d + \epsilon)] \tag{4}$$

$$[q_{2,min}, q_{2,max}] = [\frac{Q}{2}(1 + d - \epsilon), \frac{Q}{2}(1 + d + \epsilon)] \tag{5}$$

An experiment e_u, and the fluctuation intervals corresponding to this experiment, are now completely determined by the parameters d and ϵ, as we have introduced

them, and therefore we shall denote it by $e^{\epsilon}_{u,d}$. So we repeat : the experiment $e^{\epsilon}_{u,d}$ is the experiment performed by putting the particle q_1 in the point u of the sphere (and putting the particle q_2 in the diametrically opposed point $-u$), and having the parameter q_1 uniformly fluctuating in the interval $[\frac{Q}{2}(1-d-\epsilon), \frac{Q}{2}(1-d+\epsilon)]$ (and the parameter q_2 fluctuating in the interval $[\frac{Q}{2}(1+d-\epsilon), \frac{Q}{2}(1+d+\epsilon)]$).

Let us calculate now the transition probabilities corresponding to these experiments, and see that for maximal fluctuations ($q_{1,min} = 0$ and $q_{1,max} = Q$, and hence $d = 0$ and $\epsilon = 1$) our example is a model for the quantum description of the spin of a spin 1/2 quantum entity. For zero fluctuations ($q_{1,min} = q_{1,max}$, and hence $\epsilon = 0$) our example has a classical structure. The general situation ($0 < \epsilon < 1$ and $d \in [-1, 1]$) is neither quantum nor classical.

2.2 Eigenstates, Superposition States and Probabilities. For an experiments $e^{\epsilon}_{u,d}$, and a state $p_v \in \Sigma$ of the entity S, let us introduce $P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v)$ as the probability to get the outcome $o_{u,1}$ for the experiment $e^{\epsilon}_{u,d}$ if the particle q is in the state p_v, and $P(e^{\epsilon}_{u,d} = o_{u,2} \mid p_v)$ as the probability to get the outcome $o_{u,2}$ for $e^{\epsilon}_{u,d}$ if the particle q is in the state p_v. Then we can make the following calculation :

$$
\begin{aligned}
P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v) &= P(|F_1| > |F_2|) \\
&= P(C \cdot \frac{q_1 q}{r^2 - u\dot{v}} > C \cdot \frac{q_2 q}{r^2 + u \cdot v}) \\
&= P(q_1(r^2 + u \cdot v) > q_2(r^2 - u \cdot v)) \\
&= P(q_1 > \frac{Q}{2}(1 - \frac{u \cdot v}{r^2})) \\
P(e^{\epsilon}_{u,d} = o_{u,2} \mid p_v) &= P(q_1 < \frac{Q}{2}(1 - \frac{u \cdot v}{r^2}))
\end{aligned}
\tag{6}
$$

Definition 1. *If we have an entity S in a state p and an experiment e with possible outcomes $o_1, o_2, ...$, then p is an 'eigenstate' of the experiment e with 'eigen-outcome' o_k iff we can predict with certainty (probability equal to 1) that if the experiment e would be performed, the outcome o_k will occur. The region of eigenstates corresponding to an outcome o_k will be denoted by $eig(o_k)$. If a state p is not an eigenstate we will call it a superposition state and we shall denote by $sup(e)$ the region of superposition states corresponding to the experiment e.*

This definition corresponds to the traditional use of the concept of 'eigenstate' and 'eigen-outcome' in the case of the quantum formalism. Since we shall encounter situations that are neither quantum nor classical, we state this definition without referring explicitly to the mathematical Hilbert space formalism of quantum

mechanics. For our example it is easy to see that p_u and p_{-u} are eigenstates of the experiment $e^{\epsilon}_{u,d}$ for all d and ϵ. But whenever ϵ is different from 1 the experiment $e^{\epsilon}_{u,d}$ has other eigenstates. Indeed, for an experiment $e^{\epsilon}_{u,d}$ there is a region $eig^{\epsilon}_1(u,d)$ of eigenstates with eigen-outcome $o_{u,1}$ and a region $eig^{\epsilon}_2(u,d)$ of eigenstates with eigen-outcome $o_{u,2}$. Let us identify these regions for a given experiment $e^{\epsilon}_{u,d}$. As we know, the parameter q_1 is chosen at random in the fluctuation interval $[q_{1,min}, q_{1,max}] = [\frac{Q}{2}(1-d-\epsilon),\ \frac{Q}{2}(1-d+\epsilon)]$, subinterval of $[0,Q]$. The probability $P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v)$ is then given by the length of the region of the interval $[q_{1,min}, q_{1,max}]$, where

$$q_1 > \frac{Q}{2}(1 - \frac{u \cdot v}{r^2}) \tag{7}$$

(see (6)), divided by the length of the total interval $[q_{1,min}, q_{1,max}]$, which is given by ϵQ. We can then consider the following cases :

1. $d + \epsilon \leq \frac{u \cdot v}{r^2}$.

Then $P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v) = 1$ and $P(e^{\epsilon}_{u,d} = o_{u,2} \mid p_v) = 0$, because all the points of the interval $[q_{1,min}, q_{1,max}]$ satisfy the condition (7).

2. $d - \epsilon < \frac{u \cdot v}{r^2} < d + \epsilon$

Then the length of the interval where (7) is satisfied is given by $\frac{Q}{2}(1-d+\epsilon) - \frac{Q}{2}(1 - \frac{u \cdot v}{r^2}) = \frac{Q}{2}(\frac{u \cdot v}{r^2} - d + \epsilon)$, while the length of the interval where (7) is not satisfied is given by $\frac{Q}{2}(1 - \frac{u \cdot v}{r^2}) - \frac{Q}{2}(1-d-\epsilon) = \frac{Q}{2}(d + \epsilon - \frac{u \cdot v}{r^2})$. The probabilities are given by these lengths divided by ϵQ. Hence :

$$P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v) = \frac{1}{2\epsilon}(\frac{u \cdot v}{r^2} - d + \epsilon) \tag{8}$$

$$P(e^{\epsilon}_{u,d} = o_{u,2} \mid p_v) = \frac{1}{2\epsilon}(d + \epsilon - \frac{u \cdot v}{r^2}) \tag{9}$$

3. $\frac{u \cdot v}{r^2} \leq d - \epsilon$

Then $P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v) = 0$ and $P(e^{\epsilon}_{u,d} = o_{u,2} \mid p_v) = 1$, because for none of the points of the interval $[q_{1,min}, q_{1,max}]$, q_1 satisfies (7). We can now easily determine the regions of eigenstates and the region of superposition states (see fig 3) :

$$eig^{\epsilon}_1(u,d) = \{p_v \mid d + \epsilon \leq \frac{u \cdot v}{r^2}\} \tag{10}$$

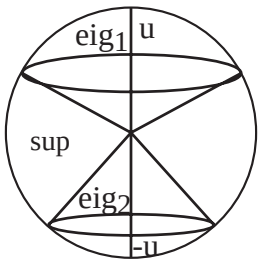

fig 3 : A representation of the regions of eigen states.

$$sup^{\epsilon}(u,d) = \{p_v \mid d - \epsilon < \frac{u \cdot v}{r^2} < d + \epsilon\} \tag{11}$$

$$eig_2^{\epsilon}(u,d) = \{p_v \mid \frac{u \cdot v}{r^2} \leq d - \epsilon\} \tag{12}$$

and write down a general expression for the transition probabilities. To do this we introduce for each subset A of the set of states Σ, the characteristic function X_A

$$X_A : \Sigma \to \{0,1\}, X_A(p) = 1 \text{ if } p \in A, \text{ and } X_A(p) = 0 \text{ if } p \notin A \tag{13}$$

We then have :

$$P(e^{\epsilon}_{u,d} = o_{u,1} \mid p_v) = 1 \cdot X_{eig_1^{\epsilon}(u,d)}(p_v) + \frac{1}{2\epsilon}(\frac{u \cdot v}{r^2} - d + \epsilon) \cdot X_{sup^{\epsilon}(u,d)}(p_v) \tag{14}$$

$$P(e^{\epsilon}_{u,d} = o_{u,2} \mid p_v) = \frac{1}{2\epsilon}(d + \epsilon - \frac{u \cdot v}{r^2}) \cdot X_{sup^{\epsilon}(u,d)}(p_v) + 1 \cdot X_{eig_2^{\epsilon}(u,d)}(p_v) \tag{15}$$

Let us analyse now the situation for different values of ϵ and d.

2.3 The Quantum Situation.

For $\epsilon = 1$ we have $d = 0$. Hence we find :

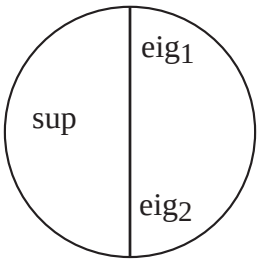

fig 4 : A representation of the quantum situation.

$$\begin{aligned} eig_1^1(u,0) &= \{p_v \mid r^2 \leq u \cdot v\} = \{p_u\} \\ sup^1(u,0) &= \{p_v \mid -r^2 < u \cdot v < r^2\} \\ eig_2^1(u,0) &= \{p_v \mid u \cdot v \leq -r^2\} = \{p_{-u}\} \end{aligned} \tag{16}$$

which shows that for experiments of the type $e^1_{u,0}$ the eigenstates are the states p_u and p_{-u}, and all the other states are superposition states (see fig. 4). If we calculate the probabilities we find :

$$\begin{aligned} P(e^1_{u,0} = o_{u,1} \mid p_v) &= \frac{1}{2}(1 + \frac{u \cdot v}{r^2}) \\ &= \frac{1}{2}(1 + cos\theta(u,v)) \\ &= cos^2(\frac{\theta(u,v)}{2}) \end{aligned} \tag{17}$$

$$\begin{aligned} P(e^1_{u,0} = o_{u,2} \mid p_v) &= \frac{1}{2}(1 - \frac{u \cdot v}{r^2}) \\ &= \frac{1}{2}(1 - cos\theta(u,v)) \\ &= sin^2(\frac{\theta(u,v)}{2}) \end{aligned} \tag{18}$$

which are the same probabilities as the ones related to the outcomes of a Stern-Gerlach spin experiment on a spin 1/2 quantum particle, of which the spin state in direction $v = (rcos\phi sin\theta, rsin\phi sin\theta, rcos\theta)$ is represented by the vector

$$(e^{-i\phi/2}cos\theta/2, e^{i\phi/2}sin\theta/2) \tag{19}$$

and the experiment corresponding to the spin measurement in direction $u = (rcos\beta sin\alpha,\ rsin\beta sin\alpha, rcos\alpha)$ by the self adjoint operator

$$\frac{1}{2}\begin{pmatrix} cos\alpha & e^{-i\beta}sin\alpha \\ e^{i\beta}sin\alpha & -cos\alpha \end{pmatrix} \tag{20}$$

in a two dimensional complex Hilbert space, which shows the equivalence between our example with $\epsilon = 1$ and the quantum model of the spin of a spin $\frac{1}{2}$ particle.

2.4 The Classical Situation.

The classical situation is the situation without fluctuations. If $\epsilon = 0$ we have

$$\begin{aligned} eig_1^0(u,d) &= \{p_v \mid r^2 d < u \cdot v\} \\ sup^0(u,d) &= \{p_v \mid r^2 d = u \cdot v\} \\ eig_2^0(u,d) &= \{p_v \mid r^2 d > u \cdot v\} \end{aligned} \tag{21}$$

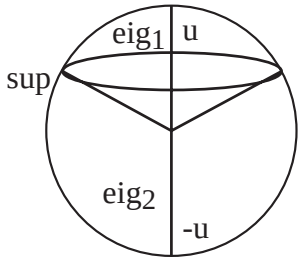

fig 5 : A representation of the classical situation.

and hence for $r^2 d < u \cdot v$

$$\begin{aligned} P(e^0_{u,d} = o_{u,1} \mid p_v) &= 1 \\ P(e^0_{u,d} = o_{u,2} \mid p_v) &= 0 \end{aligned} \tag{22}$$

and for $r^2 d > u \cdot v$

$$\begin{aligned} P(e^0_{u,d} = o_{u,1} \mid p_v) &= 0 \\ P(e^0_{u,d} = o_{u,2} \mid p_v) &= 1 \end{aligned} \tag{23}$$

For $r^2 d = u \cdot v$ the outcome of the experiment is indeterminate, and this represents a situation of 'unstable' equilibrium, typical of classical mechanics (see fig 5).

2.5 The General Situation.

To be able to give a clear picture of the intermediate situations, we shall introduce some additional concepts. First we remark that the regions of eigenstates $eig_1^\epsilon(u,d)$ and $eig_2^\epsilon(u,d)$ are determined by the points of spherical sectors of $surf$ centred around the points u and $-u$ (see fig 6). We shall denote a closed spherical sector centred around the point $u \in surf$ with angle θ by $sec(u,\theta)$. With 'closed' we

mean that also the circle on $surf$ with centre u and angle θ, that we shall denote by $circ(u,\theta)$, is contained in $sec(u,\theta)$. We remark that in the classical situation, for $\epsilon = 0$, $eig_1^0(u,d)$ and $eig_2^0(u,d)$ are given by open spherical sectors centred around u and $-u$. Let us denote such an open spherical sector centred around u and with angle θ by $sec^o(u,\theta)$.

Definition 2 : *Let us call λ_d^ϵ the angle of the spherical sectors corresponding to $eig_1^\epsilon(u,d)$ for all u, hence for $0 < \epsilon$ we have $eig_1^\epsilon(u,d) = \{p_v \mid v \in sec(u,\lambda_d^\epsilon)\}$, and $eig_1^0(u,d) = \{p_v \mid v \in sec^o(u,\lambda_d^0)\}$. We can verify easily that $eig_2(u,d)$ is determined by a spherical sector centred around the point $-u$. We call μ_d^ϵ the angle of this spherical sector, hence for $0 < \epsilon$ we have $eig_2^\epsilon(u,d) = \{p_v \mid v \in sec(-u,\mu_d^\epsilon)\}$ and $eig_2^0(u,d) = \{p_v \mid v \in sec^o(-u,\mu_d^\epsilon)\}$. Let us call σ_d^ϵ the angle of the open spherical sector determined by $sup^\epsilon(u,d)$.*

To be able to give a clear picture of the intermediate situations, we shall introduce some additional concepts. First we remark that the regions of eigenstates $eig_1^\epsilon(u,d)$ and $eig_2^\epsilon(u,d)$ and the region of superposition states $sup^\epsilon(u,d)$ are all given by spherical sectors centred around the points u and $-u$. Each of these spherical sectors is characterized by the points u and $-u$, and by a characteristic angle (see fig 6), that only depends on d and ϵ. Let us introduce systematically these angles.

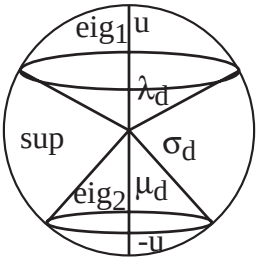

fig 6 : Representation of the general situation.

We have :

$$cos\lambda_d^\epsilon = \epsilon + d \tag{24}$$

$$cos\mu_d^\epsilon = \epsilon - d \tag{25}$$

$$\sigma_d^\epsilon = \pi - \lambda_d^\epsilon - \mu_d^\epsilon \tag{26}$$

We can see very well now in which way the spherical sectors $eig_1^\epsilon(u,d)$ and $eig_2^\epsilon(u,d)$, representing the regions of eigenstates, change with varying ϵ. For $\epsilon \to 1$ (and hence $d \to 0$) we that $\lambda_d^\epsilon \to \frac{\pi}{2}$ and $\mu_d^\epsilon \to \frac{\pi}{2}$. For $\epsilon \to 0$ we have $\lambda_d^\epsilon + \mu_d^\epsilon \to \pi$ and hence $\sigma_d^\epsilon \to 0$.

In [13] we have studied in great detail the lattice of properties of our physical entity for different and fixed values of ϵ. We show that for $\epsilon = 0$ the lattice of properties is a Boolean lattice, and for $\epsilon = 1$ is is the quantum lattice of the spin of a psin 1/2 quantum entity. The general case ($0 < \epsilon < 1$) defines a lattice of properties with a new structure, having quantum-like aspects that vanish slowly for $\epsilon \to 0$. We will not repeat the details of these calculations and refer to [13] for the interested reader, but we would like to show, as result of the theorems proved

in [13] in which way the superposition principle vanishes slowly for diminishing fluctuations.

2.6 The Vanishing Vector Space Structure.

The superposition principle is the main principle of quantum mechanics, at the origin of all the paradoxical quantum aspects. It is related directly to the vector space structure of the set of states of a quantum entity. This vector space structure is one of the main difficulties in all the attempts of reconciliation between classical and quantum theories. Indeed, it does not seem to be possible to imagine a smooth way of evolving from a vector space structure to an ordinary classical set, having no linear structure. Our approach describes such an evolution of which the details are represented in [13]. By taking into account the main theorems of [13] we want to explain in which way the vector space structure changes with ϵ. The linear structure of the quantum situation is carried by the linear closure operation that exists in a Hilbert space. It is this linear closure operation that is transformed by changing ϵ into another closure operation, that we call the ϵ-closure, to give rise to a classical topological closure in the limiting classical case. Let us explain how this comes about.

The sets of eigenstates correspond to points of spherical sectors, this are closed spherical sectors $sec(u,\alpha)$ for $0 < \epsilon$, and open spherical sectors $sec^o(u,\alpha)$ for $\epsilon = 0$. For each fixed value of ϵ we can define in the ϵ-example an angle $\alpha(\epsilon)$, which is the angle of the biggest spherical sector corresponding to a set of eigenstates $eig_1^\epsilon(u,d)$. In the quantum situation ($\epsilon = 1$ and $d = 0$) $\alpha = 0$ (see (24) and (25)), and in the classical situation ($\epsilon = 0$), $\alpha = \pi$, because we can consider the case where $d = -1$ and then from (24) follows that $cos\lambda_d^\epsilon = -1$ and hence $\lambda_d^\epsilon = \pi$. In the general case $0 < \alpha(\epsilon) < \pi$. The ϵ-closure, replacing in the general case the linear closure of the quantum case, is defined in the following way : we introduce the set $H(\alpha) = \{sec(u,\alpha) \mid u \in surf\} \cup surf$. For an arbitrary subset $K \subset surf$, we define the ϵ-closure :

$$cl^\epsilon(K) = \cup_{K \subset M, M \in H(\alpha)} M \tag{27}$$

So we consider the intersection of all closed spherical sectors with angle α that contain the points of the set K, and this is the ϵ-closure of this set K.

Quantum : In this case we have $\alpha = 0$ and hence the only spherical sectors corresponding to sets of eigenstates are the singletons of points of the sphere. Hence $H(\alpha) = \{u \mid u \in surf\} \cup surf$. Suppose that we consider a subset $K \subset surf$, then only two possibilities exists. If K contains at least two different points, we have $cl^\epsilon(K) = surf$, and if $K = \{v\}$ we have $cl^\epsilon(K) = K$. This shows that

the ϵ-closure in this case corresponds to the linear closure in the two dimensional complex Hilbert space of quantum mechanics.

Classical : In this case we have $\alpha = \pi$ and hence $H(\alpha) = \{sec^o(u,\pi) \mid u \in surf\} \cup surf$. Suppose that we consider an arbitrary subset $K \subset surf$ then $cl^\epsilon(K) = \cup_{K \subset M, M \in H(\alpha)} M$. First we remark that the complement of $sec^o(u,\pi)$ equals the sigleton $\{-u\}$. Let us consider the complement of $cl^\epsilon(K)$, then we have $cl^\epsilon(K)^C = \cap_{L \subset K^C, L \in H^C} L$, where $H^C = \{u \mid u \in surf\} \cup \emptyset$. From this follows that $cl^\epsilon(K)^C = K^C$. From this follows that $cl^\epsilon(K) = K$. Hence the ϵ-closure in the classical limit is the trivial topological closure. No states are added by the closure, hence no superpositions.

General : In this case we have a closure that is in between. Suppose that $0 < \epsilon < 1$, then $0 < \alpha(\epsilon) < \pi$, and $H(\alpha) = \{u \mid u \in surf\} \cup surf$ is well defined. If we consider a subset $K \subset surf$ and want to determine $cl^\epsilon(K)$, then we proceed in the following way. If K is not contained in a sector $sec(u,\alpha)$, we have $cl^\epsilon(K) = surf$. If there is at least one of the spherical sectors such that $K \subset sec(u,\alpha)$, we can construct the ϵ-closure for the set K by considering all of the spherical sectors $sec(u,\alpha)$ that contain the set K, and taking the intersection of all these sectors. As we have studied in detail in [13], this is indeed the generalization of a topological closure, and we have called it a PA-closure.

3. The n-Dimensional Case.

The ϵ-example in two dimensions, as it has been presented in the foregoing chapters, gives us a clear insight the classical limit process. Because of the limiting number of dimensions, it is not clear what this process becomes for an arbitrary quantum mechanical entity, of which the state is described by a wave function $\psi(x)$ element of $L^2(\Re^3)$. The aim of this section is to study the procedure in the n-dimensional situation, to prepare the general infinite dimensional situation that is presented in the next section.

3.1 The n-Dimensional Model for the Quantum Probabilities.

In [7,8] we have introduced a n-dimensional model with fluctuations on the experimental apparatuses, describing the quantum probabilities for a quantum entity represented in a n-dimensional Hilbert space. Let us repeat this model, such that we can explain in which way we will use it to construct a general procedure for the classical limit. Suppose that we have a quantum mechanical entity S described in a n-dimensional complex Hilbert space $\mathcal{H}$. An arbitrary state p of the entity is represented by a unit vector w_p of the Hilbert space $\mathcal{H}$. We consider an experiment e with set of possible outcomes $\{e_1, e_2, ..., e_n\}$, represented by a self-adjoint

operator A_e on this Hilbert space $\mathcal{H}$. To the experiment e correspond n eigen states $p_1, p_2, ..., p_n$, each eigen state p_i being represented by an eigenvector v_i of A_e. The set of eigenvectors $\{v_1, v_2, ..., v_n\}$ form a basis for the Hilbert space $\mathcal{H}$, and hence we can decompose the vector w_p in this basis and write :

$$w_p = \sum_{i=1}^{n} < w_p, v_i > v_i \tag{28}$$

then $x_i = | < v_p, v_i > |^2$ is the probability that the experiment e shall give an outcome e_i, and the state p shall be changed into the state p_i.

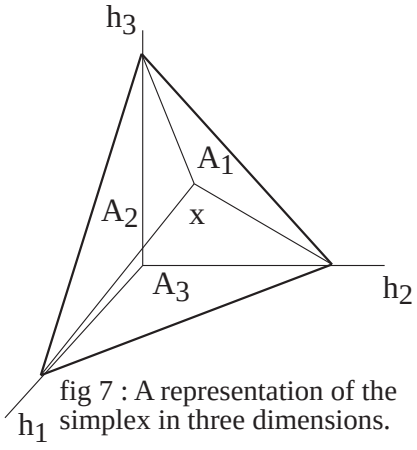

fig 7 : A representation of the simplex in three dimensions.

This means that in relation with this one experiment e we can also represent the state p of the entity S by the vector $x_p = (x_1, x_2, ..., x_n)$ with $0 \leq x_i \leq 1$ and $\sum_{i=1}^{n} x_i = 1$. These are points of a simplex $S_n \subset \Re^n$, which is the convex closure of the set of canonical base vectors $h_1 = (1, ..., 0, ...0)$, $h_2 = (0, 1, 0, ..., 0)$, ...,$h_i = (0, ..., 1, ..., 0)$,..., $h_n = (0, ..., 1)$. Hence for every state p of the entity S we have :

$$x_p = \sum_{i=1}^{n} x_i h_i \tag{29}$$

The model for the experiment as it is presented in [7,8] consists of supposing that the experiment e is in fact the 'mixture' of a set of 'pure experiments' e_λ in the following sense :

1. Each experiment e_λ has the same set of possible outcomes $\{e_1, e_2, ..., e_n\}$ and transforms the original state p of the entity S into one of the eigen states $p_1, ...p_n$ of e.

2. The experiment e consists of choosing at random (in a way that we shall point out in 3) between the experiments e_λ and performing the chosen experiment. That is the reason why we can call the experiment e a 'mixture' of the experiments e_λ.

3. This random choice is made in the following way. The parameter λ that labels the 'pure experiments' e_λ are points of the simplex S_n, hence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that $0 \leq \lambda_i \leq 1$, and $\sum_{i=1}^{n} \lambda_i = 1$. For a given state p represented by the vector $x_p = (x_1, x_2, ..., x_n)$, we call $A_{n,i}$ the convex closure of the vectors h_1, h_2,..., h_{i-1}, x, h_{i+1},..., h_n. Then clearly $S_n = \cup_{i=1}^{n} A_{n,i}$ (see fig 7).

4. We define the experiment e_λ as follows : If $\lambda \in A_{n,i}$ then the experiment e_λ gives the outcome e_i if the entity is in state p represented by x, and this experiment

e_λ transforms the state p into the state p_i represented by h_i. If λ is a point of the boundary of $A_{n,i}$ and hence also a point of $A_{n,i-1}$ or $A_{n,i+1}$, then the outcome of the experiment e_λ is indeterminate, but indeterminate in the classical sense (as for example in the case of a classical unstable equilibrium).

So we see that the 'pure experiments' e_λ are deterministic in the classical sense, namely that they contain only a 'classical indeterminism' for those states that are border-points of the $A_{n,i}$. The 'real' experiment e consists of choosing 'at random' one of the λ and performing the experiment e_λ, and it is this random choice that will give rise to 'quantum probabilities' as we shall show immediately. We can also say that the experiment e contains fluctuations that are described by the e_λ. Let us calculate the probability $P(e = e_i \mid p)$ for the experiment e to get the outcome e_i if the entity is in state p represented by x. This probability is given by :

$$P(e = e_i \mid p) = \frac{m^n(A_{n,i})}{m^n(S_n)} \tag{30}$$

where m^n is the trace of the Lebesgue measure on $\Re^n$. For example in the case $n = 3$ (see fig 7) $P(e = e_i \mid p)$ is given by the surface of $A_{3,i}$ divided by the surface of S_3. We can calculate this probability in a rather easy way, by making use of the geometrical meaning of the determinant. Let us introduce some definitions.

Definition 3. *If $y_1, y_2, ..., y_n$ are vectors of $\Re^n$, we shall denote by $M(y_1, y_2, ..., y_n)$ the $n \times n$ matrix, where $M_{i,j} = (y_i)_j$. We denote by $det(y_1, ..., y_n)$ the determinant of this matrix $M(y_1, ..., y_n)$, and by $Par(y_1, ..., y_n)$ the parallelopipedum spanned by the n vectors.*

We know that

$$|det(y_1, y_2, ..., y_n)| = m^n(Par(y_1, y_2, ..., y_n)) \tag{31}$$

and

$$m^n(S_n) = c(n)m^n(Par(h_1, ..., h_n)) \tag{31}$$

$$m^n(A_{n,i}) = c(n)m^n(Par(h_1, ..., h_{i-1}, x, h_{i+1}, ..., h_n)) \tag{32}$$

which shows that :

$$\begin{aligned} P(e = e_i \mid p) &= \frac{m^n(Par(h_1, ..., h_{i-1}, x, h_{i+1}, ..., h_n))}{m^n(Par(h_1, ..., h_n))} \\ &= \frac{|det(h_1, ..., h_{i-1}, x, h_{i+1}, ..., h_n)|}{|det(h_1, ..., h_n)|} \\ &= |det(h_1, ..., h_{i-1}, x, h_{i+1}, ..., h_n)| \\ &= x_i \end{aligned} \tag{33}$$

This shows that our model indeed gives the quantum probabilities. And we repeat that these are the consequence of our incomplete knowledge of how the measuring apparatus corresponding to the experiment e acts to produce an outcome. So the probabilities are due to a lack of knowledge about the experimental situations, or to fluctuations on the measuring apparatuses.

Let us now introduce ϵ that shall parametrize the quantity of fluctuations on the experiment e. Suppose that our knowledge about the measuring apparatus grows, then the set of 'pure experiments' e_λ, among which is chosen randomly to construct the experiment e, shall become smaller. More properly expressed, for e shall only be chosen among pure experiments e_λ, for λ in a subset $D^\epsilon \subset S_n$. And we must have $D^1 = S_n$, corresponding to the case of maximal fluctuations, and $D^0 = \{(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})\}$ corresponding to the case of zero fluctuations.

3.2 The n-Dimensional ϵ-Model.

It is clear that there are several ways to choose a collection $\{D^\epsilon |\ D^\epsilon \subset S_n, D^1 = S_n, D^0 = \{(\frac{1}{n}, ..., \frac{1}{n})\}$, and we shall start our investigation by a specific choice. Since we are most of all interested in the limiting case for $n \to \infty$ (next section), we shall investigate later the generality of the limit that we construct starting with our specific choice. We choose D^ϵ as a contraction of S_n with homothetic centre $z(n) = (\frac{1}{n}, ..., \frac{1}{n})$ and such that it is defined by the vectors $h_1^\epsilon, h_2^\epsilon, ..., h_n^\epsilon$, where

$$h_i^\epsilon = \epsilon \cdot h_i + (1 - \epsilon) \cdot z(n) \tag{34}$$

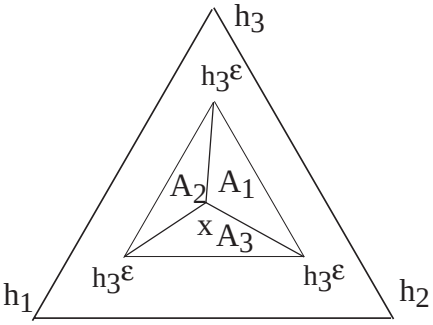

fig 8 : A representation of the pure experiments in the three dimensional case.

We see immediately that for $\epsilon = 1$ we have $h_i^\epsilon = h_i$ and we are in the situation that has been treated in [7,8], while for $\epsilon = 0$ we have $h_i = z(n)$, which means that $e = e_\lambda$ for $\lambda = z(n)$, and the situation is the one of a classical experiment, without fluctuations. We can easily calculate and see that :

$$\frac{m^n(D^\epsilon)}{m^n(S_n)} = \epsilon^{n-1} \tag{35}$$

In what follows we shall illustrate our reasoning by figures of the situation in three dimensions, but the reasoning itself will be independent of the number of dimensions (see fig 8). We can calculate the probabilities for a given ϵ, and for the case $x \in D^\epsilon$.

$$P(e^\epsilon = a_i \mid p) = \frac{m^n(A_i^\epsilon)}{m^n(D_n^\epsilon)} = \frac{|det(h_1^\epsilon, ..., h_{i-1}^\epsilon, x, h_{i+1}^\epsilon, ..., h_n^\epsilon)|}{|det(h_1^\epsilon, ..., h_n^\epsilon)|} \tag{36}$$

We have to introduce a general way of calculating the probabilities, also for the case $x \notin D^\epsilon$. To do this we introduce some additional concepts. For a given n and $x \in S_n$ we define the 'projection operator' P^n in the following way :

Definition 4. *Let us first define $\mathcal{K}_1$ as the set of indices i such that $\frac{1-\epsilon}{n} \leq x_i$ and k_1 the cardinal number of $\mathcal{K}_1$. We then introduce :*

$$P^n(x)_i = 0 \quad \text{if} \quad i \notin \mathcal{K}_1 \tag{37}$$

$$P^n(x)_i = x_i + \frac{1}{k_1}(1 - \sum_{j \in \mathcal{K}_1} x_j) \quad \text{if} \quad i \in \mathcal{K}_1 \tag{38}$$

We can easily see that $x \in D^\epsilon$ iff $P^n(x) = x$. If $P^n(x) \neq x$ this means that $k_1 < n$, and let us consider then the simplex S_{k_1} which is the convex closure of h_i such that $i \in \mathcal{K}_1$. Clearly $P(x)$ is an element of this simplex S_{k_1} because we also have :

$$\begin{aligned} \sum_{i \in \mathcal{K}_1} P^n(x)_i &= \sum_{i \in \mathcal{K}_1} x_i + \frac{1}{k}(\sum_{i \in \mathcal{K}_1} 1 - \sum_{i \in \mathcal{K}_1} (\sum_{j \in \mathcal{K}_1} x_j) \\ &= \sum_{i \in \mathcal{K}_1} x_i + 1 - \sum_{j \in \mathcal{K}_1} x_j \\ &= 1 \end{aligned} \tag{39}$$

Let us denote the vector $P^n(x)$ by x^{k_1}. For this simplex S_{k_1}, we introduce in the same way the projection operator P^{k_1}, and again we have two possibilities. If $P^{k_1}(x^{k_1}) = x^{k_1}$ we stay with the vector x^{k_1}. If $P^{k_1}(x^{k_1}) \neq x^{k_1}$ we repeat the same procedure, and define a projection P^{k_2} that shall project the vector into a simplex S_{k_2} of k_2 dimensions. This procedure must lead in a finite number of steps, $k_1, k_2, ..., k_l$ (at most n steps hence $l \leq n$) to the finding of the projection that is an element of the simplex S_{k_l} of k_l dimensions, convex closure of the base vectors h_i such that $i \in \mathcal{K}_l$. Once we have projected the original vector x into this lower dimensional simplex S_{k_l} we can now calculate the probabilities :

$$P(e = e_i \mid p) = 0 \quad \text{if} \quad i \notin \mathcal{K}_l \tag{40}$$

$$P(e = e_i \mid p) = \frac{|det(h_1^\epsilon, ..., x^{k_l}, ..., h_{k_l}^\epsilon)|}{|det(h_1^\epsilon, ..., h_{k_l}^\epsilon)|} \quad \text{if} \quad i \in \mathcal{K}_l \tag{41}$$

where h_i^ϵ is such that $i \in \mathcal{K}_l$. This final vector x^{k_l} shall be called the 'relevant projection', because it is the projection that allows us to calculate immediately the probabilities, as is shown by (40) and (41). To alleviate the notations we shall denote it by d, hence $x^{k_l} = d$, and also denote $\mathcal{K}_l$ by $\mathcal{D}$ and k_l by m.

Let us now repeat the situation : If for a given ϵ the state is represented by $x = (x_1, ..., x_n)$, we construct, as just explained, the relevant projection $d = (d_1, ..., d_m)$ of x where :

$$d_i = 0 \quad \text{if} \quad x_i + \frac{1}{m}(1 - \sum_{j \in \mathcal{D}} x_j) < \frac{1-\epsilon}{m} \tag{42}$$

$$d_i = x_i + \frac{1}{m}(1 - \sum_{j \in \mathcal{D}} x_j) \quad \text{if} \quad \frac{1-\epsilon}{m} \leq x_i + \frac{1}{m}(1 - \sum_{j \in \mathcal{D}} x_j) \tag{43}$$

and we have :

$$P(e = e_i \mid p) = 0 \quad \text{if} \quad i \notin \mathcal{D} \tag{44}$$

$$p(e = e_i \mid p) = \frac{|det(h_1^\epsilon, ..., h_{i-1}^\epsilon, d_i, h_{i+1}^\epsilon, ..., h_m^\epsilon)|}{|det(h_1^\epsilon, ..., h_m^\epsilon)|} \quad \text{if} \quad i \in \mathcal{D} \tag{45}$$

In the next section we shall investigate the limit case for $n \to \infty$ of the just introduced procedure.

4. The Classical Limit for a General Quantum Entity.

We investigate the situation where we have an experiment e that is represented by a self-adjoint operator A_e, and we suppose that the spectrum of this self adjoint operator is a subset of an interval $[a, b]$ of the set $\Re$ of real numbers. The state p of the entity S is now represented by a complex function $\psi(x)$ element of $L^2(\Re^3)$. In relation to the experiment e, we can represent the state by a function $\phi : [a, b] \to [0, 1]$, such that $\int_a^b \phi(s)ds = 1$, and $\int_c^d \phi(x)dx$ represents the probability that the experiment e gives an outcome $e_i \in [c, d]$, when the entity is in a state p. We shall make the contact with our finite dimensional case by subdividing the interval $[a, b]$ in n equal parts with length $\frac{1}{n}(b - a)$, and putting $\phi_i = \phi(\frac{i}{n}(b - a))$, and approximating the integral $\int_a^b \phi(s)ds$ by $\sum_{i=1}^n \phi_i \cdot \frac{1}{n}(b - a)$. Hence the role of the x_i in the finite dimensional case is played by $\phi_i \cdot \frac{1}{n}(b - a)$.

4.1 Construction of the Relevant Projection.

Let us now search for the 'relevant projection' in the limit case for n going to infinity. The relevant projection, the analogue of the finite dimensional d, must be a function $f : [a, b] \to [0, 1]$, such that $d_i = f_i \cdot \frac{1}{n}(b - a)$ where $f_i = f(\frac{i}{n}(b - a))$. So we must have in general :

$$f_i \cdot \frac{1}{n}(b - a) = \phi_i \cdot \frac{1}{n}(b - a) + \frac{1}{m}(1 - \sum_{j \in \mathcal{K}_m} \phi_j \cdot \frac{1}{n}(b - a)) \tag{46}$$

if

$$\frac{1}{m}(1-\epsilon) \leq \phi_i \cdot \frac{1}{n}(b-a) + \frac{1}{m}(1 - \sum_{\mathcal{K}_m} \phi_i \cdot \frac{1}{n}(b-a)) \tag{47}$$

or also

$$f_i = \phi_i + \frac{n}{m} \cdot \frac{1}{b-a}(1 - \sum_{\mathcal{D}} \frac{1}{n}(b-a)) \tag{48}$$

if

$$\frac{n}{b-a} \cdot \frac{1-\epsilon}{m} \leq f_i \tag{49}$$

Let us now identify the different concepts that we have used in the finite dimensional case for the limit case. $\mathcal{D}$ is the set of indices i of d such that $d_i > 0$, and m is the cardinal number of $\mathcal{D}$. Hence we can now write :

$$f_i \cdot \frac{1}{n}(b-a) > 0 \iff i \in \mathcal{D} \tag{50}$$

which shows that the role of $\mathcal{D}$ is plaid by the support of f. Let us denote this support by Ω, and the measure of Ω by $|\Omega|$. We know that n is the number of intervals in which we have divided the interval $[a,b]$, and m is the number of these intervals where $f(\frac{i}{n}(b-a)) > 0$. Then $\frac{n}{m}$ shall go to $\frac{b-a}{|\Omega|}$ if $n \to \infty$. Let us now write again the requirements (48) and (49) in the limit case :

$$f(x) = \phi(x) + \frac{1}{b-a} \cdot \frac{b-a}{|\Omega|}(1 - \int_{\Omega} \phi(s)ds) \tag{51}$$

if

$$\frac{1-\epsilon}{b-a} \cdot \frac{b-a}{|\Omega|} \leq f(x) \tag{52}$$

We see that the original interval $[a,b]$ disappears from the final result. So for some $\Omega \subset \Re$, and $x \in \Omega$ we have :

$$f(x) = \phi(x) + \frac{1}{|\Omega|}(1 - \int_{\Omega} \phi(s)ds) \tag{53}$$

$$\frac{1-\epsilon}{|\Omega|} \leq f(x) \tag{54}$$

and for $x \notin \Omega$ we have :

$$f(x) = 0 \tag{55}$$

These are the requirements that has to satisfy the relevant projection function $f(x)$ corresponding to a state represented by the function $\phi(x)$. Let us analyse more in detail these requirements. From (53) and (54) follows that for $x \in \Omega$ we have

$$\frac{1-\epsilon}{|\Omega|} \leq \phi(x) + \frac{1}{|\Omega|}(1 - \int_{\Omega} \phi(s)ds) \tag{56}$$

which implies that

$$\int_\Omega \phi(s)ds - \phi(x) \cdot |\Omega| \leq \epsilon \tag{57}$$

Let us now consider a point $x_1 \in \Omega$ such that

$$\frac{1-\epsilon}{|\Omega|} = f(x_1) \tag{58}$$

Then we must have :

$$\frac{1-\epsilon}{|\Omega|} = \phi(x_1) + \frac{1}{|\Omega|}(1 - \int_\Omega \phi(s)ds) \tag{59}$$

which implies that

$$\int_\Omega \phi(s)ds = \phi(x_1) \cdot |\Omega| + \epsilon \tag{60}$$

From (57) follows that for $x \in \Omega$ we have

$$\phi(x_1) \cdot |\Omega| + \epsilon - \phi(x) \cdot |\Omega| \leq \epsilon \tag{61}$$

which shows that for $x \in \Omega$ we have

$$0 \leq \phi(x) - \phi(x_1) \tag{62}$$

$$\int_\Omega (\phi(x) - \phi(x_1))dx = \epsilon \tag{63}$$

But these last relations give us an easy way to construct the relevant projection $f(x)$ and its set Ω. Indeed, consider the constant function $\phi_\Omega(x) = \phi(x_1)$, then $\Omega = \{x \mid 0 \leq \phi(x) - \phi_\Omega(x)\}$ and $\int_\Omega(\phi(x) - \phi_\Omega(x))dx = \epsilon$. Hence we have transformed our problem of finding the relevant projection in the following one. Look for a constant function ϕ_Ω such that

$$\Omega = \{x \mid 0 \leq \phi - \phi_\Omega\} \tag{64}$$

$$\int_\Omega (\phi - \phi_\Omega)(x)dx = \epsilon \tag{65}$$

If we have found Ω and ϕ_Ω we can now easily calculate the relevant projection f. Indeed for $x \in \Omega$ we have

$$f(x) = \phi(x) + \frac{1}{|\Omega|}(1 - \int_\Omega \phi(s)ds) \tag{66}$$

Hence

$$f(x) = \phi(x) - \phi_\Omega + \frac{1-\epsilon}{|\Omega|} \tag{67}$$

Let us now calculate the probabilities that correspond to a general quantum situation.

4.2 Calculation of the Probabilities.

For the finite, n-dimensional situation, we derived, for a given ϵ, the probabilities $P^\epsilon(e = e_i \mid p)$ as given by (40) and (41). In the infinite dimensional case we must look for a probability measure, which means a function $p^\epsilon : [a, b] \to [o, 1]$, such that $\int_a^b p^\epsilon(x)dx = 1$. We shall show now that we can construct this probability function as the solution of a Fredholm integral equation, if we know the relevant projection function $f(x)$. Suppose that we consider an arbitrary Fredholm integral equation $\alpha a(x) = b(x) + \int_\Omega K(x,t)a(t)dt$. If we try to solve this equation with the method of Fredholm, then we proceed in the following way : We subdivide Ω in m equal intervals of measure $\frac{|\Omega|}{m}$, and consider the value of $a(x)$ in the m points. We denote a_i and b_i the values of $a(x)$ and $b(x)$ in the point i, and $K_{i,j}$ the value of $K(x, y)$ in the point i, j. The integral can then be approximated by $\sum_{j=1}^m K_{ij}a_j \cdot \frac{|\Omega|}{m}$, and the integral equation by :

$$\alpha \cdot a_i = b_i + \sum_{j=1}^{m} K_{1j}a_j \cdot \frac{|\Omega|}{m} \tag{68}$$

Let us introduce the following notations, $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_m)$, and $c_i = (K_{i,1}, K_{i,2}, ..., K_{i,m-1}, K_{i,m})$ as vectors of an m-dimensional real vectorspace. We see that (68) can then be written asfollows :

$$(\alpha \cdot M(h_1, h_2, ..., h_m) - M(c_1, c_2, ..., c_m))a = b \tag{69}$$

and if we introduce the vectors $v_k = \alpha \cdot h_k - \frac{|\Omega|}{m} \cdot c_k$ the solution is given by

$$a_i = \frac{det(v_1, ..., v_{i-1}, b_i, v_{i+1}, ..., v_m)}{det(v_1, ..., v_m)} \tag{70}$$

We can already feel that there is an analogy between the calculation of our probabilities in the n-dimensional model, and a m-dimensional (where $m = k_l$) approximation of a Fredholm integral equation. Let us make the analogy more concrete such that we can find the integral equation that can deliver us the probability function for a given ϵ. For the n-dimensional case the probabilities are given by (44) and (45). Let us remark that if we consider the probability function $p : [a, b] \to [0, 1]$ such that $\int_a^b p(x)dx = 1$, and after having divided the interval $[a, b]$ into n equal intervals of length $\frac{1}{n}(b - a)$, and having introduced $p_i = p(\frac{i}{n}(b - a))$, we have $P(e = e_i \mid p) = p_i \cdot \frac{(b-a)}{n}$. We also know that $d_i = f_i \cdot \frac{b-a}{n}$, and $\frac{(b-a)}{n} = \frac{|\Omega|}{m}$. If we substitute all this in (44) and (45) we find :

$$p_i \cdot \frac{|\Omega|}{m} = 0 \quad \text{if} \quad i \notin \mathcal{D} \tag{71}$$

$$p_i \cdot \frac{|\Omega|}{m} = \frac{det(h_1^\epsilon, ..., h_{i-1}^\epsilon, f_i \cdot \frac{|\Omega|}{m}, h_{i+1}^\epsilon, ..., h_m^\epsilon)|}{|det(h_1^\epsilon, ..., h_m^\epsilon)} \quad \text{if} \quad i \in \mathcal{D} \tag{72}$$

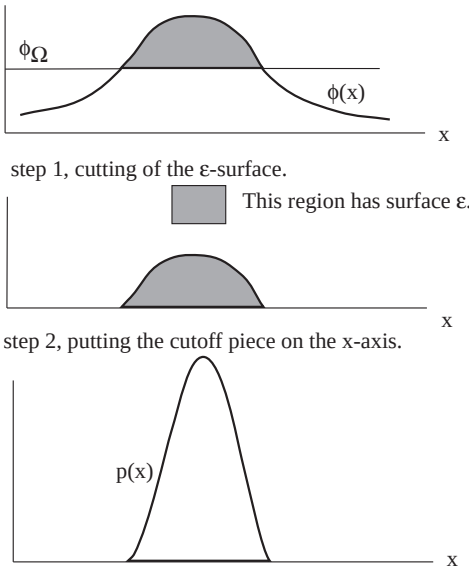

fig 9 : step 3, renormalizing by dividing by ε.

which shows that $p(x)$ is a solution of the Fredholm equation

$$\epsilon p(x) = f(x) - \int_\Omega \frac{1-\epsilon}{|\Omega|} p(t)dt \tag{73}$$

which can easily be solved

$$p(x) = \frac{1}{\epsilon}(f(x) - \frac{(1-\epsilon)}{|\Omega|}) \tag{74}$$

If we now substitute the expression for the relevant projection that we have obtained in (67) we find

$$p(x) = \frac{1}{\epsilon}(\phi - \phi_\Omega)(x) \tag{75}$$

where

$$\Omega = \{x \mid 0 \leq \phi - \phi_\Omega\} \tag{76}$$

and ϕ_Ω is a constant function such that

$$\int_\Omega (\phi - \phi_\Omega)(x)dx = \epsilon \tag{77}$$

We can see that this allows us to calculate $p(x)$ immediately from $\phi(x)$, by proceeding in the way that we explain in the following subsection.

4.3 The Classical Limit Procedure for the Case of a General Quantum Entity.
From our calculations in the two foregoing subsections follows a very simple procedure for the construction of the classical limit for a general quantum entity. Suppose that ϵ is given, and the state of the quantum entity is described by the probability distribution $\phi(x)$. We cut, by means of a constant function ϕ_Ω, a piece of the function $\phi(x)$, such that the surface contained in the cutoff piece equals ϵ (see step 1 of fig 9). We move this piece of function to the x-axis (see step 2 of fig 9). And then we renormalise by dividing by ϵ (see step 3 of fig 9). If we proceed in this way for smaller values of ϵ, we shall finally arrive at a delta-function for the

classical limit $\epsilon \to 0$, and the delta-function is located in the original maximum of the quantum probability distribution.

5. Non-Locality in the Classical Limit.

We want to investigate what becomes of the non-local behaviour of quantum entities taking into account the classical limit procedure that we propose in this paper. Suppose that we consider a double slit experiment, then the state p of a quantum entity having passed the slits can be represented by a probability function $p(x)$ of the form represented in fig 10. We can see that the non-locality presented by this probability function gradually disappears when ϵ becomes smaller, and in the case where $p(x)$ has only one maximum finally disappears completely.

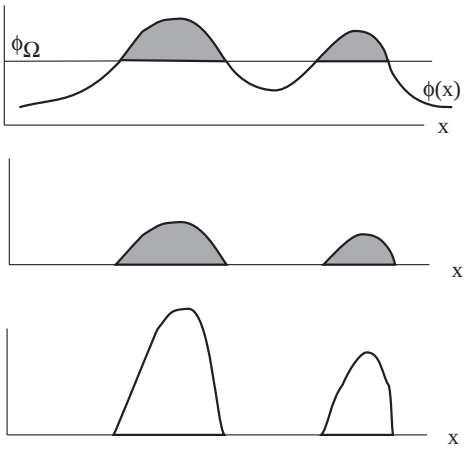

fig 10 : the classical limit procedure in the situation of a non-local quantum state.

When there are no fluctuations on the measuring apparatus used to detect the particle, it shall be detected with certainty in one of the slits, and always in the same one. If $p(x)$ has two maxima (one behind slit 1, and the other behind slit 2) that are equal, the non-locality does not disappear. Indeed, in this case the limit-function is the sum of two delta-functions (one behind slit 1 and one behind slit 2). So in this case the non-locality remains present even in the classical limit. If our procedure for the classical limit is a correct one, also macroscopical classical entities can be in non-local states. How does it come that we don't find any sign of this non-locality in the classical macroscopical world? This is due to the fact that the set of states, representing a situation where the probability function has more than one maximum, has measure zero, compared to the set of all possible states, and moreover these state are 'unstable'. The slightest perturbation will destroy the symmetry of the different maxima, and hence shall give rise to one point of localization in the classical limit. Also classical macroscopical reality is non-local, but the local model that we use to describe it gives the same statistical results, and hence cannot be distinguished from the non-local model.

6. Conclusion.

The approach that we present here, although it still has to be developed in many aspects, provides an explanation to the problems that we have pointed out in the introduction. It explains the origin of the quantum probabilities by the existence

of fluctuations of internal variables of the experimental apparatuses. We see this 'contextual' aspect of our approach as an important philosophical step, because it introduces explicitly the unavoidable influence of the environment on the physical entity under consideration. It would be too simple to interpret these variables as the 'hidden variables' of a 'classical hidden variable theory', because they do not deliver an 'additional deeper' description of the reality of the physical entity. Their presence, as variables of the experimental apparatuses, has a well defined philosophical meaning, and expresses that we, human beings, want to construct a model of reality, independent of the fact that we experience this reality. The reason is that we look for 'properties' or 'relations of properties', and they are defined by our ability to make predictions independent of experience. Since we don't control these variables in the experimental apparatuses, we shall not allow them in our model of reality, and the probability introduced by them cannot be eliminated from a predictive human model. This is in our opinion the real meaning of the relation between man and reality. Man looks for a predictive model and therefore has to cut pieces of the world such that they can fit this model. In the macroscopical world, because of the availability of many experiments with neglectable fluctuations, this cutting delivers a 'almost' deterministic model. In other regions of reality, where these kind of experiments are not available, man's model shall be indeterministic. The pre-material world is such a region. There are other regions of the world where the same kind of unavoidable indeterminism appears. For example : the economic region, the psychological region, the social region, the cultural region, where in general all available experiments have uncontrolable variables introducing probabilities. Man's models of these regions of the world shall contain the quantum kind of indeterminism. And this is the point where introvert and extrovert physics meet, and where man is an observer of existing reality as well as a creator of new reality, and participates in everything 'that is the case'.

The explicit classical limit for an arbitrary quantum entity in our approach (see 4.3) explains why non-local states become un-detectable in the classical world (see 5). Philosophically speaking however non-locality does not disappear, and is a fundamental property of nature. From this follows that we have to belief that our model of space, as the theatre in which all entities are present and move around, should be considered partly as a human construction, due to our human experience with the macroscopical material entities. Space is the garden of man, in which he encounters and creates all what is close to him, but this garden, as quantum mechanics has shown us, is not the world.

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