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Quantum, Classical and Intermediate, an illustrative example

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Abstract. We present a model that allows to build structures that evolve continuously from classical to quantum, and we study the intermediate situations, giving rise to structures that are neither classical nor quantum. We construct the closure structure corresponding to the collection of eigenstate sets of these intermediate situations, and demonstrate how the superposition principle disappears during the transition from quantum to classical. We investigate the validity of the axioms of quantum mechanics for the intermediate situations.

1. Introduction.

We use the approach where the quantum structures arise as a consequence of the presence of fluctuations on the measurement situations, and where the indeterminism of the measurement process is due to the fact that an experiment e consists of different possible sub-experiments $e(\lambda)$ that are not distinguished macroscopically. During an experimental process corresponding to the experiment e, one of these hidden experiments $e(\lambda)$ actually occurs, and each one of these $e(\lambda)$ is deterministic. Therefore the probabilistic aspect of the experiment e finds its origin in the 'lack of knowledge' about which one of the hidden experiments $e(\lambda)$ takes place.

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Earlier we have been able to show that the introduction of fluctuations on the measuring apparatuses allows us to build models for an arbitrary finite-dimensional Hilbert space quantum entity, and we have proposed a mechanistic physical entity that delivers a model for a two-dimensional Hilbert space quantum entity, namely a spin 1/2 model [(1)]. The quantum probability model of this example has been studied in detail [(2,3,4)], and this result has meanwhile been generalized to the case of an infinite dimensional Hilbert space [(5)]. We have also been able to show that these models, with fluctuations on the measuring apparatuses, give rise to a non classical structure for the sets of their propositions [(6,7)].

If the quantum structure can be explained by the presence of fluctuations on the measuring apparatuses, we can go a step further, and wonder what types of structure arise when we consider the original models, with fluctuations on the measuring apparatuses, and introduce a variation of the magnitude of these fluctuations. We have studied the 'sphere-model', that we shall introduce again in this paper, under varying fluctuations, parameterizing these variations by a number $\epsilon \in [0, 1]$, such that $\epsilon = 1$ corresponds to the situation of maximal fluctuations, giving rise to a quantum structure, and $\epsilon = 0$ corresponds to the situation of zero fluctuations, generating a classical structure, and other values of ϵ correspond to intermediate situations, giving rise to a structure that is neither quantum nor classical [(5,8)].

The aim of this paper is to study in detail the sphere-model that we have introduced in the earlier papers, and to see that the intermediate situations give rise to closure structures that are neither quantum nor classical. We use the theoretical framework presented in [(9)], which means that we describe a physical entity S by a set of states $\Sigma = \{p, q, r, ...\}$, a set of relevant experiments $\mathcal{E} = \{e, f, g, ...\}$, where each experiment e has an outcome set $O_e = \{x_e^1, ..., x_e^i, ...\}$, and a set of probabilities $\{P(e = x_e^i \mid p) \mid e \in \mathcal{E}, x_e^i \in O_e, p \in \Sigma\}$.

2. The sphere-model.

The physical entity S that we consider is a point particle P that can move on the surface of a sphere, denoted $surf$, with center O and radius 1. The unit-vector v where the particle is located on $surf$ represents the state p_v of the particle. Hence the set of states that we consider is given by $\Sigma = \{p_v \mid v \in surf\}$.

To introduce the experiments, we consider two diametrically opposite points u and $-u$ on the surface of the sphere, and denote by $[-1, +1]_u$ the interval of real numbers $[-1, +1]$ coordinating the points of the line between u and $-u$, in such a way that -1 coordinates

$-u$ and $+1$ coordinates u (see Figure 1). We introduce two real parameters $\epsilon \in [0, 1]$, and $d \in [-1+\epsilon, 1-\epsilon]$, and consider the subinterval $[d-\epsilon, d+\epsilon]_u \subset [-1, +1]_u$. The experiment $e^\epsilon_{u,d}$ consists of the particle P falling from its original place v orthogonally onto the line between u and $-u$, and arriving in a point, coordinated in the interval $[-1, 1]_u$ by the real number $v \cdot u$. In the interval $[d - \epsilon, d + \epsilon]_u$ we consider a uniformly distributed random variable κ, and the experiment proceeds as follows. If $\kappa \in [d - \epsilon\ ,\ v \cdot u[$, the particle P moves to the point u, and the experiment $e^\epsilon_{u,d}$ gives outcome x^1_u. If $\kappa \in]v \cdot u\ ,\ d + \epsilon]$, it moves to the point $-u$, and the experiment $e^\epsilon_{u,d}$ gives outcome x^2_u. If $\kappa = v \cdot u$ it moves with probability $\frac{1}{2}$ to the point u, and the experiment $e^\epsilon_{u,d}$ gives outcome x^1_u, and it moves with probability $\frac{1}{2}$ to the point $-u$, and then the experiment e_u gives outcome x^2_u. This completes the description of the experiment $e^\epsilon_{u,d}$.

We shall consider now different situations labeled by the parameter ϵ. The entity $S(\epsilon)$ is described by a set of states $\Sigma = \{p_v \mid v \in surf\}$, a set of experiments $\mathcal{E}(\epsilon) = \{e^\epsilon_{u,d} \mid u \in surf, d \in [-1 + \epsilon, 1 - \epsilon]\}$, and a set of probabilities $\{P(e^\epsilon_{u,d} = x^i_u \mid p_v) \mid e^\epsilon_{u,d} \in \mathcal{E}(\epsilon), x^i_u \in O_{e^\epsilon_{u,d}}, p_v \in \Sigma\}$. To lighten the notation we denote the probability $P(e^\epsilon_{u,d} = x^1_u \mid p_v)$ by $P^\epsilon_d(p_u \mid p_v)$, and the probability $P(e^\epsilon_{u,d} = x^2_u \mid p_v)$ by $P^\epsilon_d(p_{-u} \mid p_v)$. We have the following cases:

1. $d + \epsilon \leq v \cdot u$.

Then $P^\epsilon_d(p_u \mid p_v) = 1$ and $P^\epsilon_d(p_{-u} \mid p_v) = 0$.

2. $d - \epsilon < v \cdot u < d + \epsilon$

$$P^\epsilon_d(p_u \mid p_v) = \frac{1}{2\epsilon}(v \cdot u - d + \epsilon) \tag{1}$$

$$P^\epsilon_d(p_{-u} \mid p_v) = \frac{1}{2\epsilon}(d + \epsilon - v \cdot u) \tag{2}$$

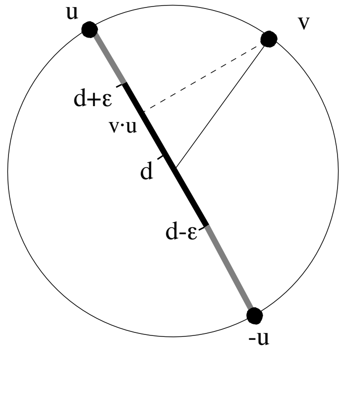

Fig. 1 : A representation of the experiment $e^\epsilon_{u,d}$. We have chosen a case where d=0.2, and ϵ=$\frac{1}{2}$. If we consider the elastic-sphere-example realizing this situation, then the elastic breaks uniformly inside the interval $[d-\epsilon, d+\epsilon]$, and is unbreakable outside this interval, in the points of the set $[-1, d-\epsilon] \cup [d+$

ϵ,1].

3. $v \cdot u \leq d - \epsilon$

Then $P_d^\epsilon(p_u \mid p_v) = 0$ and $P_d^\epsilon(p_{-u} \mid p_v) = 1$.

The sphere model can be realized in the laboratory by means of a physical entity and experiments to be performed on this entity. Therefore we suppose that the particle P is a material point particle, located on the sphere. The experiment $e^\epsilon_{u,d}$ happens by means of a piece of elastic of length 2. The piece of elastic is fixed, with one of its end-points in the point u of the surface and the other end-point in the diametrically opposite point $-u$. Once the elastic is placed, the material particle P falls from its original place v orthogonally onto the elastic, and sticks on it. The piece of elastic consists of three different parts: one part, coordinated by the interval $[-1, d - \epsilon]_u$, where it is unbreakable, a second part, coordinated by the interval $]d - \epsilon, d + \epsilon[_u$ where it breaks uniformly, and a third part, coordinated by the interval $[d + \epsilon, +1]_u$ where it is again unbreakable (see Figure 1). The experiment proceeds by the elastic that breaks such that the particle that sticks to it is moved to one of the two endpoints u or $-u$. If it breaks in the part coordinated by $]d - \epsilon, v \cdot u[_u$, the particle P will be drawn to the point u by the elastic still connected to it, and we will say that the experiment $e^\epsilon_{u,d}$ gives outcome x^1_u. If it breaks in the part coordinated by $]v \cdot u, d + \epsilon[_u$, the particle P will be drawn to the point $-u$ by the elastic still connected to it, and we will say that the experiment $e^\epsilon_{u,d}$ gives outcome x^2_u. If it breaks exactly in the point $v \cdot u$ itself, the particle P can go both ways. We suppose that in this case the particle P goes to the point u with probability $\frac{1}{2}$, and to the point $-u$ with probability $\frac{1}{2}$. It is clear that this 'sphere-elastic-example' is a physical realization of the sphere model. It is important to know that such a realization exists, because it shows that the sphere model is not just a mathematical construction. Other physical realizations can be proposed, such as the charge-example put forward in [1,2].

3. The eigen-closure structure of the sphere-model.

As put forward in [9], we now study the collection of eigenstate sets, that we shall denote by $\mathcal{F}(\epsilon)$ corresponding to the entity $S(\epsilon)$, and the eigen-closure structure related to this collection of eigen state sets. First we do this for the primitive experiments, and denote the eigenstate set corresponding to the outcome x_u^1 of the experiment $e_{u,d}^\epsilon$ by $eig_{u,d}^\epsilon(1)$, and the eigenstate set corresponding to the outcome x_u^2 of the experiment $e_{u,d}^\epsilon$ by $eig_{u,d}^\epsilon(2)$. We have:

$$eig_{u,d}^\epsilon(1) = \{p_v \mid d + \epsilon \leq v \cdot u\} \tag{3}$$

$$eig_{u,d}^\epsilon(2) = \{p_v \mid v \cdot u \leq d - \epsilon\} \tag{4}$$

We have the following three situations:

3.1 The quantum situation ($\epsilon = 1$).

For $\epsilon = 1$ we always have $d = 0$, and the sphere model becomes a quantum model for the spin of a spin 1/2 quantum entity. Indeed, the transition probabilities are then the same as the ones related to the outcomes of a Stern-Gerlach spin experiment on a spin 1/2 quantum particle, of which the quantum-spin-state in direction $v = (cos\phi sin\theta, sin\phi sin\theta, cos\theta)$, denoted by $\bar{\psi}_v$, is represented by the vector

$$\psi_v = (e^{-i\phi/2} cos\theta/2, e^{i\phi/2} sin\theta/2) \tag{5}$$

of the two-dimensional complex Hilbert space, and the experiment e_u corresponding to the spin experiment in direction $u = (cos\beta sin\alpha, sin\beta sin\alpha, cos\alpha)$ by the self adjoint operator

$$H_u = \frac{1}{2}\begin{pmatrix} cos\alpha & e^{-i\beta} sin\alpha \\ e^{i\beta} sin\alpha & -cos\alpha \end{pmatrix} \tag{6}$$

on this Hilbert space. For the eigenstate sets we find:

$$eig_{u,0}^1(1) = \{p_v \mid +1 \leq v \cdot u\} = \{p_u\} \tag{7}$$

$$eig_{u,0}^1(2) = \{p_v \mid v \cdot u \leq -1\} = \{p_{-u}\} \tag{8}$$

which shows that the eigenstates are the states p_u and p_{-u}, and all the other states are superposition states.

3.2 The classical situation ($\epsilon = 0$).

The classical situation is the situation without fluctuations. If $\epsilon = 0$, then d can take any value in the interval $[-1, +1]$, and we have:

$$eig_{u,d}^0(1) = \{p_v \mid d < v \cdot u\} \tag{9}$$

$$eig^0_{-u,d}(2) = \{p_v \mid v \cdot u < d\} \quad (10)$$

which shows that for the classical situation, the only superposition states are the states p_v such that $v \cdot u = d$, and all the other states are eigenstates.

3.3 The general situation.

To give a clear picture of the intermediate situations, we introduce additional concepts. First we remark that the regions of eigenstates $eig^\epsilon_{u,d}(1)$ and $eig^\epsilon_{-u,d}(2)$ are determined by the points of spherical sectors of $surf$ centered around the points u and $-u$. We denote a closed spherical sector centered around the point $u \in surf$ with angle θ by $sec(u, \theta)$. We remark that in the classical situation, for $\epsilon = 0$, $eig^0_{u,d}(1)$ and $eig^0_{u,d}(2)$ are given by open spherical sectors centered around u and $-u$. We denote an open spherical sector centered around u and with angle θ by $sec^o(u, \theta)$. We call λ^ϵ_d the angle of the spherical sectors corresponding to $eig^\epsilon_{u,d}(1)$ for all u, hence for $0 \neq \epsilon$ we have $eig^\epsilon_{u,d}(1) = \{p_v \mid v \in sec(u, \lambda^\epsilon_d)\}$, and $eig^0_{u,d}(1) = \{p_v \mid v \in sec^o(u, \lambda^0_d)\}$. We can verify easily that $eig^\epsilon_{u,d}(2)$ is determined by a spherical sector centered around the point $-u$. We call μ^ϵ_d the angle of this spherical sector, hence for $0 < \epsilon$ we have $eig^\epsilon_{u,d}(2) = \{p_v \mid v \in sec(-u, \mu^\epsilon_d)\}$ and $eig^0_{u,d}(2) = \{p_v \mid v \in sec^o(-u, \mu^\epsilon_d)\}$. We denote by σ^ϵ_d the angle of superposition states. We have:

$$cos\lambda^\epsilon_d = \epsilon + d \quad (11)$$

$$cos\mu^\epsilon_d = \epsilon - d \quad (12)$$

$$\sigma^\epsilon_d = \pi - \lambda^\epsilon_d - \mu^\epsilon_d \quad (13)$$

$$\lambda^\epsilon_{-d} = \mu^\epsilon_d \text{ and } \sigma^\epsilon_{-d} = \sigma^\epsilon_d \quad (14)$$

3.4 Construction of $\mathcal{F}(\epsilon)$.

We consider the surface of the sphere, denoted by $surf$, of the sphere-model and an arbitrary subset Q, $Q \subset surf$, and introduce the set

$$H(\alpha) = \{sec(u, \alpha) \mid u \in surf\} \cup \{surf\} \quad (15)$$

We also introduce for $Q \subset surf$ the following:

$$cl^\alpha(Q) = \cap_{Q \subset R, R \in H(\alpha)} R \quad (16)$$

Theorem 1 *cl^α is a closure for $0 \leq \alpha < \pi$ on the set $surf$.*

Theorem 2 *For every subset $Q \subset surf$ we have $cl^\alpha(Q) \subset cl^\beta(Q)$ if $\beta \leq \alpha$.*

Proof : Suppose that $\beta \leq \alpha$. We remark that in this case $sec(u,\beta) = cl^\alpha(sec(u,\beta))$. Let us consider $cl^\beta(Q) = \cap_{Q \subset R, R \in H(\beta)} R$. We can substitute each spherical sector $sec(u,\beta)$ of this intersection by $cl^\alpha(sec(u,\beta)$, and then we have that $cl^\beta(Q)$ is given by an intersection of elements of $H(\alpha)$. This shows that $cl^\alpha(Q) \subset cl^\beta(Q)$.

This theorem allows us to construct in a more explicit way the eigen-closed subsets of our model. Indeed, we remark again that for a fixed ϵ the set of generating subsets of the eigen-closure structure on Σ defined by the collection of eigenstate-sets consists of the set $\{H(\lambda_d^\epsilon) \mid d \in [-1+\epsilon, 1-\epsilon]\}$. The foregoing theorem shows that we only have to take into account the $H(\lambda_d^\epsilon)$ for a maximum of the λ_d^ϵ. A maximum of λ_d^ϵ for a given ϵ, means a minimum of $cos(\lambda_d^\epsilon)$, hence a minimum of $\epsilon + d$. Let us denote the minimum of d by d_m, then clearly $d_m = -1+\epsilon$, and let us denote the corresponding maximum of λ_d^ϵ by λ_{max}^ϵ.

Theorem 3 *Consider an arbitrary subset $A \subset \Sigma$, and let $Q = \{u \mid p_u \in A\}$. Then :*

$$cl_{eig}(A) = \{p_w \mid w \in \cap_{Q \subset R, R \in H(\lambda_{max}^\epsilon)} R, \text{ where } \lambda_{max}^\epsilon = \lambda_{d_m}^\epsilon, \text{ with } d_m = -1+\epsilon\} \quad (17)$$

Proof : Immediate consequence of theorem 2.

This theorem allows us to construct explicitly the eigen- closure of arbitrary subsets of the set of states. Let us use these results to show that the closure structure of the sphere model satisfies the two first axioms, namely the axiom of state determination and the axiom of atomicity (see ref [(9)] section 2.3).

Theorem 4 *The entity $S(\epsilon)$ satisfies the axiom of state determination and the axiom of atomicity*

Proof : If we consider an arbitrary state p_v of the sphere-example, we have to show that $cl_{eig}(\{p_v\}) = \{p_v\}$. Consider an arbitrary value of ϵ, and an arbitrary state p_w different from p_v. Since v is different from w, the angle $\theta(v,w)$ between the vectors v and w is strictly greater then zero. We can then construct an experiment $e_{u,d}^\epsilon$ such that p_v is an eigenstate of $e_{u,d}^\epsilon$, and p_w is not. This shows that $p_w \notin cl_{eig}(\{p_v\})$. Since p_w is an arbitrary state different from p_v, we conclude that $cl_{eig}(\{p_v\}) = \{p_v\}$.

4. Different situations and examples.

In this section we investigate in detail the eigen-closure structure and show in which way the possible superpositions disappear when we go from quantum to classical.

4.1 The quantum case.

We choose $\epsilon = 1$. Then $d_m = -1 + \epsilon = 0$.

We have $cos(\lambda^1_{max}) = 1$, hence $\lambda^1_{max} = 0$. We have $H(\lambda^\epsilon_{max}) = \{\{p_u\} \mid u \in surf\} \cup \{surf\}$.

Consider an arbitrary $A \subset \Sigma$, and $Q = \{u \mid p_u \in A\}$. Then $cl_{eig}(A) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\lambda^1_{max})} R\}$.

Let us consider some examples of subsets $A \subset \Sigma$. We know from theorem 4 that each singleton $\{p_u\}$ is a closed subset of Σ, and hence $cl_{eig}(\{p_u\}) = \{p_u\}$. So let us take an example of subsets containing at least two different states.

a. $A = \{p_x, p_y\}$, for $x \neq y$., hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\lambda^1_{max})} R\} = \Sigma$. The closure of the set consisting of two different states is Σ, which is an expression of the superposition principle in quantum mechanics.

b. A is an arbitrary subset of Σ different from $\emptyset$ and different from a singleton.

Since in this case there exist $x \neq y$ such that $\{p_x, p_y\} \subset A$, we have $\Sigma \subset cl_{eig}(\{p_x, p_y\}) \subset cl_{eig}(A) \subset \Sigma$, which shows that $cl_{eig}(A) = \Sigma$. The only closed subset different from $\emptyset$ or from a singleton is Σ. Hence the collection of eigenstate-sets $\mathcal{F}(\epsilon = 1) = \{\emptyset, \{p_u\}, \Sigma\}$, a situation that corresponds to what we know in quantum mechanics.

4.2 An intermediate near-to-quantum case.

We choose $\epsilon = \frac{3}{4}$. Then $d_m = -\frac{1}{4}$.

We have $cos(\lambda^{\frac{3}{4}}_{max}) = +\frac{1}{2}$. Hence $\lambda^{\frac{3}{4}}_{max} = \frac{\pi}{3}$. So we have $H(\frac{\pi}{3}) = \{sec(u, \frac{\pi}{3}) \mid u \in surf\} \cup \{surf\}$.

a. $A = \{p_x, p_y\}$, for $x \neq y$ hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\frac{\pi}{3})} R\}$. Let us construct $cl_{eig}(A)$.

1. $\frac{2\pi}{3} < \theta(x, y)$.

Then there exists no $sec(u, \frac{\pi}{3})$ such that $\{p_x, p_y\} \subset sec(u, \frac{\pi}{3})$. Hence $cl_{eig}(\{p_x, p_y\}) = \Sigma$.

2. $\theta(x, y) \leq \frac{2\pi}{3}$.

Consider the two points v and w, such that $\theta(x, w) = \theta(y, w) = \theta(x, v) = \theta(y, v) = \frac{\pi}{3}$, and the spherical sectors $sec(v, \frac{\pi}{3})$ and $sec(w, \frac{\pi}{3})$. Our claim is that $cl_{eig}(\{p_x, p_y\}) = \{p_u \mid u \in sec(w, \frac{\pi}{3}) \cap sec(v, \frac{\pi}{3})\}$. Indeed if $sec(u, \frac{\pi}{3})$ is an arbitrary spherical sector such that $x \in sec(u, \frac{\pi}{3})$ and $y \in sec(u, \frac{\pi}{3})$, then clearly $sec(w, \frac{\pi}{3}) \cap sec(v, \frac{\pi}{3}) \subset sec(u, \frac{\pi}{3})$.

b. $A = \{p_x, p_y, p_z\}$, for $x \neq y$, $x \neq z$ and $y \neq z$, hence $Q = \{x, y, z\}$.

1. $\nexists\ sec(u, \frac{\pi}{3})$ such that $Q \subset sec(u, \frac{\pi}{3})$.

In this case $cl_{eig}(\{p_x, p_y, p_z\}) = \Sigma$.

2. $\exists\ sec(u, \frac{\pi}{3})$ such that $Q \subset sec(u, \frac{\pi}{3})$.

Consider three points u, v and w such that $\theta(u, x) = \theta(u, y) = \theta(v, x) = \theta(v, z) = \theta(w, z) = \theta(w, y) = \frac{\pi}{3}$, and the spherical sectors $sec(u, \frac{\pi}{3})$, $sec(v, \frac{\pi}{3})$ and $sec(w, \frac{\pi}{3})$. $cl_{eig}(\{p_x, p_y, p_z\}) = \{p_v \mid v \in sec(u, \frac{\pi}{3}) \cap sec(v, \frac{\pi}{3}) \cap sec(w, \frac{\pi}{3})\}$.

4.3 The in-between case.

We choose $\epsilon = \frac{1}{2}$. Then $d_m = -\frac{1}{2}$.

We have $cos(\lambda_{max}^{\frac{1}{2}}) = 0$. Hence $\lambda_{max}^{\frac{1}{2}} = \frac{\pi}{2}$. So we have $H(\frac{\pi}{2}) = \{sec(u, \frac{\pi}{2}) \mid u \in surf\} \cup \{surf\}$.

a. $A = \{p_x, p_y\}$, for $x \neq y$ hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\frac{\pi}{2})} R\}$. Let construct $cl_{eig}(A)$.

We remark that there always exists a spherical sector $sec(u, \frac{\pi}{2})$ such that $Q \subset sec(u, \frac{\pi}{2})$. Indeed we only have to consider the big circle through x and y, and take u the North Pole of this big circle. This shows that for $\epsilon \leq \frac{1}{2}$ the property-closure of two states cannot be equal anymore to Σ as it is in the quantum case, and as it can be for $\frac{1}{2} < \epsilon \leq 1$.

The construction of v and w, and $sec(v, \frac{\pi}{2})$ and $sec(w, \frac{\pi}{2})$, that we explained in detail for the case $\epsilon = \frac{3}{4}$ also works in this case. But since the contouring circles $circ(u, \frac{\pi}{2})$ of spherical sectors with angle $\frac{\pi}{2}$ are big circles of $surf$, we have that $sec(v, \frac{\pi}{2}) \cap sec(w, \frac{\pi}{2})$ is the big circle through x and y. Hence $cl_{eig}(\{x, y\})$ is the portion of the big circle from x to y.

b. $A = \{p_x, p_y, p_z\}$, for $x \neq y$, $x \neq z$ and $y \neq z$, hence $Q = \{x, y, z\}$.

Again we remark that always $\exists\ sec(u, \frac{\pi}{2})$ such that $Q \subset sec(u, \frac{\pi}{2})$, hence even the property-closure of three different states cannot result in Σ if $\epsilon \leq \frac{1}{2}$. An analogous reasoning than the one we have made for the two states shows that $cl_{eig}(\{x, y, z\})$ is given by the sphere-triangle of $surf$ formed by x, y and z.

4.4 An intermediate near-to-classical case.

We choose $\epsilon = \frac{1}{4}$. Then $d_m = -\frac{3}{4}$.

We have $cos(\lambda^{\frac{1}{4}}_{max}) = -\frac{1}{2}$. Hence $\lambda^{\frac{1}{2}}_{max} = \frac{2\pi}{3}$. So we have $H(\frac{2\pi}{3}) = \{sec(u, \frac{2\pi}{3}) \mid u \in surf\} \cup \{surf\}$.

a. $A = \{p_x, p_y\}$, for $x \neq y$ hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\frac{2\pi}{3})} R\}$. Let construct $cl_{eig}(A)$.

Again there always exists a spherical sector $sec(u, \frac{2\pi}{3})$ such that $Q \subset sec(u, \frac{2\pi}{3})$. Let us call $cl_{eig}(Q)^C$ the set theoretical complement of $cl_{eig}(Q)$. We remark that $sec(u, \frac{2\pi}{3})^C = sec^o(-u, \frac{\pi}{3})$, which is the open spherical sector centered around $-u$ with angle $\frac{\pi}{3}$. It can be shown that $cl_{eig}(Q)^C = \cup_{sec^o(u, \frac{\pi}{3}) \subset Q^C, u \in surf} sec^o(u, \frac{\pi}{3})$ and Q^C is the complement of $\{x, y\}$. It is easy to see that we can overlap the whole of Q^C with open spherical sectors $sec^o(u, \frac{\pi}{3})$, which means that $cl_{eig}(Q)^C = Q^C$ and hence $cl_{eig}(Q) = Q$. This shows that in this case of $\epsilon = \frac{1}{4}$ for two different states there is no superposition, in the sense that $cl_{eig}(\{p_x, p_y\}) = \{p_x, p_y\}$, and this is generally true for $\epsilon < \frac{1}{2}$. This however does not mean that we have already reached a classical situation once $\epsilon < \frac{1}{2}$. This becomes clear in the next case.

b. $A = \{p_x, p_y, p_z\}$, for $x \neq y$, $x \neq z$ and $y \neq z$, hence $Q = \{x, y, z\}$.

Again we remark that always $\exists\ sec(u, \frac{2\pi}{3})$ such that $Q \subset sec(u, \frac{2\pi}{3})$, hence even the eigen-closure of three different states cannot result in Σ if $\epsilon \leq \frac{1}{2}$. But in this case we have again additional states coming in in the closure. Indeed $cl_{eig}(Q)$ is the triangle with points x, y and z and sides pieces of circles with angle $\frac{\pi}{3}$.

4.5 The classical case.

We choose $\epsilon = 0$. Then $d_m = -1$.

We have $cos(\lambda^0_{max}) = -1$. Hence $\lambda^0_{max} = \pi$. So we have $H(\pi) = \{sec^o(u,\pi) \mid u \in surf\} \cup \{surf\}$.

As we shall show, we only have to consider the case where A is an arbitrary subset of Σ. Let us construct $cl_{eig}(A)$.

We remark that $sec^o(u,\pi) = \{u\}^C$, hence the open spherical sectors in this case are just the complements of the points of $surf$. Clearly there always exists a spherical sector $sec^o(u,\pi)$ such that $Q \subset sec^o(u,\pi)$. If we introduce again $cl_{eig}(Q)^C$ as the set theoretical complement of $cl_{eig}(Q)$, and remark that any set equals the union of its points, we have $cl_{eig}(Q)^C = \cup_{u \in Q^C, u \in surf}\{u\} = Q^C$. Hence $cl_{eig}(Q) = Q$ for any Q. This shows that in this case of $\epsilon = 0$ there is never a superposition since the closure of any subset of the set of states equals this subset. With other words, if $\epsilon = 0$ we have $\mathcal{F}(\epsilon = 0) = \mathcal{P}(\Sigma)$, and the collection of the eigenstate-sets is isomorphic to $\mathcal{P}(\Sigma)$.

It is interesting to remark that for all the cases $\epsilon \neq 0$ the closure $cl_{eig}(Q)$ of an arbitrary subset $Q \subset surf$ is a topologically closed subset, since it is the intersection of closed sets. Only for the special case of $\epsilon = 0$ the closure $cl_{eig}(Q)$ of an arbitrary subset $Q \subset surf$ is equal to Q, and this is possible since it is the intersection of topologically open sets, and an infinite intersection of open sets can be open or closed.

5. The axioms of quantum mechanics.

We have shown already (see theorem 4) that the entity $S(\epsilon)$ satisfies the axiom of state determination and the axiom of atomicity. Let us investigate the other axioms of quantum mechanics.

Theorem 5 *For a given ϵ, the superposition angle σ^ϵ_d is minimal iff $d = 0$. Let us denote this minimal superposition angle by σ^ϵ. We have:*

$$cos\sigma^\epsilon = 1 - 2\epsilon^2 \tag{18}$$

Proof: We know that $\sigma^\epsilon_d = \pi - \lambda^\epsilon_d - \mu^\epsilon_d$, and hence $cos\sigma^\epsilon_d = -cos(\lambda^\epsilon_d + \mu^\epsilon_d) = sin(\lambda^\epsilon_d)sin(\mu^\epsilon_d) - cos(\lambda^\epsilon_d)cos(\mu^\epsilon_d) = \sqrt{(1-(\epsilon+d)^2)(1-(\epsilon-d)^2)} - \epsilon^2 + d^2$. It is easy to verify that this function has only one maximum for $d = 0$ in the interval $[-1+\epsilon, 1-\epsilon]$. For $d = 0$ we have $\lambda^\epsilon_d = \mu^\epsilon_d$, and hence $cos\sigma^\epsilon = cos(\pi - 2\lambda^\epsilon_0) = 1 - 2cos^2\lambda^\epsilon_0 = 1 - 2\epsilon^2$.

For two states p_v and p_w we introduce $\theta(v,w)$, the angle between the two vectors v and w, and we prove the following lemma:

Lemma 1 *Suppose that u and $-u$ are two diametrically opposed points of $surf$, and v and w are two arbitrary points of $surf$. We then have:*

$$\pi \leq \theta(u,v) + \theta(v,w) + \theta(w,-u) \tag{19}$$

the equality being satisfied only in the case that u, v, w, and $-u$ are in the same plane.

Theorem 6 *Consider two states p_v and p_w of the entity $S(\epsilon)$ then we have (see ref [9], definition 9):*

$$p_v \perp p_w \Leftrightarrow \sigma^\epsilon \leq \theta(v,w) \Leftrightarrow v \cdot w \leq 1 - 2\epsilon^2 \tag{20}$$

Proof: Suppose that $p_v \perp p_w$, then there exists an experiment $e^\epsilon_{u,d}$ such that $p_v \in eig^\epsilon_{u,d}(1)$ and $p_w \in eig^\epsilon_{u,d}(2)$, which means that $\theta(u,v) \leq \lambda^\epsilon_d$ and $\theta(w,-u) \leq \mu^\epsilon_d$. We also have $\lambda^\epsilon_d + \sigma^\epsilon_d + \mu^\epsilon_d = \pi \leq \theta(u,v) + \theta(v,w) + \theta(w,-u) \leq \lambda^\epsilon_d + \theta(v,w) + \mu^\epsilon_d$. From this follows that $\sigma^\epsilon_d \leq \theta(v,w)$, and hence also $\sigma^\epsilon \leq \theta(v,w)$. Suppose now that we have $\sigma^\epsilon \leq \theta(v,w)$. Consider then the plane formed by the vectors v and w. In this plane we can choose a vector u and its diametrically opposed vector $-u$ and the experiment $e^\epsilon_{u,0}$, such that $v \in eig^\epsilon_{u,d}(1)$ and $w \in eig^\epsilon_{-u,d}$.

Theorem 7 *The entity $S(\epsilon)$ satisfies the axiom ortho 1 (see ref [9]), iff $\epsilon = 1$ or $\epsilon = 0$.*

Proof: Suppose that the entity $S(\epsilon)$ satisfies axiom ortho 1. Consider $eig^\epsilon_{u,d}(1)$ for some experiment $e^\epsilon_{u,d}$, and a state $p_v \perp eig^\epsilon_{u,d}(1)$. From axiom ortho 1 follows that $p_v \in eig^\epsilon_{u,d}(2)$. This means that $\pi - \lambda^\epsilon_d - \sigma^\epsilon \leq \mu^\epsilon_d$. From this follows that $\pi - \lambda^\epsilon_d - \mu^\epsilon_d \leq \sigma^\epsilon$, which implies that $\sigma^\epsilon_d \leq \sigma^\epsilon$, for all d. This can only be satisfied if $\sigma^\epsilon_d = \sigma^\epsilon = 0$, which is equivalent to $\epsilon = 0$, or if $\sigma^\epsilon_d = \sigma^\epsilon = \pi$, which is equivalent to $\epsilon = 1$.

Theorem 8 *If the entity $S(\epsilon)$ satisfies the axiom ortho 2 (see ref [9]), then $\epsilon = 1$ or $\epsilon = 0$.*

Proof: Suppose that axiom ortho 2 is satisfied, and consider p_u. The collection of states $\{p_u\}^\perp$ is a closed spherical sector with center $-u$ and angle $\pi - \sigma^\epsilon$. From axiom ortho 2 follows that this spherical sector must be contained in some $eig^\epsilon_{u,d}(1)$ with $d = -1 + \epsilon$, and hence with an angle λ^ϵ_{max}. Hence we have $\pi - \sigma^\epsilon \leq \lambda^\epsilon_{max}$. From this follows that $cos\lambda^\epsilon_{max} \leq cos(\pi - \sigma^\epsilon)$. We have $cos\lambda^\epsilon_{max} = 2\epsilon - 1$, and $cos(\pi - \sigma^\epsilon) = 2\epsilon^2 - 1$. This proves that $2\epsilon - 1 \leq 2\epsilon^2 - 1$, which implies that $\epsilon \leq \epsilon^2$. This can only be satisfied for $\epsilon = 0$ or $\epsilon = 1$.

These two theorems show that the axioms ortho 1 and ortho 2 are not satisfied in the intermediate situations. Also the axioms of weak modularity and the covering law are not satisfied in the intermediate situations.

Theorem 9 *If the entity $S(\epsilon)$ satisfies the axiom of weak modularity then $\epsilon = 1$ or $\epsilon = 0$.*

Proof: If $\epsilon = 1$, in the quantum situation, the axiom of weak modularity is satisfied. Suppose that $\epsilon \neq 1$. Then for an arbitrary $d \in [-1+\epsilon, 1-\epsilon]$ we have that $\lambda_d^\epsilon > 0$ or $\mu_d^\epsilon > 0$. Suppose that $\lambda_d^\epsilon > 0$, and consider $e_{u,d}^\epsilon$. We have that $\{p_u\} \subset eig_{u,d}^\epsilon(1)$. If the axiom of weak modularity is satisfied, then there exists an element $F \in \mathcal{F}(\epsilon)$ such that $F \perp \{p_u\}$ and $cl_{eig}(\{p_u\} \cup F) = eig_{u,d}^\epsilon(1)$. This means that $F \subset eig_{u,d}^\epsilon(1)$, and hence there is at least one state $p_w \in F$ such that $p_w \subset eig_{u,d}^\epsilon(1)$ and $p_w \perp p_u$. From this follows that $\sigma^\epsilon \leq \theta(u,w) \leq \lambda_d^\epsilon$, and this for an arbitrary $d \in [-1+\epsilon, 1-\epsilon]$. Hence $cos\lambda_d^\epsilon \leq cos\sigma^\epsilon$ for all d, which implies that $\epsilon + d \leq 1 - \epsilon^2$ for $d \in [-1+\epsilon, 1-\epsilon]$. Take $d = 1-\epsilon$, then we have $1 \leq 1 - 2\epsilon^2$, which implies that $\epsilon = 0$.

Theorem 10 *If the axiom of the covering law is satisfied then $\epsilon = 1$ or $\epsilon = 0$.*

Proof: Suppose that $\epsilon \neq 1$ and $\epsilon \neq 0$. Then we have $0 \neq \lambda_{max}$ and $\pi \neq \lambda_{max}$. Let us call $\phi = min(\lambda_{max}, \pi - \lambda_{max})$, and choose $\eta < \phi$. Consider the following two elements of the eigenclosure structure $F_1 = sec(u, \alpha)$ with $\alpha = \lambda_{max} - \frac{\eta}{2}$ and $F_2 = sec(u, \lambda_{max})$. Choose a point $v \in surf$ such that $\theta(u,v) = \lambda_{max} + \eta$. We then have $2\lambda_{max} = \theta(u,v) - \eta + \alpha + \frac{\eta}{2} = \theta(u,v) + \alpha - \frac{\eta}{2} < \theta(u,v) + \alpha$. This shows that F_1 and p_v cannot together be contained in a spherical sector $sec(w, \lambda_{max})$. As as consequence we have $cl_{eig}(F_1 \cup \{p_v\}) = \Sigma$. If the axiom of the covering law would be satisfied, it should follow from the fact that $cl_{eig}(F_1 \cup \{p_v\}) = \Sigma$ that for every state $p_w \neq p_v$ contained in Σ we have $p_w \in F_1$. This is clearly not the case, for example already the element $F_2 = sec(u, \lambda_{max})$ contains a lot of states that are not contained in F_1. This shows that the covering law is not satisfied.

6. A general study of the classical and non-classical situations of the sphere-model.

We would like to return now to the physical meaning of the ϵ-example. The probabilities in the model find their origin in the presence of fluctuations on the experimental apparatuses. The quantum situation appears if the magnitude of the fluctuations on the experiment apparatuses is maximal. In this case the elastic can break uniformly in all of its points. We have proposed to characterize the classical situation as being the situations with zero fluctuations, where the elastic breaks in one predetermined point. This leads us to a collection of eigenstate-sets $\mathcal{F}(\epsilon = 0)$ that is Boolean and isomorphic to $\mathcal{P}(\Sigma)$. We have two remarks in relation with this finding. 1) A situation of zero fluctuations on the experiment apparatuses is a very strong idealization. It would be much more realistic to describe the classical situation as being the situation where the fluctuations can be minimized as much as one wants. Let us see whether we can construct the eigen-closure structure corresponding to this slightly more realistic situation. 2) We have remarked already that a closure structure is a generalization of an ordinary topology. For $\epsilon = 0$ each subset K of Σ is closed for cl_{eig} and hence $\mathcal{F}(\epsilon = 0) = \mathcal{P}(\Sigma)$, which shows that closure structure for the case $\epsilon = 0$ corresponds to the discrete topology on Σ. This result is in some sense not very satisfactory as to the possibility of characterizing the eigenstate sets of an entity by means of the closure structure. We would rather have liked the classical situation to correspond to a closure structure corresponding with the ordinary standard topology on the sphere.

To answer both questions we want to investigate the classical situation in more detail, and therefore we define $\mathcal{E}_{st}$ as the set of all $e^{\epsilon}_{u,d}$ such that $0 < \epsilon$, and $\mathcal{F}_{st}$ as the collection of eigenstate sets generated by the experiments $\mathcal{E}_{st}$

We know that $\mathcal{B}(\epsilon) = H(\lambda^{\epsilon}_{max})$ is a generating set for the collection of closed subspaces $\mathcal{F}(\epsilon)$. From this follows that $\mathcal{B}_{st} = \cup_{0<\epsilon}\mathcal{B}(\epsilon)$ is a generating set for $\mathcal{F}_{st}$, and if we consider an arbitrary subset $K \subset \Sigma$, and $Q = \{u \mid p_u \in K\}$, we can find the closure structure corresponding to this collection of eigenstate-sets. Let us denote the closure generating this closure structure by cl_{st}

Theorem 11 *Consider an arbitrary subset $K \subset \Sigma$, and let $Q = \{u \mid p_u \in K\}$. Then :*

$$cl_{st}(K) = \{p_w \mid w \in \cap_{Q \subset R, R \in \mathcal{B}_{st}} R\} \tag{21}$$

Theorem 12 *The eigen-closure structure introduced by cl_{st}, with collection of eigen-closed subsets $\mathcal{F}_{st}$ is the ordinary topology on the surface of the sphere $surf$, and as a consequence we have for $K, L \subset \Sigma$:*

$$cl_{st}(K \cup L) = cl_{st}(K) \cup cl_{st}(L) \tag{22}$$

and from this follows that $\mathcal{F}_{st}$ is a Boolean lattice.

Proof : Consider $\mathcal{B}_{st}$, and $R \in \mathcal{B}_{st}$, R different from $surf$. Then R is a closed spherical sector $sec(u, \lambda^{\epsilon}_{max})$. If we consider R^C, then it is an open spherical sector $sec^o(-u, \pi - \lambda^{\epsilon}_{max})$. If we choose an arbitrary $0 < \delta$ we can always find an $0 < \epsilon$ such that $\pi - \lambda^{\epsilon}_d < \delta$. This proves that $Z = \{R^C \mid R \in \mathcal{B}_{st}\}$ is a base for the standard topology on the surface of the sphere, and as a consequence $\mathcal{F}_{st}$ corresponds to the standard topology on the sphere. For $Q \subset surf$ we denote by $\bar{Q}$ the closure of Q in this standard topology of the sphere. Then follows that for $K \subset \Sigma$ we have $cl_{st}(K) = \{p_v \mid v \in \bar{Q}\}$, if $K = \{p_v \mid v \in Q\}$. Since for the standard topology on the surface of the sphere we have that the closure of the union of two subsets is the union of the closures of the subsets, we have $cl_{st}(K \cup L) = cl_{st}(K) \cup cl_{st}(L)$. Let us show now that this is sufficient to make the closure structure Boolean. We have to show that for three elements $K, L, M \in \mathcal{F}_{st}$ we have the distrubitive law to be valid. So we have to prove that $cl_{st}(K \cup L) \cap M = cl_{st}((K \cap M) \cup (L \cap M))$. We have $cl_{st}(K \cup L) \cap M = (cl_{st}(K) \cup cl_{st}(L)) \cap M = (K \cup L) \cap M = (K \cap M) \cup (L \cap M) = cl_{st}(K \cap M) \cup cl_{st}(L \cap M) = cl_{st}((K \cap M) \cup (L \cap M))$. We also have $cl_{st}((K \cap L) \cup M)) = cl_{st}(K \cap L) \cup cl_{st}(M) = (K \cap L) \cup M = (K \cup M) \cap (L \cup M) = (cl_{st}(K) \cup cl_{st}(M)) \cap (cl_{st}(L) \cup cl_{st}(M)) = cl_{st}(K \cup M) \cap cl_{st}(L \cup M)$, which shows the other distributive law to be true.

7. Conclusion.

We want to make some remarks about the novelty of the investigation of the intermediate situations as they are presented in this paper. It is well known that classical theories and quantum theories are structurally completely different (Boolean versus non-Boolean, commutative versus non-commutative, Kolmogorovian versus non-Kolmogorovian). In the field of the more general approaches (lattice theories, *algebra's, probability models), the classical situation can be considered to be a 'special' case of the general situation. A classical entity is described by a Boolean lattice, a commutative *algebra and a Kolmogorovian probability model, while for a general entity the lattice is not necessarily Boolean, the *algebra not always commutative, and the probability model not necessarily Kolmogorovian. Hence these general approaches can study the quantum as well as the classical and also a mixture of both cases. These type of general situations have been studied intensively in all the categories that we have mentioned (e.g. [10,11,12,13]). For a complete orthocomplemented weakly modular atomic lattice it follows from Piron's representation theorem that the lattice is isomorphic to the direct union of a set of irreducible lattices (see [10,11,12,13]), such that the classical properties of the entity are represented by subsets of the index set, while the quantum properties are represented by elements of the irreducible components. This theorem shows that the structure that we have called the intermediate situation in this paper must not be confused with the mixture of classical and quantum properties that can be described in structures satisfying the quantum axioms. Indeed, in the traditional approach the mixture contains only classical properties (described by subsets of the index set of the direct union) or pure quantum properties (described by elements of the irreducible components). In the intermediate situation, as presented in this paper, we describe properties that are neither classical nor quantum, but in between. That is the reason that the structure that we have studied here does not satisfy the quantum axioms: it is not a complete, orthocomplemented, weakly modular atomic lattice, satisfying the covering law. This result explains why the intermediate situations considered in this paper could not be described by the earlier general approaches.

With the sphere model corresponds a realisable physical entity, e.g. the elastic-sphere example, whereon we have defined realisable physical experiments, that could eventually be executed in the laboratory, and will then generate experimental data fitting into a quantum mechanical structure. We are well aware of the strong existing paradigm stating that quantum-reality is intrinsically different from classical macroscopic reality, and in this way forbidding to imagine quantum structures existing inside this macroscopic reality. It seems that our elastic example confronts this paradigm, and indeed it does, and to avoid prejudices rising immediately we must explain exactly in which way it does. It is difficult in principle to imagine the physical entity S, the point particle P, at any state being located in a point of the sphere, as a quantum entity. But this difficulty comes from the fact that we, in our imagination, in preparing and identifying the state of this physical entity S automatically use experiments other than the $e^{\epsilon}_{u,d}$ that we have introduced. Usually we 'see' the point on the sphere and suppose unconsciously that the 'seeing' experiment is an experiment that is available to us for the study of the entity S that is the point particle P on the surface of the sphere. What we should do is to imagine the physical entity S, hence the point particle P on the sphere, as only knowable to us by means of the experiments $e^{\epsilon}_{u,d}$ that we have introduced. This is indeed exactly the situation that we encounter when we make investigations about quantum entities in the micro-world. We cannot 'see' or 'touch' these entities, and have only knowledge about them by means of the experiments that we can carry out on them. This remark makes it possible for us to state exactly the philosophy of our approach. If we state that the entity S is described by the sets Σ and $\mathcal{E}$, then we also suppose that the elements of $\mathcal{E}$ are the only experiments available to us for the identification and preparation of the states Σ of the entity S. It is in this light that our classification scheme, quantum, classical and intermediate, has to be understood. And we believe that it is the correct way of approaching the knowledge that we have about the reality of the micro-world.

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