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QUANTUM, CLASSICAL AND INTERMEDIATE: A MEASUREMENT MODEL.

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Abstract: We present a measurement model where the origin of the quantum probabilities lies in the presence of fluctuations between the measurement apparatus and the physical system. First we make a reasoning where we show that the measurement process cannot be described by the unitary Schrödinger evolution only. Afterwards we present our model of measurement and show the necessity of developping a more general structure than orthodox Hilbert space quantum mechanics to resolve the measurement problem.

1. Introduction.

In conventional textbooks on quantum mechanics, the classical limit is often presented through Ehrenfest's theorem: if the wave function representing a quantum

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system evolves according to the Schrödinger equation:

$$i\hbar\,\partial_t\,\psi(q) = -\frac{\hbar^2}{2m}\Delta\psi(q) + V(q)\psi(q) \tag{1}$$

where q represents the position of the system, then, the average position and momentum obey the classical dynamical equations of Hamilton:

$$\begin{cases} m\,\dfrac{d\,\langle q\rangle}{d_t} = \langle p\rangle \\ \dfrac{d}{d_t}\,\langle p\rangle = \left\langle \dfrac{\partial\, V}{\partial_q} \right\rangle \end{cases} \tag{2}$$

This is usually understood as: "for classical systems, we can neglect fluctuations of quantum nature, and identify the system's behavior with its average behavior, which is classical".

Unfortunately, this line of thougth presents severe drawbacks:

- no quantitative criterion allows us to neglect quantum fluctuations,
- the analogy with hamiltonian equations is purely formal, because $\left\langle \frac{\partial V}{\partial q} \right\rangle \neq \frac{\partial V}{\partial q}\left(\langle q\rangle\right)$ except for an harmonic oscillator,
- experimentally, measurements are necessary to get information about the system, but the theory of measurement is not defined in a classical context.

Effectively, although at a microscopical level any measurement is generally strongly disturbing and probabilistic, at a macroscopical level, it is non invasive and deterministic. In his derivation of uncertainty relations for instance, Heisenberg used the idea that a measurement is accompanied by an exchange of action comparable to h, the Planck constant, in an uncontrollable way.

It is sometimes believed that the problem of measurement can be solved with as a single ingredient the Schrödinger local interaction between the measuring apparatus and theobserved system. We shall show here (section 2) that such an approach c ontradicts well established experimental facts, with as consequence that something new must happen during the measurement, and we shall present a model where it is possible to describe this "something new".

This model allows us to describe a continuous transition from classical to quantum. In the model the amount of quantum-behaviour is coupled to the amount of fluctuations of a hidden variable introduced in the interaction between the measurement apparatus and the physical entity. We shall show how the quantum case and the classical case are both limits of a more general 'intermediate' case. (section 3). The limit of zero fluctuations is classical and the limit of maximal fluctuations is quantum.

If we consider the structure of the intermediate case, we can show that the Hilbert space axioms of quantum mechanics are no longer valid [1)], but are replaced by a more general structure, and this explains why it is not possible to have a continuous transition between quantum and classical within the orthodox Hilbert space quantum mechanics. We shall show (section 4) that for the intermediate case, the probability model is not quantum (representable by a Hilbertian probability model). Both results indicate that the fundamental difficulty of describing the measurement process might be due to a structural shortcoming of the available physical theories (quantum mechanics and classical mechanics).

2. Local Schrödinger evolution is not enough.

In his experiments testing Bell's inequalities, A. Aspect [2)] disposed of a pair of correlated photons, that were locally separated and described by the following quantum state:

$$\frac{1}{\sqrt{2}}\Big(|+\rangle_{\rm L}\otimes|+\rangle_{\rm R}\;+\;|-\rangle_{\rm L}\otimes|-\rangle_{\rm R}\Big) \tag{3}$$

where $|\pm\rangle_{\rm L,R}$ describe mutually exclusive states of polarization of the photons in the distants regions Left(L) and Right(R). If the experimenter places parallel polarizers in Left and Right, then, according to standard quantum mechanics, these are perfectly correlated. This was observed:up to little experimental errors,whenever two photons were simultaneously detected,they shared the same polarization. Let us suppose now that all what happens is described by a local Schrödinger interaction between the incoming photons and the detectors and that we perform first a measurement in Left. If we denote $|\,0\,\rangle_{\rm PL}$ the state of the Left polarizer prior to the measurement, then the initial state

$$|\,0\,\rangle_{\rm PL} \otimes \frac{1}{\sqrt{2}}\Big(|\,+\,\rangle_{\rm L} \otimes |\,+\,\rangle_{\rm R} \;+\; |\,-\,\rangle_{\rm L} \otimes |\,-\,\rangle_{\rm R}\Big) \tag{4}$$

will evolve to

$$\frac{1}{\sqrt{2}}\Big(|\,+\,+\rangle_{\rm PL} \otimes |\,+\,\rangle_{\rm R} \;+\; |\,-\,-\rangle_{\rm PL} \otimes |\,-\,\rangle_{\rm R}\Big) \tag{5}$$

where $|{+}{+}\rangle >_{\rm PL}$ and $|{-}{-}\rangle >_{\rm PL}$ express a very complex entangled state between the incoming photon's field and the Left polarizer. If thereafter we perform a measurement in Right, we have analogously that, prior to the measurement, the state of the whole system is given by

$$\frac{1}{\sqrt{2}}\Big(|\,+\,+\rangle_{\rm PL} \otimes |\,+\,\rangle_{\rm R} \;+\; |\,-\,-\rangle_{\rm PL} \otimes |\,-\,\rangle_{\rm R}\Big) \otimes |\,0\,\rangle_{\rm PR} \tag{6}$$

whatever the result of the Left measurement was. Effectively, the Schrödinger evolution does not suffice to break the superposition between different polarizations ("to kill the cat"). But then, the polarizations in Left and Right are not correlated, in contradiction with, among others, Aspect's experimental results. This is in fact not astonishing: the act of measurement implies memory and irreversible processes which are absents in the naive evolution postulated here. In conclusion, we need "something more" to characterize the measurement process. We will present in the next part a model aimed to do this task, with the help of some new ingredients.

3. A measurement model allowing a continous limit between classical and quantum.

3a. Spin measurement.

We will adopt as a guiding example the measurement of a spin 1/2:

- for an individual spin it is in principle possible to use a Stern-Gerlach apparatus, after which the particle collapses along one of the two outcome channels,
- for an assembly of spins, we will consider a model of a ferro-magnet, where we can measure the global spin via the magnetic field that it induces in its surrounding. This field can be measured classically, for instance by the torque undergone by a ring conducting an electrical current. Ideally, this ring can be considered to be little, and far from the magnet so that it does not exert any back-action on it and reacts deterministically to the local field.

We will consider that during each measurement the result is pre-determined, so that it is essentially a classical measurement, but fluctuations of the measuring device cause repeated experiments to be of probabilistic nature. This idea was already exploited in 1966 by Bohm and Bub [3), where a nonlinear evolution allowed to describe a collapsing process in any finite-dimensional Hilbert space. In 1981, Gisin [4), using the model of Bohm and Bub described the interaction between a classical pointer and a spin 1/2 occuring during a measurement. We will consider the Gisin equations and introduce the classical limit by a modification of it in terms of N, the amount of particules in the system. For $N = 1$, we are in the quantum case; for N tending to infinity, we recover the classical limit.

Prior to this, let us recall some properties of a spin 1/2:

- it is defined in a two-dimensional Hilbert space and can always be expressed as a superposition of a spin-up and a spin-down state along an a priori direction z:

$$\big|\psi\big\rangle = \alpha\big| + \big\rangle \ + \ \beta\big| - \big\rangle, \tag{7}$$

- then, the spin-measurement with a Stern-Gerlach device oriented along z is described by the observable $\frac{\hbar}{2}\,\sigma_z$, where

$$\sigma_z = \big| + \big\rangle\big\langle + \big| \ - \ \big| - \big\rangle\big\langle - \big|, \tag{8}$$

- the Pauli mapping maps bijectively the set of physical states onto the 3-sphere unity, by:

$$\big|\psi\big\rangle \rightarrow \vec{n} = \big(2\mathrm{Re}\,\alpha^*\beta, 2\mathrm{Im}\,i\beta^*\alpha, |\alpha|^2 - |\beta|^2\big) \tag{9}$$

(in Cartesian coordinates) and conversely:

$$\alpha = \cos\frac{\theta}{2}e^{-\frac{i\varphi}{2}} \qquad \beta = \sin\frac{\theta}{2}e^{\frac{i\varphi}{2}}, \tag{10}$$

where θ, φ are the polar angles of $\vec{n}$.

This allows us to visualize all states on the sphere, for instance the up-state is sent on the north pole, the down-state on the south pole. This being done, let us write the equations of Gisin:

$$\begin{cases} \partial_t \left|\psi\right\rangle = (x - r)\big(\langle P_z\rangle - P_z\big)\left|\psi\right\rangle \\ \dot{r} = j \\ \dot{j} = -\omega^2\big(r - \langle P_z\rangle\big) - \lambda j. \end{cases} \tag{11}$$

where x is a real parameter randomly distributed in $[0, 1]$, r is the position of a classical pointer which undergoes a damped oscillation around the average value of the projector on the up-state:

$$\langle P_z \rangle = \langle \psi | + \rangle \langle + | \psi \rangle = |\alpha|^2 = \cos^2 \frac{\theta}{2}. \tag{12}$$

When the coupling constants ω and λ are sufficiently high, we can consider that the pointer directly follows the value of $\langle P_z \rangle$ so that we find back a closed equation for the state (in fact equivalent to the Bohm and Bub equation):

$$\partial_t |\psi\rangle = \big(x - \langle P_z \rangle\big)\big(\langle P_z \rangle - P_z\big)|\psi\rangle. \tag{13}$$

This equation presents two attractors, one in the up-state, one in the down-state, where the initial state will asymptotically collapse, depending on the initial value of $x - \langle P_z \rangle$: if this quantity is positive (negative) initially, it will collapse to the down-state (up-state). x being at random in $[0, 1]$, the probability of measuring spin-up (-down) is given by:

$$\langle P_z \rangle = \cos^2 \frac{\theta}{2} \left(1 - \langle P_z \rangle = \sin^2 \frac{\theta}{2}\right), \tag{14}$$

and we recover the quantum probabilities.

Let us now take into account the fact that when we measure the spin of an assembly of N spinors, the pointer is sensitive to the average individual spin multiplied by N. The closed spin equation then becomes:

$$\partial_t |\psi\rangle = \big(x - N\langle P_z \rangle\big)\big(\langle P_z \rangle - P_z\big)|\psi\rangle. \tag{15}$$

This equation is strongly disturbing, even when N goes to infinity, so that it is necessary to introduce one term more, in order to stabilize the spins in a vicinity of their initial values. This can be done for instance by multiplying the righthand part of the equality (15) by the factor $\cos N\langle P_z \rangle$ which freezes the spin in a vicinity of opening $\frac{2\pi}{N}$ around its initial value. If we do not introduce this term, we have an equation describing a dichotomic measurement, for all values of N, allowing a characterization in the propositional formalism of Piron.

The probability distribution induced by the dynamics is in fact the same as the distribution of the ϵ-model studied by us precedently (see section 4a). We will now show how this probability evolves continously, from a quantum one when $N = 1$, to a deterministic one when N increases. Afterwards, we will discuss the applicability of the representation theorem of Piron in function of N.

3b. Probability distributions.

When $N = 1$, we recover the equation (13), for which the probability of having spin-up is, as already shown, the quantum probability $\cos^2 \frac{\theta}{2}$. For all values of N, we will have spin-up if:

$$x - N\langle P_z \rangle < 0 \tag{16}$$

but x is upperly bounded by 1, so that we will have spin-up with certainty one if:

$$\langle P_z \rangle = \cos^2 \frac{\theta}{2} > 1/N. \tag{17}$$

This defines a spherical sector of opening $\theta_{\mathrm{up}}(\cos^2 \frac{\theta_{\mathrm{up}}}{2} = \frac{1}{N})$ around the upper pole composed by eigenstates of the result "up". The down-state is an equilibrium-state, for all values of N, so it will have zero-probability of collapsing to the up-state. Between the south pole and the spherical sector around the north pole, the probability will continously evolve from 0 to 1. Here are shown the probabilities of having the answer "up" in function of the azimuthal angle for three values of N (1, 4 and ∞)*. For the limiting case $N = \infty$, the mesasurement is deterministic (the probability is 1 or 0 everywhere).

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* This is the same probability as in ref. 5, with $\epsilon = 1/N$.

4. Why Hilbert space is not sufficient.

4a. Connection with the ϵ-model and the representation theorem of Piron.

The model that we have introduced here is equivalent with the ϵ-model that we have already presented on several occasions [1,5] in a way that we shall explain now. The set of states of the physical entity that we consider is the set of points of the unity sphere in three dimensional real space. The identification that we express in (9) and (10) defines an isomorphism between the state space of the ϵ-model with the state space of the measurement model presented in this paper. In the ϵ-model appear two parameters, $\epsilon \in [0, 1]$ and $d \in [-1 + \epsilon, 1 - \epsilon]$, such that the size of ϵ describes the amount of fluctuations on the interaction between the measurements and the entity, and d describes the possible assymetries of the measurements introduced in the ϵ-model (for a detailed introduction of the ϵ-model we refer to [1,5]). The measurements and the probabilities of the here presented measurement model are equivalent with the measurements and probabilities of the ϵ-model for the case where $d = \epsilon - 1$, and when $\epsilon = \frac{1}{N}$. We find then for two states p and q, represented by points u and v of the unity sphere, that the transition probability $P(p|q)$ is given by:

$$P(p|q) = N cos^2(\frac{\theta}{2}) \tag{18}$$

where θ is the angle between the two vectors u and v, and $P(p|q) = 1$ when (17) is fulfilled.

Since the ϵ-model is defined independently of the quantum mechanical apparatus (we don't use a Hilbert space for the set of states) and also independently of the classical mechanical apparatus, we can investigate whether Piron's axioms [6,7] for quantum and classical mechanics are satisfied on this model. We have shown in [1] that for the intermediate situation ($0 < \epsilon < 1$), some of the axioms (weak modularity and the covering law) are not satisfied. Since we can make the just mentioned identification of the two models, we can use the results that have been proved for the ϵ-model, also for the measurement model presented here. This means that for $1 < N < \infty$ the measurement model that we present cannot be described by a Hilbert space, neither by a classical phase space.

4b. Impossibility to represent the probabilities in a Hilbert space.

We shall demonstrate in the particular case $N = 4$ (see Figure) that the probability model of our measurement model cannot be represented in a Hilbert space. To do this let us take three coplanar states p, q and r, of which p and q make an angle $\frac{\pi}{3}$ and p and r an angle $\frac{5\pi}{6}$. Then the transition probability $P(p|q)$ equals $+1$, $P(q|r)$ equals $+1$, and $P(p|r)$ equals 0.56. If this probability model would be representable in a Hilbert space, then p, q and r would correspond to unit vectors of this Hilbert space, and then from $P(p|q) = 1$ would follow that $p = q$, similarly $q = r$. But then $p = r$ which which is a contradiction with $P(p|r) = 0.56$.

This shows that the considered probabilities cannot be described by orthodox quantum mechanics.

Conclusion.

The philosophical idea underlying this article is that, in order to describe the whole process of measurement, we must also describe the observer himself. In section 2, it is shown that we must introduce new tools, different from the usual unitary evolution inside the Hilbert space. A naive model was presented in section 3, for which we show that we must introduce a structure more general than the Hilbertspace. Quantum theory of open systems [8] and stochastic quantum mechanics [9,10,11] allow to introduce non-unitary evolutions but remain inside the Hilbert space. They could maybe form a first step towards a general theory of measurement englobing the classical world too.

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