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Quantum structures, separated physical entities and probability.

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Abstract. We prove that if the physical entity S consisting of two separated physical entities S_1 and S_2 satisfies the axioms of orthodox quantum mechanics, then at least one of the two subentities is a classical physical entity. This theorem implies that separated quantum entities cannot be described by quantum mechanics. We formulate this theorem in an approach where physical entities are described by the set of their states, and the set of their relevant experiments. We also show that the collection of eigenstate sets forms a closure structure on the set of states, that we call the eigen-closure structure. We derive another closure structure on the set of states by means of the orthogonality relation, and call it the ortho-closure structure, and show that the main axioms of quantum mechanics can be introduced in a very general way by means of these two closure structures. We prove that for a general physical entity, and hence also for a quantum entity, the probabilities can always be explained as being due to the presence of a lack of knowledge about the interaction between the experimental apparatus and the entity .

1. Introduction.

We have developed the description of a compound entity S consisting of two separated entities S_1 and S_2 in detail in [(1)], and want to propose here a new description of the entity S consisting of two separated entities S_1 and S_2, that takes into account the critique put forward in [(2)]. To do this we have to treat the situation where we consider explicitly experiments with more than two outcomes for the empirical basis of the theory. The description in [(1)] was given in the theory originated by Piron [(3)] and elaborated by Piron and myself [(1,4,5,6)]. The basic philosophy of the description that we present here is rooted in the approach developed in [(1,3,4,5,6)], but now we allow explicitly experiments with more than two outcomes for the empirical basis of the theory. In this paper we shall, apart from developing the general description of the entity consisting of separated entities, incorporating the critique of [(2)], also develop the mathematical framework of this approach with experiments with more than two outcomes. We particularly want to investigate the different topological structures that are generated on the state space. We show that some of the original axioms of the theory in [(1,4,5,6)], are identical with well known properties of the topological structure, and we shall see that by means of the introduction of these topological structures the description of the entity consisting of separated entities turns out to be much easier, and founded on the collection of 'all' available experiments. In this way the critique put forward in [(2)] is explicitly taken into account. The main theorem of [(1)], namely "If the entity S describing the two separated entities S_1 and S_2 satisfies the axioms of orthodox quantum mechanics, then at least one of the subentities S_1 or S_2 is a classical entity" (which implies that two separated quantum entities cannot be described by quantum mechanics) remains true in this new description.

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2 The description of one entity.

The basic concepts are 'experiments', denoted by $e, f, g, ...$, with outcome sets denoted by O_e, O_f, $O_g, ...$, to be performed on a physical entity S and 'states' of this physical entity, denoted by $p, q, r, ...$. For a given experiment e we denote a possible outcome by x_e^i, and the probability that the experiment e gives the outcome x_e^i, when the entity is in state p, by $P(e = x_e^i \mid p)$. The set of states of S is denoted by Σ, and the set of experiments by $\mathcal{E}$. In principle the outcome set O_e of an experiment e can be a finite or an infinite set, and the approach that we want to present is valid for both cases. As is well known, the case where the outcome set O_e of an experiment e is an infinite set has to be treated with care and easily gives rise to extended notations. Since we are interested mainly in the problem of the description of the compound entity S, consisting of two separated entities S_1 and S_2, and don't want to complicate the main results by a burden of notations, we treat only the case of finite outcome sets. The reader can check that the results remain valid and that the theorems can be proved for the case of infinite outcome sets in an analogous way. Hence for each experiment e, the outcome set is given by $O_e = \{x_e^1, x_e^2, ..., x_e^n\}$.

Let us indicate how these concepts are encountered in Hilbert space quantum mechanics. A state p of S is represented by a ray (one dimensional subspace) of the Hilbert space $\mathcal{H}$, denoted by $\bar{\psi}_p$, where ψ_p is the unit vector in this ray. We'll denote the set of all rays by $\Sigma_{\mathcal{H}}$. An experiment e is represented by a self-adjoint operator H_e on $\mathcal{H}$, and the outcome-set O_e of e is given by the spectrum σ_e of H_e. We'll denote the collection of all self-adjoint operators by $\mathcal{E}_{\mathcal{H}}$.

2.1 The entity, the set of states and the set of relevant experiments.

In the following we always consider an entity S, described by a set of states Σ, a set of experiments $\mathcal{E}$, and a set of probabilities $\{P(e = x_e^i \mid p) \mid e \in \mathcal{E}, x_e^i \in O_e, p \in \Sigma\}$. We introduce the following map:

Definition 1 : *When $A \subset O_e$, we define $eig_e(A) \subset \Sigma$ such that when $p \in eig_e(A)$, and S is in the state p, the outcome of the experiment e is with certainty (probability equal to 1) in A. We say that $eig_e(A)$ is an eigenstate set of the experiment e with eigenoutcome set A.*

The connection between the eigenstate set $eig_e(A)$ of an entity and a property of the entity, as introduced in [1,3,4,5,6], is easy to point out. Therefore we shortly explain the concept of 'property' and 'test' (experimental project or question) as used in [1,3,4,5,6].

A test is an experiment that can be performed on the entity and where one has agreed in advance what is the positive result. In this way, each test α defines a particular property a of the entity. We say that a property a is 'actual' for a given entity S if in the event of the test α the positive result would be certainly obtained. Other properties which are not actual are said to be potential. We suppose the set Q of all tests that can be performed on the entity to be known. A test $\alpha \in Q$ is said to be 'true' if the corresponding property a is actual, which means that if we would perform α the positive outcome would come out with certainty. A test α is said to be stronger than a test β (we denote $\alpha < \beta$), iff whenever α is true also β is true. A test α and a test β are said to be equivalent iff $\alpha < \beta$ and $\beta < \alpha$. We identify the equivalence classes containing α with its corresponding property a. The set $\mathcal{L}$ of all properties is a complete lattice with the inherited order relation. We can represent each property a by the set $\mu(a)$ of all states of the entity S that make this property actual, which defines a map $\mu : \mathcal{L} \to \mathcal{P}(\Sigma)$, which is called the Cartan map.

We can see that $eig_e(A)$ is the Cartan image $\mu(a_e^A)$ of the property a_e^A, defined by means of the test α_e^A, consisting of performing the experiment e and giving the positive answer if the outcome is an element of A. Since the Cartan map is an isomorphism on its image, the property lattice $\mathcal{L}$ of the entity S is isomorphic to the collection of eigenstate sets. The statement 'the property a_e^A is actual' is equivalent to the statement 'the state of S is in $eig_e(A)$' and to the statement 'the experiment e gives with certainty an outcome in A'.

In the quantum case the eigenstate sets are obviously represented by the closed subspaces of the Hilbert space. We have the following theorem:

Theorem 1 *Let us introduce the following notations: $\mathcal{P}(O_e)$ is the collection of all subsets of O_e, and $\mathcal{P}(\Sigma)$ is the collection of all subsets of Σ. Then the map*

$$eig_e : \mathcal{P}(O_e) \to \mathcal{P}(\Sigma) \tag{1}$$

satisfies the following properties:

$$eig_e(\emptyset) = \emptyset \tag{2}$$

$$eig_e(O_e) = \Sigma \tag{3}$$

$$eig_e(\cap_i A_i) = \cap_i eig_e(A_i) \tag{4}$$

Proof : $p \in eig_e(\cap_i A_i) \iff$ S being in state p, an execution of e gives with certainty an outcome in $\cap_i A_i \iff$ S being in state p, an execution of e gives with certainty an outcome in A_i for each i $\iff p \in eig_e(A_i)$ for each $i \iff p \in \cap_i eig_e(A_i)$.

We remark that $eig_e(A \cup B)$ is in general not equal to $eig_e(A) \cup eig_e(B)$ for $A, B \in \mathcal{P}(O_e)$. The existence of superposition states is the origin of the failure of this equality in the quantum case.

We want to introduce now a specific operation on the set of experiments. Let us put forward a general definition:

Definition 2 *Let us consider an arbitrary subset $E \subset \mathcal{E}$, and introduce the experiment, corresponding to this subset E, that we shall denote by e_E, in the following way: the experiment e_E consists of performing one of the experiments $f \in E$.*

The experiment e_E has an outcome set $O_E = \cup_{f \in E} O_f$. The enlarged set of experiments $\{e_E |\ E \subset \mathcal{E}\}$ shall be denoted by $cl(\mathcal{E})$, and this notation shall become clear in a moment. If we introduce $O_\mathcal{E} = \cup_{f \in \mathcal{E}} O_f$, then $O_\mathcal{E}$ is the outcome set of the experiment $e_\mathcal{E}$, and the outcome set of any experiment $e_E \in cl(\mathcal{E})$ is contained in $O_\mathcal{E}$. To distinguish we shall call the original experiments $f \in \mathcal{E}$ 'primitive' experiments, and the newly introduced experiments 'union' experiments.

What is interesting to note is the fact that a certain structure presents itself naturally in relation with the enlarged set of experiments; it is the structure of a 'closure'. Let us give some definitions:

Definition 3 *Consider a set X. We say that cl is a closure iff, for $K, L \subset X$, we have:*

$$K \subset cl(K) \tag{5}$$

$$K \subset L \Rightarrow cl(K) \subset cl(L) \tag{6}$$

$$cl(cl(K)) = cl(K) \tag{7}$$

$$cl(\emptyset) = \emptyset \tag{8}$$

A set X with a closure cl satisfying (5), (6), (7) and (8) shall be called a 'closure-structure'.

Definition 4 *Suppose that we have an entity S, with a set of states Σ and a set of experiments $\mathcal{E}$. If $E \subset \mathcal{E}$, we define by $cl(E)$ the collection of all 'union experiments' constructed by means of the experiments of E. Hence $cl(E) = \{e_D \mid D \subset E\}$.*

For some experiments $e, f, g \in \mathcal{E}$ we shall also denote the experiment $e_{\{e,f,g\}}$ by $e \cup f \cup g$.

Theorem 2 *We have $\cup_i e_{E_i} = e_{\cup_i E_i}$, and the operation cl defines a closure on the set $\mathcal{E}$.*

Proof: For an arbitrary subset $E \subset \mathcal{E}$ we have $E \subset cl(E)$. Suppose that we consider $E, D \subset \mathcal{E}$, such that $E \subset D$ then $cl(E) \subset cl(D)$. We have $cl(cl(E)) = cl(E)$ for an arbitrary E. Clearly also $cl(\emptyset) = \emptyset$.

Definition 5 *In the same way as we have introduced the eigenstate set corresponding to a set of outcomes for a primitive experiment (definition 1) we can introduce the eigenstate set corresponding to a set of outcomes of a general union experiment. Suppose that e_E is a union experiment and O_E its outcome set. For $A \subset O_E$ we define $eig_E(A)$ to be the set of all states, such that when S is in one of these states, the experiment e_E gives with certainty an outcome in A. We denote the collection of eigenstate sets $eig_e(\mathcal{P}(O_e))$, corresponding to a primitive experiment e, by $\mathcal{F}_e$, and the collection of eigenstate sets corresponding to an experiment E by $\mathcal{F}_E$.*

Theorem 3 *Suppose that e_E is an experiment, O_E its outcome set, and $A \subset O_E$, and $eig_E(A)$ the eigenstate-set corresponding to A. Then we can consider the sets $A \cap O_e$, and the corresponding eigenstate sets $eig_e(A \cap O_e)$. We have :*

$$eig_E(A) = \cap_{e \in E} eig_e(A \cap O_e) \tag{9}$$

Proof : $p \in eig_E(A) \iff S$ being in state p, the experiment e_E gives with certainty an outcome in A $\iff$ each primitive experiment $e \in E$, S being in state p, gives with certainty an outcome in $A \cap O_e$ $\iff p \in eig_e(A \cap O_e)$ for each $e \in E \iff p \in \cap_{e \in E} eig_e(A \cap O_e)$.

This theorem shows that each eigenstate set is the set theoretical intersection of eigenstate sets corresponding to primitive experiments.

The structure of the property lattice as constructed in [1,3,5,6] is that of a complete lattice. Since the collection of eigenstate sets is connected to the property lattice by the Cartan map, this complete lattice structure should be present in some way or another in the collection of eigenstate sets. By means of the introduction of the 'union' experiment, we can identify what is at the origin of the completeness of the property lattice. To do this we have to reflect about the relation between the outcome sets O_e and O_f of different experiments e and f.

Definition 6. *We call two experiments $e, f \in \mathcal{E}$ 'distinguishable' iff we can define O_e and O_f in such a way that $O_e \cap O_f = \emptyset$.*

Two experiments e and f are distinguishable if they can be distinguished from each other by means of their outcomes. Experiments that we 'consciously' perform on the entity S are always distinguishable, because we can 'name' the outcome corresponding to the given experiment (when a certain outcome occurs with an experiment e, even if physically equivalent with the outcome of another experiment, we can distinguish it if we know that we were performing the experiment e and not this other experiment). We want to introduce however this concept of 'distinguished experiments' explicitly, because we shall see in section 4 that when we consider the situation of 'hidden experiments' it is a valuable concept. Suppose that we consider a test α_e^A, consisting of performing the experiment e and giving the positive answer 'yes' if the outcome is in A, and a test α_f^B, consisting of performing the experiment f and giving a positive answer 'yes' if the outcome is in B. Piron has introduced in [3] the concept of 'product test', and if α_e^A tests whether the property a_e^A is actual and α_f^B tests whether the property a_f^B is actual, then $\alpha_e^A \cdot \alpha_f^B$ tests whether both properties a_e^A and a_f^B are actual. It is by requiring that the collection of tests of the entity S contains all the product questions, that the lattice of properties becomes a complete lattice. The product test is defined by means of the experiment $e \cup f$, and is given by $\alpha_{e \cup f}^{A \cup B}$, consisting of performing the experiment $e \cup f$ and giving a positive answer 'yes' if the outcome is in $A \cup B$. We remark that, although the product test can always be defined, it only tests whether the two properties a_e^A and a_f^B are actual, if e and f are distinguishable experiments. Indeed, suppose that e and f are not distinguishable, then $O_e \cap O_f \neq \emptyset$, which means that there is at least one outcome $x_e^i = x_f^j \in O_e \cap O_f$. Suppose that A does not contain this outcome while B does, then it is possible that the entity S is in a state p such that e has as possible outcomes the set $A \cup \{x_e^i\}$, which is a state where a_e^A is not actual, and where f has as possible outcomes B. Then $e \cup f$ has as possible outcomes $A \cup B$, which means that in this state p the test $\alpha_{e \cup f}^{A \cup B}$ gives with certainty a positive outcome. This shows that in this case of non distinguishable experiments, $\alpha_{e\cup f}^{A\cup B}$ does not test the actuality of both properties.

Theorem 4 *Let us denote by $\mathcal{F}$ the collection of eigenstate sets of the experiment $e_{\mathcal{E}}$. If all the experiments are distinguishable then $\mathcal{F}_E \subset \mathcal{F}$ for $E \subset \mathcal{E}$.*

Proof: Consider an arbitrary element $F \in \mathcal{F}_E$. Then there exists $A \subset O_E$ such that $F = eig_E(A)$. Consider $A' = A \cup (\cup_{e \in E^C} O_e)$, then we have $eig_{\mathcal{E}}(A') = eig_E(A)$, which shows that $F \in \mathcal{F}$.

Corollary 1 *When all the experiments are distinguishable then the collection of all eigenstate sets is given by $\mathcal{F}$.*

From now on we shall consider the experiments $e \in \mathcal{E}$ to be 'distinguishable' experiments, which corresponds to the situation where we consciously can choose between the different experiments, and in this way distinguish between the outcomes. Hence we have $O_e \cap O_f = \emptyset$ for $e, f \in \mathcal{E}$.

Theorem 5 *We have :*

$$\emptyset, \Sigma \in \mathcal{F} \tag{10}$$

$$F_i \in \mathcal{F} \Rightarrow \cap_i F_i \in \mathcal{F} \tag{11}$$

Proof : Suppose that $F_i \in \mathcal{F}\ \forall\ i$, then there exists $A_i \subset O_{\mathcal{E}}$ such that $F_i = eig_{\mathcal{E}}(A_i)\ \forall\ i$. We have $eig_{\mathcal{E}}(A_i) = \cap_{e \in \mathcal{E}} eig_e(A_i \cap O_e)$. From this follows that $\cap_i F_i = \cap_i \cap_{e \in \mathcal{E}} eig_e(A_i \cap O_e) = \cap_{e \in \mathcal{E}} \cap_i eig_e(A_i \cap O_e) = \cap_{e \in \mathcal{E}} eig_e(\cap_i A_i \cap O_e) = eig_{\mathcal{E}}(\cap_i A_i)$.

2.2 The eigen-closure structure.

In this section we shall introduce the closure structure that enables us to characterize the collection of eigenstate sets $\mathcal{F}$, and hence the property lattice $\mathcal{L}$, for an arbitrary entity S.

Theorem 6 *If X is a set equipped with a closure cl, and we define a subset $F \subset X$ to be closed iff $cl(F) = F$, then the family $\mathcal{F}$ of closed subsets of X satisfies :*

$$\emptyset \in \mathcal{F}, X \in \mathcal{F} \tag{12}$$

$$F_i \in \mathcal{F} \Rightarrow \cap_i F_i \in \mathcal{F} \tag{13}$$

This theorem shows that a closure structure generates a collection of closed subsets $\mathcal{F}$ that satisfy (12) and (13). Since these properties are satisfied in the collection of eigenstate sets corresponding to a general entity S (see theorem 5), we can wonder whether this collection of eigenstate sets can be characterized by a closure structure. The following theorem shows that this is indeed the case.

Theorem 7 *Suppose we have a set X and a family $\mathcal{F}$ of subsets of X that satisfy (12) and (13), and for an arbitrary subset $K \subset X$ we define $cl(K) = \cap_{K \subset F, F \in \mathcal{F}} F$, then cl is a closure and $\mathcal{F}$ is the set of closed subsets of X defined by this closure.*

Proof : Let $K, L \subset X$. Clearly $K \subset cl(K)$. If $K \subset L$ then $cl(K) \subset cl(L)$. If $F \in \mathcal{F}$ then $cl(F) = F$. Since $cl(K) \in \mathcal{F}$ we have $cl(cl(K)) = cl(K)$. This shows that cl is a closure. Consider a set K such that $cl(K) = K$, then $K = \cap_{K \subset F, F \in \mathcal{F}} F$, and hence $K \in \mathcal{F}$.

Theorem 7 and theorem 5 show that for an entity S, with a set of states Σ, the collection of eigenstate sets $\mathcal{F}$ is the collection of closed subsets corresponding to a closure structure, and theorem 7 shows us how to define this closure structure.

Definition 7 *Let us consider an entity S, with a set of states Σ, and a set of experiments $\mathcal{E}$. For an arbitrary set of states $K \subset \Sigma$ we define :*

$$cl_{eig}(K) = \cap_{K \subset F, F \in \mathcal{F}} F \tag{14}$$

where $\mathcal{F}$ is the collection of eigenstate sets, then cl_{eig} is the closure that defines the closure structure corresponding to this collection of eigenstate sets. We shall call it the eigen-closure structure of S.

To characterize more easily this eigen-closure structure cl_{eig} of S we introduce the following definition:

Definition 8 *Suppose we have a set X and $\mathcal{F}$ is the set of closed subsets corresponding to a closure cl on X. We say that the collection $\mathcal{B} \subset \mathcal{F}$ is a 'generating set' for $\mathcal{F}$ iff for each subset $F \in \mathcal{F}$ we have a family $B_i \in \mathcal{B}$ such that $F = \cap_i B_i$.*

Theorem 8 *Suppose we have a set X equipped with a closure cl and $\mathcal{B}$ is a generating set for the set of closed subsets $\mathcal{F}$. Then for an arbitrary subset $K \subset X$ we have :*

$$cl(K) = \cap_{K \subset B, B \in \mathcal{B}} B \tag{15}$$

Proof : We know that $cl(K) = \cap_{K \subset F, F \in \mathcal{F}} F$. Because $\mathcal{B}$ is a generating set for $\mathcal{F}$ we have $F = \cap_{F \subset B, B \in \mathcal{B}} B$. Hence $cl(K) = \cap_{K \subset F}(\cap_{F \subset B} B) = \cap_{K \subset B, B \in \mathcal{B}} B$.

Theorem 9 *The collection of eigenstate sets corresponding to primitive experiments $\cup_{e \in \mathcal{E}} \mathcal{F}_e$, is a generating set for the eigen-closure structure of the entity S.*

Theorem 6 also shows that a closure structure is a generalization of an ordinary topology. From this we can understand why the concepts of closure that are well known in physics, e.g. the topological closure of subsets of a phase-space in classical mechanics, and the linear closure of subsets of a vector space in quantum mechanics, are both examples of closures.

We mentioned already that the property lattice $\mathcal{L}$ of the entity S (as introduced in [1,3,5,6]) is isomorphic to the closure structure $\mathcal{F}$ of eigenstate sets of the entity S. The isomorphism is the Cartan map $\mu : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$, such that $a_e^A < a_f^B \Leftrightarrow eig_e(A) \subset eig_f(B)$, and $\mu(\wedge_i a_{e_i}^{A_i}) = \cap_i eig_{e_i}(A_i)$, and $\mu(\vee_i a_{e_i}^{A_i}) = \cap_{\cup_i eig_{e_i}(A_i) \subset F, F \in \mathcal{F}} F = cl_{eig}(\cup_i eig_{e_i}(A_i))$.

2.3 State-determination and atomicity: the $\mathbf{T}_0$ and $\mathbf{T}_1$ topological separation axioms.

Suppose that the entity is in a state p. We can then consider the collection of all the eigenstate sets that contain the state p, and we know then that also the intersection of all these eigenstate sets is an eigenstate set, and that it is the eigen-closure of the singleton $\{p\}$. We'll denote this set by $s(p)$.

$$s(p) = cl_{eig}(\{p\}) = \cap_{F \in \mathcal{F}, p \in F} F \tag{16}$$

Suppose that $A \subset O_{\mathcal{E}}$ such that $eig_{\mathcal{E}}(A) = s(p)$. We have $A = \cup_{e \in \mathcal{E}}(A \cap O_e)$, and since $p \in s(p)$, we know that if the entity is in the state p, an experiment $e \in \mathcal{E}$ shall give with certainty an outcome in the set $A \cap O_e$. The set:

$$\{A \cap O_e \mid e \in \mathcal{E}\} \tag{17}$$

is the set of 'certain outcomes' for an entity S in state p, and $\{a_e^{A \cap O_e} \mid e \in \mathcal{E}\}$ is the set of 'actual' properties of the entity S. We know that we express the statement 'if S is in state p, then the experiment e gives with certainty an outcome in A' by means of '$p \in eig_e(A)$' or by means of the statement 'the property a_e^A is actual'. We can prove the following theorem:

Theorem 10 *We have:*

$$p \in eig_E(A) \quad \Leftrightarrow \quad s(p) \subset eig_E(A) \quad \Leftrightarrow a_E^A \text{ is actual} \tag{18}$$

The foregoing theorem is in fact the reason that we can decide to represent the states of the entity by the elements $s(p)$, which is what is done in ref [1,3,5,6], where a state is identified with the set of all actual properties, and represented by the intersection of all these properties, which is the inverse Cartan image of $s(p)$. We come now to the point where we can find an unexpected connection between the axioms of our physical theory, and some well known axioms of topology. Let us repeat some definitions:

Definition 9 Suppose that we have a set X with a closure cl and a collection of closed subsets $\mathcal{F}$. We say that the closure structure defined in this way has the T_0-separation property iff for $x, y \in X$ we have $x \neq y$ implies $cl(x) \neq cl(y)$. We say that the closure structure has the T_1 separation property iff for $x \in X$ we have $\{x\} \in \mathcal{F}$.

That the state of an entity is determined by the collection of actual properties is equivalent with the fact that the eigen-closure structure has the T_0 separation property. Let us express this in the following axiom:

Axiom of state determination ($\mathbf{T}_0$). *We say that for an entity S, with set of states Σ and set of experiments $\mathcal{E}$, the axiom of 'state determination' is satisfied iff $s(p)$ determines p, which means that for $p \neq q$ we have $s(p) \neq s(q)$.*

Theorem 11 *For an entity S, the axiom of state determination is satisfied iff the eigen-closure structure satisfies the T_0 separation property.*

Let us think again of the quantum case where $\mathcal{F}(\Sigma_{\mathcal{H}})$ is the collection of eigenstate-sets. If we consider a state $\bar{\psi}$ of the quantum entity we can wonder what is $s(\bar{\psi}) = cl_{eig}(\{\bar{\psi}\})$.

Theorem 12 *Suppose that we have a quantum entity and $\bar{\psi}_p \in \Sigma_{\mathcal{H}}$, is an arbitrary state of this quantum entity, then we have :*

$$s(\bar{\psi}) = cl_{eig}(\{\bar{\psi}_p\}) = \{\bar{\psi}_p\} \tag{19}$$

Proof : Since $cl_{eig}(\{\bar{\psi}_p\}) \in \mathcal{F}(\Sigma_{\mathcal{H}})$ there is an orthogonal projection $Proj$ such that $Proj(\psi_q) = \psi_q$ iff $\bar{\psi}_q \in cl_{eig}(\{\bar{\psi}_p\})$. From (16) follows that this orthogonal projector is the smallest one with this property, and this proves that $Proj = Proj_{\bar{\psi}_p}$, where $Proj_{\bar{\psi}_p}$ is the orthogonal projector on the ray $\bar{\psi}_p$. From this follows that $cl_{eig}(\{\bar{\psi}_p\}) = \{\bar{\psi}_p\}$.

The foregoing theorem shows that for a quantum entity, the axiom of state determination is satisfied. But more, we see that $cl_{eig}(\{p\}) = \{p\}$. This extra property, which leads to atomicity of the property lattice, is equivalent to the T_1 separation property of topology.

Axiom of atomicity ($\mathbf{T}_1$). *We say that for an entity S, with set of states Σ and set of experiments $\mathcal{E}$, the axiom of atomicity is satisfied iff for each $p \in \Sigma$ we have $cl_{eig}(\{p\}) = \{p\}$.*

Theorem 13 *For an entity S, the axiom of atomicity is satisfied iff the eigen-closure structure satisfies the T_1 separation property.*

2.4 The ortho-closure structure.

Entities that satisfy the 'axiom of state determination', and hence have a property lattice corresponding to a closure structure satisfying the T_0 separation property, and the 'axiom of atomicity', and hence having a property lattice corresponding to a closure structure satisfying the T_1 separation property, satisfy already some of the axioms needed for the representation theorem of Piron (ref [3,7]). They have a property lattice full of atoms. The next axioms should lead to an orthocomplementation on the property lattice. By means of the closure structure we can approach the problem of the orthocomplementation in a very general way. The reason is that the orthogonality relation that is introduced in ref [1,5] gives rise to another closure structure. For an entity described by quantum mechanics the two closure structures are identical. Two states $\bar{\psi}_p$ and $\bar{\psi}_q$ of a quantum entity are orthogonal iff $< \psi_p | \psi_q >= 0$. This defines an orthogonality relation on the set of states $\Sigma_{\mathcal{H}}$ of the quantum entity. The collection of eigenstate sets $\mathcal{F}(\Sigma_{\mathcal{H}})$ corresponding to the quantum entity is a closure structure, but also the orthogonality relation makes it possible to identify the same closure structure. Indeed, if we consider an arbitrary set K of states of the quantum entity, and we define $K^\perp$ to be the set of states orthogonal to each state of K, then $K^\perp = F_{proj}$ for some orthogonal projector of $\mathcal{H}$. Moreover for each $F_{proj} \in \mathcal{F}(\Sigma_{\mathcal{H}})$ we can find a set $K \subset \Sigma$, such that $K^\perp = F_{proj}$.

As we have done for the eigenstate sets of a general physical entity, we now want to investigate in which way we can construct for a general physical entity a closure structure by means of an orthogonality relation. Let us first introduce the necessary definitions.

Definition 10 *For two states p and q of an entity S we say that p is orthogonal to q, and we denote $p \perp q$, iff there exists an experiment $e \in \mathcal{E}$ and subsets $A, B \subset O_e$, such that $A \cap B = \emptyset$ and $p \in eig_e(A)$ and $q \in eig_e(B)$. For two sets $K, L \subset \Sigma$ we say that $K \perp L$ iff for each $p \in K$ and $q \in L$ we have $p \perp q$.*

Theorem 14 *For two states p and q of an entity S we have:*

$$p \perp q \quad \Leftrightarrow \quad s(p) \perp s(q) \tag{19}$$

Clearly this orthogonality relation satisfies the following properties. For $p, q \in \Sigma$ we have:

$$p \not\perp p \tag{20}$$

$$p \perp q \Rightarrow q \perp p \tag{21}$$

which means that it is an antireflexive and symmetric relation. Let us put forward the well known procedure to construct a closure structure with this relation.

Theorem 15 *Consider the set of states Σ equipped with the orthogonality relation $\perp$, and let us define for $K \subset \Sigma$ the set $K^\perp = \{p |\ p \perp q, q \in K\}$, and $cl_{orth}(K) = (K^\perp)^\perp$, then cl_{orth} is a closure.*

Proof: see [8].

Theorem 16 *Let us denote the collection of all ortho-closed subsets by $\mathcal{F}_{orth}$, then it can easily be shown that this closure structure is orthocomplemented, which means that the map $^\perp : \mathcal{F}_{orth} \rightarrow \mathcal{F}_{orth}$ satisfies:*

$$K \subset L \Rightarrow L^\perp \subset K^\perp \tag{22}$$

$$K^{\perp\perp} = K \tag{23}$$

$$K \cap K^\perp = \emptyset \tag{24}$$

Corollary 2 *The following formulas are satisfied in $\mathcal{F}_{orth}$, for $F_i \in \mathcal{F}_{orth}$:*

$$(\cap_i F_i)^\perp = cl_{orth}(\cup_i F_i^\perp) \tag{25}$$

$$(\cup_i F_i)^\perp = \cap_i F_i^\perp \tag{26}$$

$$cl_{orth}(F \cup F^\perp) = \Sigma \tag{27}$$

Proof: (1) $cl_{orth}(\cup_i F_i^\perp) \subset F \Leftrightarrow \cup_i F_i^\perp \subset F \Leftrightarrow F_i^\perp \subset F\ \forall\ i \Leftrightarrow F^\perp \subset F_i \quad \forall\ i \Leftrightarrow F^\perp \subset \cap_i F_i \Leftrightarrow (\cap_i F_i)^\perp \subset F$. From this follows that $cl_{orth}(\cup_i F_i^\perp) = (\cap_i F_i)^\perp$. (2) $F \subset \cap_i F_i^\perp \Leftrightarrow F_i \subset F^\perp\ \forall\ i \Leftrightarrow \cup_i F_i \subset F^\perp \Leftrightarrow cl_{orth}(\cup_i F_i) \subset F^\perp \Leftrightarrow F \subset (\cup_i F_i)^\perp$. From this follows that $(\cup_i F_i)^\perp = \cap_i F_i^\perp$. (3) $cl_{orth}(F \cup F^\perp) = (F^\perp \cap F)^\perp = \emptyset^\perp = \Sigma$.

The ortho-closure has a simple generating set of elements.

Theorem 17 *The set* $\mathcal{B} = \{\{p\}^\perp | \ p \in \Sigma\}$ *is a generating set for the set of ortho-closed subsets* $\mathcal{F}_{orth}$.

Proof: Consider an arbitrary element $F \in \mathcal{F}_{orth}$. We have $F^\perp = \cup_{p \in F^\perp}\{p\}$, and hence $F = F^{\perp\perp} = \cap_{p \in F^\perp}\{p\}^\perp$.

We are now in a position to formulate the axioms that make the property lattice into an orthocomplemented lattice. In general, as we show explicitly by means of examples in [9], the ortho-closure structure is different from the eigen-closure structure. They coincide for the quantum and the classical case, and we want to formulate now some axioms that make them coincide.

Axiom ortho 1. *If* S *is an entity,* Σ *its set of states and* $\mathcal{E}$ *its set of experiments, we say that axiom ortho 1 is satisfied iff for* $e \in \mathcal{E}$ *and* $A \subset O_e$ *we have that* $p \perp eig_e(A)$ *implies* $p \in eig_e(A^C)$.

Theorem 18 *If axiom ortho 1 is satisfied for an entity* S*, then* $\mathcal{F}_{eig} \subset \mathcal{F}_{orth}$.

Proof: Suppose that axiom ortho 1 is satisfied, and consider $eig_e(A)$. We have $eig_e(A)^\perp = eig_e(A^C)$, and as a consequence $eig_e(A)^{\perp\perp} = eig_e(A^C)^\perp = eig_e(A)$. This shows that $eig_e(A) \in \mathcal{F}_{orth}$. Because $\{eig_e(A) \mid e \in \mathcal{E}, A \subset O_e\}$ is a generating set for $\mathcal{F}_{eig}$, we have $\mathcal{F}_{eig} \subset \mathcal{F}_{orth}$.

Axiom ortho 2. *If* S *is an entity,* Σ *its set of states and* $\mathcal{E}$ *its set of experiments, we say that 'axiom ortho 2' is satisfied iff for each state* $p \in \Sigma$ *we have that* $\{p\}^\perp \in \mathcal{F}_{eig}$.

Theorem 19 *If 'axiom ortho 2' is satisfied for an entity* S*, then* $\mathcal{F}_{orth} \subset \mathcal{F}_{eig}$.

Proof: Since $\{p\}^\perp \in \mathcal{F}_{eig}$, and because $\{\{p\}^\perp \mid p \in \Sigma\}$ is a generating set for $\mathcal{F}_{orth}$ we have $\mathcal{F}_{orth} \subset \mathcal{F}_{eig}$.

Corollary 3 *If axiom ortho 1 and axiom ortho 2 are satisfied, the two closure structures,* cl_{eig} *and* cl_{orth} *are equal, and* $\mathcal{F}_{eig} = \mathcal{F}_{orth}$ *is orthocomplemented.*

An entity satisfying the axiom of state determination and atomicity and the axioms ortho 1 and ortho 2, has a complete orthocomplemented atomic property lattice. We analyze in ref [9] the way in which we have introduced an orthocomplementation on the property lattice in a more detailed way.

2.5 Classical entities.

We want to put forward a characterization of classical entities in this approach, and therefore introduce the following definition:

Definition 11 *We say that an experiment* $e \in \mathcal{E}$ *is a 'classical' experiment iff for every state* $p \in \Sigma$ *the experiment* e *gives with certainty an outcome* $x_p \in O_e$.

Let us remark that if e and f are classical experiments of the entity S, then $e \cup f$ is in general not a classical experiment.

Definition 12 *We say that an entity* S *is a classical entity iff each experiment* $e \in \mathcal{E}$ *is a classical experiment.*

Theorem 20 *Suppose that the axiom of state determination and axiom ortho 1 is satisfied, then we have:* S *is a classical entity iff two different states of* S *are always orthogonal.*

Proof: Let us suppose that S is a classical entity and consider two states $p, q \in \Sigma$ such that $p \neq q$. Since the axiom of state determination is satisfied we have $s(p) \neq s(q)$. This means that $s(p) \not\subset s(q)$ or $s(q) \not\subset s(p)$. Let us suppose that $s(p) \not\subset s(q)$. Consider $A \subset O_\mathcal{E}$ such that $s(q) = eig_\mathcal{E}(A)$. This means that when the entity S is in state q, each experiment $e \in \mathcal{E}$ gives an outcome in A. Let us suppose that the entity S is in state p. Then there is at least one experiment $f \in \mathcal{E}$ that gives an

outcome in A^C, because if this would not be the case we would have $s(p) \subset s(q)$. Since f is a classical experiment, we have $p \in eig_f(A^C)$. We also have $q \in eig_f(A)$, which shows that $p \perp q$.
Let us now prove the inverse implication. Suppose that S is an entity such that all different states are orthogonal. We have to show that S is a classical entity. Consider $e \in \mathcal{E}$ and $A \subset O_e$. If $p \notin eig_e(A)$ we have that $p \neq q$ for all $q \in eig_e(A)$. This implies that $p \perp q$ for all $q \in eig_e(A)$. Since axiom ortho 1 is satisfied this implies $p \in eig_e(A^C)$. So we have shown that for an arbitrary $e \in \mathcal{E}$ and an arbitrary $p \in \Sigma$ we have $p \in eig_e(A)$ or $p \in eig_e(A^C)$. Let us show that this implies that every experiment $e \in \mathcal{E}$ is a classical experiment. Consider $e \in \mathcal{E}$, and suppose that x_e^i is a possible outcome of e when S is in state p. Then, since p is not contained in $eig_e(\{x_e^i\}^C)$, we have $p \in eig_e(\{x_e^i\})$.

2.6 Weak modularity and the covering law.

We know that the lattice of closed subspaces satisfies two more properties, it is weakly modular, and satisfies the covering law. These are the two properties that are necessary for Piron's representation theorem (ref [3,7]).

Axiom of weak modularity. *If S is an entity with set of states Σ and set of experiments $\mathcal{E}$, and $F, G \in \mathcal{F}_{eig}$ such that $F \subset G$, then it is possible to find a $H \in \mathcal{F}_{eig}$ such that $H \perp F$ and $cl_{eig}(F \cup H) = G$.*

Axiom of the covering law. *If S is an entity with set of states Σ and set of experiments $\mathcal{E}$, and $F \in \mathcal{F}_{eig}$, and $p \in \Sigma$ such that $p \notin F$, then $cl_{eig}(F \cup \{p\})$ covers F, which means that if $G \in \mathcal{F}_{eig}$ such that $F \subset G \subset cl_{eig}(F \cup \{p\})$, we have $F = G$ or $G = cl_{eig}(F \cup \{p\})$.*

The axioms that we have formulated (a. state determination, b. atomicity, c. ortho 1 and ortho 2, d. weak modularity, e. covering law) are satisfied in quantum mechanics and also in classical mechanics. By a celebrated representation theorem of Piron [3,7], also the inverse is in a certain sense true. These five axioms force our formalism to be equivalent to quantum mechanics with superselection rules. If there are no superselection rules we get ordinary quantum mechanics in one Hilbert space, and if all states are separated by a superselection rule we get classical mechanics. This situation has been analyzed in detail and we refer the reader to ref [1,3,5,6,7].

3 Separated entities.

Let us put forward the situation that we want to describe. We suppose that we have an entity S consisting of two entities S_1 and S_2. The entity S has a set of states Σ, a set of distinguishable experiments $\mathcal{E}$, and a set of probabilities $\{P(e = x_e^k \mid p) \mid e \in \mathcal{E}, x_e^k \in O_e, p \in \Sigma\}$, and the entities S_1 and S_2 have respectively sets of states Σ_1 and Σ_2, sets of distinguishable experiments $\mathcal{E}_1$ and $\mathcal{E}_2$, and sets of probabilities $\{P(e_1 = x_{e_1}^i \mid p_1) \mid e_1 \in \mathcal{E}_1, x_{e_1}^i \in O_{e_1}, p_1 \in \Sigma_1\}$, and $\{P(e_2 = x_{e_2}^j \mid p_2) \mid e_2 \in \mathcal{E}_2, x_{e_2}^j \in O_{e_2}, p_2 \in \Sigma_2\}$. The outcomes of experiments $e \in \mathcal{E}$, $e_1 \in \mathcal{E}_1$ and $e_2 \in \mathcal{E}_2$ are respectively denoted by $O_e = \{x_e^1, x_e^2, ..., x_e^k, ..., x_e^r\}$, $O_{e_1} = \{x_{e_1}^1, x_{e_1}^2, ..., x_{e_1}^i, ..., x_{e_1}^n\}$ and $O_{e_2} = \{x_{e_2}^1, x_{e_2}^2, ..., x_{e_2}^j, ..., x_{e_2}^m\}$. We also introduce respectively for $E \subset \mathcal{E}$, $E_1 \subset \mathcal{E}_1$ and $E_2 \subset \mathcal{E}_2$ the experiments e_E, e_{E_1} and e_{E_2}, as explained in section 2, with outcome sets $O_E = \cup_{e \in E} O_e$, $O_{E_1} = \cup_{e_1 \in E_1} O_{e_1}$ and $O_{E_2} = \cup_{e_2 \in E_2} O_{e_2}$.
We want to express now that S_1 and S_2 are 'separated' entities, and that S 'just' consists of these two entities S_1 and S_2. We do this by requiring that the collection of experiments $\mathcal{E}$ of the entity S consists of experiments $e_1 \times e_2$, where e_1 is an experiment of S_1 and e_2 is an experiment of S_2. The experiment $e_1 \times e_2$ is the experiment that consists of performing the experiments e_1 and e_2 together, so it has an outcome set $O_{e_1} \times O_{e_2}$. We have to explain now what we mean by 'separated' entities. Suppose that we consider an experiment e_1 of the entity S_1. Then we can perform this experiment alone, without doing anything on entity S_2, or we can perform the experiment together with an experiment e_2 on S_2, and then in fact we are performing the experiment $e_1 \times e_2$. Suppose that we consider a well defined situation of reality, hence a state of the entity S, then it should be so that the outcomes that we can obtain for the experiment e_1, the entity S being in this state, do not depend on what we do on entity S_2 (whether we perform the experiment e_2 or not, or whether we perform any arbitrary other

experiment f_2 on S_2). This is the idea of 'separation' that we want to put forward: the results of an experiment on one of the entities is not influenced by what is done on the other entity.

Let us remark that this is the same idea of separation that was put forward by Einstein, Podolsky and Rosen, and let us also remark that it is a very weak requirement. When two entities S_1 and S_2 are separated (in the sense that we have introduced), this does not mean that there are no interactions between S_1 and S_2 (this is clearly not well understood, because often the concept of 'separation' is confused with 'no interaction'). If we think of examples of two entities in macroscopic reality, then, although interaction exists between them, they are in general separated. Two material bodies B_1 and B_2 that move in space under influence of the gravitation force between them constitute a compound entity B consisting of 'separated' entities B_1 and B_2. Indeed, the outcomes of an experiment e_1 performed on B_1 do not depend on what kind of experiment e_2 is performed on B_2. This shows that the vast majority of 'two entity' situations considered in ordinary classical physics are situations of 'separated' entities. Let us now introduce these notions in a formal way.

Definition 13 *We say that two experiments e_1 and e_2, with outcome sets O_{e_1} and O_{e_2}, that can be performed together, and this experiment is denoted $e_1 \times e_2$, and has outcome set $O_{e_1} \times O_{e_2}$, are 'separated' experiments iff:*

$(x^i_{e_1}, x^j_{e_2})$ is a possible outcome of $e_1 \times e_2 \quad \Leftrightarrow \quad x^i_{e_1}$ is a possible outcome of e_1 and $x^j_{e_2}$ is a possible outcome of e_2

The entity S, with set of states Σ and set of experiments $\mathcal{E}$, is said to consist of the two separated entities S_1 and S_2, with sets of states Σ_1 and Σ_2 and sets of experiments $\mathcal{E}_1$ and $\mathcal{E}_2$, iff:

e_1 and e_2 are separated experiments for all $e_1 \in \mathcal{E}_1$ and $e_2 \in \mathcal{E}_2$ and $\mathcal{E} = \{e_1 \times e_2 \mid e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2\}$

We remark that this definition of the separation of experiments is very weak. In fact we could also express this idea of separation by means of the probabilities by requiring that:

$P(e_1 \times e_2 = (x^i_{e_1}, x^j_{e_2}) \mid p) = P(e_1 = x^i_{e_1} \mid p) \cdot P(e_2 = x^j_{e_2} \mid p)$

This requirement on the probabilities implies that the experiments are separated in the weak sense, as in definition 13. We want to work with definition 13 and show that this is sufficient to derive the closure structure of the entity S. When we study the probability structure of the separated entities, we shall use the requirement on the probabilities.

3.1 The eigen-closure structure.

We can now proceed by constructing the eigen-closure structure (and hence the property lattice) of the entity S consisting of the two separated entities S_1 and S_2. We'll denote by $\mathcal{F}$ the collection of all eigen-closed subsets of Σ corresponding to the experiments $\mathcal{E}$, and by $\mathcal{F}_1$ and $\mathcal{F}_2$ the collection of eigen-closed subsets of respectively Σ_1 and Σ_2 corresponding to the experiments $\mathcal{E}_1$ and $\mathcal{E}_2$. We know that $\mathcal{F}$ represents the property lattice of S, and $\mathcal{F}_1$ and $\mathcal{F}_2$ represent the property lattices of S_1 and S_2.

Let us consider $E_1 \subset \mathcal{E}_1$ and $E_2 \subset \mathcal{E}_2$, and the experiment $e_{E_1} \times e_{E_2}$ as the experiment that consists of performing both experiments e_{E_1} and e_{E_2} together. Clearly if each $e_1 \in E_1$ is separated from $e_2 \in E_2$, also e_{E_1} is separated from e_{E_2}, and moreover we have $e_{E_1 \times E_2} = e_{E_1} \times e_{E_2}$.

Theorem 21 *We have the following equivalences:*
S is in a state of $eig_{E_1 \times E_2}(A_1 \times A_2)$
$\Leftrightarrow$ $e_{E_1 \times E_2}$ gives with certainty a outcome in $A_1 \times A_2$
$\Leftrightarrow$ e_{E_1} gives with certainty an outcome in A_1 and e_{E_2} gives with certainty an outcome in A_2
$\Leftrightarrow$ S_1 is in a state of $eig_{E_1}(A_1)$ and S_2 is in a state of $eig_{E_2}(A_2)$.

Proof: A direct consequence of the definition of separated experiments.

It is interesting to remark that we can find back the elements of the closure structures $\mathcal{F}_1$ and $\mathcal{F}_2$ of S_1 and S_2 as subsets of Σ. If we consider again the foregoing theorem we can proof the following theorem:

Theorem 22 *For $F_1 \in \mathcal{F}_1$, such that $F_1 = eig_{\mathcal{E}_1}(A_1)$, and $F_2 \in \mathcal{F}_2$, such that $F_2 = eig_{\mathcal{E}_2}(A_2)$, we define a map $\nu_1 : \mathcal{F}_1 \to \mathcal{F}$ such that $\nu_1(F_1) = eig_{\mathcal{E}}(A_1 \times O_{\mathcal{E}_2})$, and a map $\nu_2 : \mathcal{F}_2 \to \mathcal{F}$ such that $\nu_2(F_2) = eig_{\mathcal{E}}(O_{\mathcal{E}_1} \times A_2)$. The maps ν_1 and ν_2 are injective morphisms between the closure structures, and we have for $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$:*

$$S \text{ is in a state } p \in \nu_1(F_1) \Leftrightarrow S_1 \text{ is in a state } p_1 \in F_1 \tag{28}$$

$$S \text{ is in a state } p \in \nu_2(F_2) \Leftrightarrow S_2 \text{ is in a state } p_2 \in F_2 \tag{29}$$

Proof: Consider $F_1^i \in \mathcal{F}_1$ such that $F_1^i = eig_{\mathcal{E}_1}(A_1^i)$, then $\cap_i eig_{\mathcal{E}_1}(A_1^i) = eig_{\mathcal{E}_1}(\cap_i A_1^i)$. We have $\nu_1(\cap_i F_1^i) = \nu_1(eig_{\mathcal{E}_1}(\cap_i A_1^i)) = eig_{\mathcal{E}}(\cap_i A_1^i \times O_{\mathcal{E}_2}) = eig_{\mathcal{E}}(\cap_i (A_1^i \times O_{\mathcal{E}_2})) = \cap_i eig_{\mathcal{E}}(A_1^i \times O_{\mathcal{E}_2}) = \cap_i \nu_1(F_1^i)$. We have $\nu_1(\Sigma_1) = \nu_1(eig_{\mathcal{E}_1}(O_{\mathcal{E}_1})) = eig_{\mathcal{E}}(O_{\mathcal{E}_1} \times O_{\mathcal{E}_2}) = \Sigma$, and $\nu_1(\emptyset) = \nu_1(eig_{\mathcal{E}_1}(\emptyset)) = eig_{\mathcal{E}}(\emptyset \times O_{\mathcal{E}_2}) = \emptyset$. Suppose that $\nu_1(F_1) = \nu_1(G_1)$, and hence $eig_{\mathcal{E}}(A_1 \times O_{\mathcal{E}_2}) = eig_{\mathcal{E}}(B_1 \times O_{\mathcal{E}_2})$. From this follows that $eig_{\mathcal{E}_1}(A_1) = eig_{\mathcal{E}_1}(B_1)$ and hence $F_1 = G_1$, which shows that ν_1 is an injective map. In an analogous way we show that ν_2 is an injective morphism. We consider $F_1 \in \mathcal{F}_1$, such that $F_1 = eig_{\mathcal{E}_1}(A_1)$, and suppose that $p \in \nu_1(F_1)$, which means that $p \in eig_{\mathcal{E}}(A_1 \times O_{\mathcal{E}_2})$. Then $p \in eig_{\mathcal{E}_1 \times \mathcal{E}_2}(A_1 \times O_{\mathcal{E}_2})$ which means that if S is in p the experiment $e_{\mathcal{E}_1}$ gives with certainty an outcome in A_1. If S_1 is in state p_1, then we can conclude that $p_1 \in eig_{\mathcal{E}_1}(A_1) = F_1$.

Corollary 4 S is in a state of $\nu_1(F_1) \cap \nu_2(F_2) \Leftrightarrow S_1$ is in a state of F_1 and S_2 is in a state of F_2.

We now want to investigate the structure of $\mathcal{F}$ and try to characterize it in function of the closure structures $\mathcal{F}_1$ and $\mathcal{F}_2$.

Theorem 23 *Suppose that $\{(x_{E_1}^{i_1}, x_{E_2}^{j_1}), (x_{E_1}^{i_2}, x_{E_2}^{j_2}), ..., (x_{E_1}^{i_k}, x_{E_2}^{j_k}), ...\}$ is a collection of possible outcomes, for the experiment $e_{E_1} \times e_{E_2}$, with E_1 and E_2 separated, the entity S being in a state p, then all the outcomes of the set $A_1 \times A_2$ where:*

$$A_1 = \{x_{E_1}^i \mid i \in \{i_1, i_2, ..., i_k, ...\}\} \tag{30}$$

$$A_2 = \{x_{E_2}^j \mid j \in \{j_1, j_2, ..., j_k, ...\}\} \tag{31}$$

are possible outcomes of the experiment $e_{E_1} \times e_{E_2}$.

Proof: If the elements of $\{(x_{E_1}^{i_1}, x_{E_2}^{j_1}), (x_{E_1}^{i_2}, x_{E_2}^{j_2}), ..., (x_{E_1}^{i_k}, x_{E_2}^{j_k}), ...\}$ are possible outcomes of the experiment $e_{E_1} \times e_{E_2}$, then, since e_{E_1} and e_{E_2} are separated experiments, we know that $\{x_{E_1}^i \mid i \in \{i_1, i_2, ..., i_k, ...\}\} = A_1$ are possible outcomes of the experiment e_{E_1} and $\{x_{E_2}^j \mid j \in \{j_1, j_2, ..., j_k, ...\}\} = A_2$ are possible outcomes of the experiment e_{E_2}. Then follows, again because e_{E_1} and e_{E_2} are separated experiments, that all elements of $A_1 \times A_2$ are possible outcomes of $e_{E_1} \times e_{E_2}$.

By means of this theorem we can characterize a general property of the entity S in function of properties of S_1 and S_2. Some elements of $\mathcal{F}$ are easy to characterize in function of elements of $\mathcal{F}_1$ and $\mathcal{F}_2$, as shows the next theorem.

Theorem 24 *If $A_1 \subset O_{E_1}$ and $A_2 \subset O_{E_2}$, then:*

$$eig_E(A_1 \times A_2) = \nu_1(eig_{E_1}(A_1)) \cap \nu_2(eig_{E_2}(A_2)) \tag{32}$$

where $E = E_1 \times E_2$.

Proof: Consider $p \in eig_E(A_1 \times A_2)$. Suppose that the entity S is in state p, and we perform the experiment e_E, then the outcome is with certainty an element of $A_1 \times A_2$. From theorem 21 we know that $e_E = e_{E_1} \times e_{E_2}$, and that e_{E_1} and e_{E_2} are separated. This means that if we perform the experiment e_{E_1}, the entity being in the same state p, the outcome is with certainty in A_1, and if we perform the experiment e_{E_2}, the entity being in state p, the outcome is with certainty in A_2. This means that $p \in \nu_1(eig_{E_1}(A_1)) \cap \nu_2(eig_{E_2}(A_2))$. So we have shown that $eig_E(A_1 \times A_2) \subset \nu_1(eig_{E_1}(A_1)) \cap \nu_2(eig_{E_2}(A_2))$. The other inclusion is proven in an analogous way, again making use of the fact that e_{E_1} and e_{E_2} are separated experiments.

This theorem allows us to characterize easily the elements of $\mathcal{F}$ that are connected with product sets $A_1 \times A_2 \subset O_\mathcal{E}$ of outcomes. This is however but a small number of all the elements of $\mathcal{F}$. We shall try now to find a characterization of a general element of $\mathcal{F}$. This is certainly not straightforward, but making use very explicitly of the separation of the two entities we can proof the following theorem:

Theorem 25 *Consider a general element $F \in \mathcal{F}$, then we know that this is equal to $eig_\mathcal{E}(A)$ for some $A \subset O_\mathcal{E}$. Let $\{A_1^i \times A_2^i \mid i \in I\}$ be the collection of product sets, such that each $A_1^i \times A_2^i \subset A$, and $A_1^i \subset O_{\mathcal{E}_1}$ and $A_2^i \subset O_{\mathcal{E}_2}$, hence $A = \cup_i(A_1^i \times A_2^i)$. We then have:*

$$eig_\mathcal{E}(A) = \cup_i eig_\mathcal{E}(A_1^i \times A_2^i) \tag{33}$$

Proof: Clearly $\cup_i eig_\mathcal{E}(A_1^i \times A_2^i) \subset eig_\mathcal{E}(A)$. Consider $p \in eig_\mathcal{E}(A)$. Suppose that the entity S is in state p, and let $\{(x_{E_1}^{i_1}, x_{E_2}^{j_1}), (x_{E_1}^{i_2}, x_{E_2}^{j_2}), ..., (x_{E_1}^{i_k}, x_{E_2}^{j_k}), ...\}$ be the set of possible outcomes for the experiment $e_\mathcal{E}$. Then obviously $\{(x_{E_1}^{i_1}, x_{E_2}^{j_1}), (x_{E_1}^{i_2}, x_{E_2}^{j_2}), ..., (x_{E_1}^{i_k}, x_{E_2}^{j_k}), ...\} \subset A$. From theorem 23 follows that all outcomes of the set $A_1 \times A_2$ are possible outcomes if S is in state p, where $A_1 = \{x_{E_1}^i \mid i \in \{i_1, i_2, ..., i_k, ...\}\}$ and $A_2 = \{x_{E_2}^j \mid j \in \{j_1, j_2, ..., j_k, ...\}\}$. From this follows that $p \in eig_\mathcal{E}(A_1 \times A_2)$. Since all elements of $A_1 \times A_2$ are possible outcomes, we must have $A_1 \times A_2 \subset A$, because if this would not be the case, there would be an outcome outside A possible, which would mean that $p \notin eig_\mathcal{E}(A)$. From this follows that $p \in \cup_i(A_1^i \times A_2^i)$. Since we made this reasoning for an arbitrary $p \in eig_\mathcal{E}(A)$ it follows that $eig_\mathcal{E}(A) \subset \cup_i eig_\mathcal{E}(A_1^i \times A_2^i)$.

Corollary 5 *From theorem 24 and theorem 25 follows that an arbitrary element $F \in \mathcal{F}$ can always be written as follows:*

$$F = \cup_i(\nu_1(F_1^i) \cap \nu_2(F_2^i)) \tag{34}$$

where $F_1^i \in \mathcal{F}_1$ and $F_2^i \in \mathcal{F}_2$.

We can calculate some examples for some specific forms of A.

Theorem 26 *For $A_1, B_1 \in O_{\mathcal{E}_1}, A_2, B_2 \in O_{\mathcal{E}_2}$ we have:*

$$eig_\mathcal{E}(A_1 \times O_{\mathcal{E}_2} \cup O_{\mathcal{E}_1} \times A_2) = eig_\mathcal{E}(A_1 \times O_{\mathcal{E}_1}) \cup eig_\mathcal{E}(O_{\mathcal{E}_1} \times A_2) \tag{35}$$

$$eig_\mathcal{E}(A_1 \times A_2 \cup B_1 \times B_2) = eig_\mathcal{E}(A_1 \cap B_1 \times A_2 \cup B_2) \cup eig_\mathcal{E}(A_1 \times A_2) \cup eig_\mathcal{E}(B_1 \times B_2) \cup eig_\mathcal{E}(A_1 \cup B_1 \times A_2 \cap B_2) \tag{36}$$

Proof: We note that from $C_1 \times C_2 \subset A_1 \times O_{\mathcal{E}_2} \cup O_{\mathcal{E}_1} \times A_2$ follows that $C_1 \times C_2 \subset A_1 \times O_{\mathcal{E}_2}$ or $C_1 \times C_2 \subset O_{\mathcal{E}_1} \times A_2$. This means that the only $C_1 \times C_2$ such that $C_1 \times C_2 \subset A_1 \times O_{\mathcal{E}_2} \cup O_{\mathcal{E}_1} \times A_2$ are $C_1 \times C_2$ that are contained in $A_1 \times O_2$ or in $O_1 \times A_2$. Using the general result of theorem 25 we conclude that $eig_\mathcal{E}(A_1 \times O_{\mathcal{E}_2} \cup O_{\mathcal{E}_1} \times A_2) = eig_\mathcal{E}(A_1 \times O_{\mathcal{E}_1}) \cup eig_\mathcal{E}(O_{\mathcal{E}_1} \times A_2)$. To derive the second equality we have to look for the elements $C_1 \times C_2$ that are contained in $A_1 \times A_2 \cup B_1 \times B_2$. Here we can have $C_1 \times C_2$ that are contained in $A_1 \times A_2 \cup B_1 \times B_2$, but are not contained in $A_1 \times A_2$ and not in $B_1 \times B_2$. For example $A_1 \cap B_1 \times A_2 \cup B_2$ and $A_1 \cup B_1 \times A_2 \cap B_2$. It is easy to prove that if $C_1 \times C_2 \subset A_1 \times A_2 \cup B_1 \times B_2$, then $C_1 \times C_2 \subset A_1 \cap B_1 \times A_2 \cup B_2$ or $C_1 \times C_2 \subset A_1 \times A_2$ or

$C_1 \times C_2 \subset B_1 \times B_2$ or $C_1 \times C_2 \subset A_1 \cup B_1 \times A_2 \cap B_2$. This proves that $eig_\mathcal{E}(A_1 \times A_2 \cup B_1 \times B_2) = eig_\mathcal{E}(A_1 \cap B_1 \times A_2 \cup B_2) \cup eig_\mathcal{E}(A_1 \times A_2) \cup eig_\mathcal{E}(B_1 \times B_2) \cup eig_\mathcal{E}(A_1 \cup B_1 \times A_2 \cap B_2)$.

3.2 The set of states and the orthogonality relation.

We shall now proceed by characterizing the set of states Σ of the entity S in function of the sets of states Σ_1 and Σ_2 of the entities S_1 and S_2. As we have defined in section 2, for an arbitrary state $p \in \Sigma$, we introduce $s(p) = \cap_{p \in F, F \in \mathcal{F}} F$, $s(p_1) = \cap_{p_1 \in F_1, F_1 \in \mathcal{F}_1} F_1$ and $s(p_2) = \cap_{p_2 \in F_2, F_2 \in \mathcal{F}_2} F_2$. We also introduce $s_1(p) = \cap_{p \in \nu_1(F_1), F_1 \in \mathcal{F}_1} \nu_1(F_1)$ and $s_2(p) = \cap_{p \in \nu_2(F_2), F_2 \in \mathcal{F}_2} \nu_2(F_2)$. Let us see in which way $s(p)$, $s(p_1)$, $s(p_2)$, $s_1(p)$ and $s_2(p)$ are related.

Theorem 27 *For $p \in \Sigma$, $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ we have:*

$$s_1(p) = \nu_1(s(p_1)) \tag{37}$$

$$s_2(p) = \nu_2(s(p_2)) \tag{38}$$

Proof: We have $\nu_1(s(p_1)) = \nu_1(\cap_{p_1 \in F_1} F_1) = \nu_1(\cap_{p \in \nu_1(F_1)} F_1) = \cap_{p \in \nu_1(F_1)} \nu_1(F_1) = s_1(p)$, and in an analogous way we show that $s_2(p) = \nu_2(s(p_2))$.

We have now sufficient material to show in the next theorem that a state of S must essentially be a product of a state of S_1 and a state of S_2.

Theorem 28 *For any arbitrary state p of S we have:*

$$s(p) = s_1(p) \cap s_2(p) \tag{39}$$

Proof: We clearly have $s(p) \subset s_1(p) \cap s_2(p)$, so let us proof that also the other inclusion holds. Since $s(p) \in \mathcal{F}$, we know that there exists $A \subset O_\mathcal{E}$ such that $s(p) = eig_\mathcal{E}(A)$, and also that $eig_\mathcal{E}(A) = \cup_i eig_\mathcal{E}(A_1^i \times A_2^i)$, with $A_1^i \times A_2^i \subset A$. We have $p \in s(p)$ and hence $p \in \cup_i eig_\mathcal{E}(A_1^i \times A_2^i)$. This means that there is one k such that $p \in eig_\mathcal{E}(A_1^k \times A_2^k)$, but then $s(p) \subset eig_\mathcal{E}(A_1^k \times A_2^k)$. On the other hand, since $A_1^k \times A_2^k \subset A$ we have $eig_\mathcal{E}(A_1^k \times A_2^k) \subset eig_\mathcal{E}(A) = s(p)$. This shows that $s(p) = eig_\mathcal{E}(A_1^k \times A_2^k) = eig_\mathcal{E}(A_1^k \times O_{\mathcal{E}_2}) \cap eig_{\mathcal{E}_1}(O_{\mathcal{E}_1} \times A_2^k)$. Since $p \in eig_\mathcal{E}(A_1^k \times O_{\mathcal{E}_2})$, and $p \in eig_\mathcal{E}(O_{\mathcal{E}_1} \times A_2^k)$ we have $s_1(p) \subset eig_\mathcal{E}(A_1^k \times O_{\mathcal{E}_2})$ and $s_2(p) \subset eig_\mathcal{E}(O_{\mathcal{E}_1} \times A_2^k)$, which implies that $s_1(p) \cap s_2(p) \subset eig_\mathcal{E}(A_1^k \times O_{\mathcal{E}_2}) \cap eig_\mathcal{E}(O_{\mathcal{E}_1} \times A_2^k) = s(p)$.

Let us now investigate the orthogonality relations. We want to show that the morphisms ν_1 and ν_2 also conserve the orthogonality relations. We first remark again that for each of the entities S, S_1 and S_2, and $p, q \in \Sigma$, $p_1, q_1 \in \Sigma_1$ and $p_2, q_2 \in \Sigma_2$ we have:

$$p \perp q \quad \Leftrightarrow \quad s(p) \perp s(q) \tag{40}$$

$$p_1 \perp_1 q_1 \quad \Leftrightarrow \quad s(p_1) \perp_1 s(q_1) \tag{41}$$

$$p_2 \perp_1 q_2 \quad \Leftrightarrow \quad s(p_2) \perp_2 s(q_2) \tag{42}$$

Theorem 29 *Consider $p_1, q_1 \in \Sigma_1$ and $p_2, q_2 \in \Sigma_2$, we have:*

$$p_1 \perp_1 q_1 \Leftrightarrow \nu_1(s(p_1)) \perp \nu_1(s(q_1)) \tag{43}$$

$$p_2 \perp_2 q_2 \Leftrightarrow \nu_2(s(p_2)) \perp \nu_2(s(q_2)) \tag{44}$$

Proof: Suppose that $p_1 \perp_1 q_1$. This means that we have sets $A_1, B_1 \subset O_{\mathcal{E}_1}$ such that $A_1 \cap B_1 = \emptyset$ and $s(p_1) \subset eig_{\mathcal{E}_1}(A_1)$ and $s(q_1) \subset eig_{\mathcal{E}_1}(B_1)$. From this follows that $\nu_1(s(p_1)) \subset eig_\mathcal{E}(A_1 \times O_{\mathcal{E}_2})$

and $\nu_1(s(q_1)) \subset eig_{\mathcal{E}}(B_1 \times O_{\mathcal{E}_2})$, and $A_1 \times O_{\mathcal{E}_2} \cap B_1 \times O_{\mathcal{E}_2} = \emptyset$. This shows that $\nu_1(s(p_1)) \perp \nu_1(s(q_1))$. Suppose now that $\nu_1(s(p_1)) \perp \nu_1(s(q_1))$. Then there exists $A_1 \times O_{\mathcal{E}_2}$ and $B_1 \times O_{\mathcal{E}_2}$ such that $A_1 \times O_{\mathcal{E}_2} \cap B_1 \times O_{\mathcal{E}_2} = \emptyset$, and $\nu_1(s(p_1)) \subset eig_{\mathcal{E}}(A_1 \times O_{\mathcal{E}_2})$ and $\nu_1(s(q_1)) \subset eig_{\mathcal{E}}(B_1 \times O_{\mathcal{E}_2})$. From this follows that $s(p_1) \subset eig_{\mathcal{E}_1}(A_1)$ and $s(q_1) \subset eig_{\mathcal{E}_1}(B_1)$, and $A_1 \cap B_1 = \emptyset$. This shows that $p_1 \perp_1 q_1$. In an analogous way we prove that $p_2 \perp q_2$ iff we have $\nu_2(s(p_2)) \perp \nu_2(s(q_2))$.

Theorem 30 *We have:*

$$p \perp q \quad \Leftrightarrow \quad p_1 \perp_1 q_1 \quad 'or' \quad p_2 \perp_2 q_2 \tag{45}$$

Proof: Suppose that $p_1 \perp_1 q_1$ then $\nu_1(s(p_1)) \perp \nu_1(s(q_1))$. Taking into account that $s(p) = \nu_1(s(p_1)) \cap \nu_2(s(p_2))$ and $s(q) = \nu_1(s(q_1)) \cap \nu_2(s(q_2))$, it follows that $s(p) \perp s(q)$, and hence $p \perp q$. In an analogous way show that if $p_2 \perp_2 q_2$ it follows that $p \perp q$. The other side of the implication is harder to proof. Suppose that $p \perp q$, then there exists an experiment $e_1 \times e_2$ and sets $A, B \subset O_e$ such that $A \cap B = \emptyset$, and such that $s(p) \subset eig_{e_1 \times e_2}(A)$ and $s(q) \subset eig_{e_1 \times e_2}(B)$. We know that $eig_{e_1 \times e_2}(A) = \cup_i eig_{e_1 \times e_2}(A_1^i \times A_2^i)$, where for all i we have $A_1^i \times A_2^i \subset A$. In the same way we have $eig_{e_1 \times e_2}(B) = \cup_j eig_{e_1 \times e_2}(B_1^j \times B_2^j)$, where for all j we have $B_1^j \times B_2^j \subset B$. Hence there exists a k and a l such that $s(p) \subset eig_{e_1 \times e_2}(A_1^k \times A_2^k)$ and $s(q) \subset eig_{e_1 \times e_2}(B_1^l \times B_2^l)$. From this follows that $\nu_1(s(p_1)) \cap \nu_2(s(p_2)) \subset eig_{e_1 \times e_2}(A_1^k \times A_2^k)$ and $\nu_1(s(q_1)) \cap \nu_2(s(q_2)) \subset eig_{e_1 \times e_2}(B_1^l \times B_2^l)$, and hence that $s(p_1) \subset eig_{e_1}(A_1^k)$, $s(p_2) \subset eig_{e_2}(A_2^k)$, $s(q_1) \subset eig_{e_1}(B_1^l)$ and $s(q_2) \subset eig_{e_2}(B_2^l)$. Since $A \cap B = \emptyset$ and $A_1^k \times A_2^k \subset A$ and $B_1^l \times B_2^l \subset B$, we have that $A_1^k \times A_2^k \cap B_1^l \times B_2^l = \emptyset$, and from this follows that $A_1^k \cap B_1^l = \emptyset$ or $A_2^k \cap B_2^l = \emptyset$. So we have proven that $p_1 \perp_1 q_1$ or $p_2 \perp_2 q_2$.

3.3 The axioms and the entity consisting of separated entities.

We want to investigate now the different axioms that are satisfied for a quantum situation, and see which ones are satisfied and which ones are not for the description of the entity consisting of two separated entities. We'll suppose the 'state determination axiom' to be satisfied for all three entities. Hence for $p, q \in \Sigma$, $p_1, q_1 \in \Sigma_1$ and $p_2, q_2 \in \Sigma_2$, we have:

$$s(p) = s(q) \quad \Rightarrow \quad p = q \tag{46}$$

$$s(p_1) = s(q_1) \quad \Rightarrow \quad p_1 = q_1 \tag{47}$$

$$s(p_2) = s(q_2) \quad \Rightarrow \quad p_2 = q_2 \tag{48}$$

Theorem 31 *The axiom of atomicity is satisfied for the entity S iff it is satisfied for the entity S_1 and for the entity S_2.*

Proof: Suppose that the axiom of atomicity is satisfied for S_1 and for S_2. Consider $s(p)$ for some state $p \in \Sigma$, and $q \in s(p)$. Then we have $s(q) \subset s(p)$, and hence $\nu_1(s(q_1)) \cap \nu_2(s(q_2)) \subset \nu_1(s(p_1)) \cap \nu_2(s(p_2))$. From this follows that $s(q_1) \subset s(p_1)$ and $s(q_2) \subset s(p_2)$. Since the axiom of atomicity is satisfied for S_1 and S_2, we have $\{q_1\} \subset \{p_1\}$ and $\{q_2\} \subset \{p_2\}$, which gives $q_1 = p_1$ and $q_2 = p_2$. From this follows that $\nu_1(s(q_1)) = \nu_1(s(p_1))$ and $\nu_2(s(q_2)) = \nu_2(s(p_2))$, which implies that $s(q) = s(p)$. Since the axiom of state determination is satisfied we have $p = q$. As a consequence we have proven that $s(p) = \{p\}$, and hence the axiom of atomicity is satisfied for the entity S. Suppose now that the axiom is satisfied for the entity S. Consider $p_1 \in \Sigma_1$. We want to show that $s(p_1) = \{p_1\}$. Consider $q_1 \in s(p_1)$ then $s(q_1) \subset s(p_1)$ and $\nu_1(s(q_1)) \subset \nu_1(s(p_1))$. Consider an arbitrary state $p_2 \in \Sigma_2$. Then $\nu_1(s(q_1)) \cap \nu_2(s(p_2)) \subset \nu_1(s(p_1)) \cap \nu_2(s(p_2))$, which implies that $s(q) = s(p)$. But since the axiom of atomicity is satisfied for S, we have $\{q\} \subset \{p\}$, and hence $q = p$. But then $s(q) = s(p)$ which makes that $\nu_1(s(q_1)) \cap \nu_2(s(p_2)) = \nu_1(s(p_1)) \cap \nu_2(s(p_2))$, and hence $\nu_1(s(q_1)) = \nu_1(s(p_1))$. This gives $s(q_1) = s(p_1)$, and since the axiom of state determination is satisfied we have $q_1 = p_1$. From this result follows that $s(p_1) = \{p_1\}$. In this way we have proven that the axiom of atomicity is satisfied for S_1. In an analogous way we can show that the axiom of atomicity is satisfied for the entity S_2.

Theorem 32 *The axiom ortho 1 is satisfied for the entity S iff it is satisfied for the entities S_1 and S_2.*

Proof: Suppose that axiom ortho 1 is satisfied for S_1 and for S_2. We want to proof that axiom ortho 1 is satisfied for S. Consider the entity S to be in state p such that $p \perp eig_{e_1 \times e_2}(A)$. We have to prove that this is equivalent to $p \in eig_{e_1 \times e_2}(A^C)$. As we know from the foregoing, we shall use the fact that we can write $A = \cup_i (A_1^i \times A_2^i)$ for $A_1^i \times A_2^i \subset A$. Let us calculate A^C. We have $A^C = \cap_i (A_1^i \times A_2^i)^C = \cap_i (A_1^{iC} \times O_{e_2} \cup O_{e_1} \times A_2^{iC})$. Let us consider the following equivalences:
$p \perp eig_{e_1 \times e_2}(A)$
$\Leftrightarrow p \perp \cup_i eig_{e_1 \times e_2}(A_1^i \times A_2^i)$
$\Leftrightarrow p \perp \nu_1(eig_{e_1}(A_1^i)) \cap \nu_2(eig_{e_2}(A_2^i))$ for all i
$\Leftrightarrow p_1 \perp_1 eig_{e_1}(A_1^i)$ or $p_2 \perp_2 eig_{e_1}(A_2^i)$ for all i
$\Leftrightarrow p_1 \in eig_{e_1}(A_1^{iC})$ or $p_2 \in eig_{e_2}(A_2^{iC})$ for all i.
$\Leftrightarrow \nu_1(s(p_1)) \subset eig_{e_1 \times e_2}(A_1^{iC} \times O_{e_2})$ or $\nu_2(s(p_2)) \subset eig_{e_1 \times e_2}(O_{e_1} \times A_2^{iC})$ for all i
$\Leftrightarrow \nu_1(s(p_1)) \cap \nu_2(s(p_2)) \subset eig_{e_1 \times e_2}(A_1^{iC} \times O_{e_2})$ or $\nu_1(s(p_1)) \cap \nu_2(s(p_2)) \subset eig_{e_1 \times e_2}(O_{e_1} \times A_2^{iC})$ for all i
$\Leftrightarrow \nu_1(s(p_1)) \cap \nu_2(s(p_2)) \subset eig_{e_1 \times e_2}(A_1^{iC} \times O_{e_2}) \cup eig_{e_1 \times e_2}(O_{e_1} \times A_2^{iC})$ for all i
$\Leftrightarrow \nu_1(s(p_1)) \cap \nu_2(s(p_2)) \subset \cap_i (eig_{e_1 \times e_2}(A_1^{iC} \times O_{e_2}) \cup eig_{e_1 \times e_2}(O_{e_1} \times A_2^{iC}))$
$\Leftrightarrow p \in \cap_i (eig_{e_1 \times e_2}(A_1^{iC} \times O_{e_2} \cup O_{e_1} \times A_2^{iC})$
$\Leftrightarrow p \in eig_{e_1 \times e_2} \cap_i (A_1^{iC} \times O_{e_2} \cup O_{e_1} \times A_2^{iC})$
$\Leftrightarrow p \in eig_{e_1 \times e_2}(A^C)$.
Suppose now that axiom ortho 1 is satisfied for S. Let us show that it is also satisfied for S_1 and S_2. We have $p_1 \perp_1 eig_{e_1}(A_1) \Rightarrow p \perp eig_{e_1 \times e_2}(A_1 \times O_{e_2}) \Leftrightarrow p \in eig_{e_1 \times e_2}(A_1^C \times O_{e_2}) \Leftrightarrow p_1 \in eig_{e_1}(A_1^C)$. In an analogous way we show that axiom ortho 1 is satisfied for S_2.

Theorem 33 *The axiom ortho 2 is satisfied for the entity S iff it is satisfied for the entities S_1 and S_2.*

Proof: Suppose that the axiom ortho 2 is satisfied for S_1 and S_2. This means that for $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ we have $\{p_1\}^{\perp_1} \in \mathcal{F}_1$ and $\{p_2\}^{\perp_2} \in \mathcal{F}_2$. Consider $p \in \Sigma$. We must show that $\{p\}^\perp \in \mathcal{F}$. Let us first try to show that $\{p\}^\perp = \nu_1(\{p_1\}^{\perp_1}) \cup \nu_2(\{p_2\}^{\perp_2})$. Consider $q \in \{p\}^\perp$ such that $s(q) = \nu_1(s(q_1)) \cap \nu_2(s(q_2))$. We have:
$q \in \{p\}^\perp \Leftrightarrow q_1 \perp_1 p_1 \;\; or \;\; q_2 \perp_2 p_2$
$\Leftrightarrow s(q_1) \subset \{p_1\}^{\perp_1} \;\; or \;\; s(q_2) \subset \{p_2\}^{\perp_2}$
$\Leftrightarrow \nu_1(s(q_1)) \subset \nu_1(\{p_1\}^{\perp_1}) \;\; or \;\; \nu_2(s(q_2)) \subset \nu_2(\{p_2\}^{\perp_2})$
$\Leftrightarrow s(q) \subset \nu_1(\{p_1\}^{\perp_1}) \cup \nu_2(\{p_2\}^{\perp_2})$
$\Leftrightarrow q \in \nu_1(\{p_1\}^{\perp_1}) \cup \nu_2(\{p_2\}^{\perp_2})$
We have also $\{p_1\}^{\perp_1} = eig_{\mathcal{E}_1}(A_1)$ for some $A_1 \subset O_{\mathcal{E}_1}$, and $\{p_2\}^{\perp_2} = eig_{\mathcal{E}_2}(A_2)$ for some $A_2 \subset O_{\mathcal{E}_2}$. Let us then consider $A = A_1 \times O_{\mathcal{E}_2} \cup O_{\mathcal{E}_1} \times A_2$. Then we have $eig_{\mathcal{E}}(A) = eig_{\mathcal{E}}(A_1 \times O_{\mathcal{E}_2}) \cup eig_{\mathcal{E}}(O_{\mathcal{E}_1} \times A_2) = \nu_1(\{p_1\}^{\perp_1}) \cup \nu_2(\{p_2\}^{\perp_2}) = \{p\}^\perp$. This shows that $\{p\}^\perp \in \mathcal{F}$.
Suppose now that S satisfies the axiom ortho 2. Let us show that then S_1 and S_2 satisfy this axiom. For an arbitrary $p_1 \in \Sigma_1$ we have, because axiom ortho 2 is satisfied for S, that $(\nu_1(s(p_1)))^\perp \in \mathcal{F}$. This means that $(\nu_1(s(p_1)))^\perp = eig_{\mathcal{E}}(A)$ for some experiment $e_{\mathcal{E}}$ and some outcome set A. We also have $q \in (\nu_1(s(p_1))^\perp) \Leftrightarrow s(q) \subset (\nu_1(s(p_1))^\perp) \Leftrightarrow \nu_1(s(q_1)) \cap \nu_2(s(q_2)) \perp \nu_1(s(p_1)) \Leftrightarrow \nu_1(s(q_1)) \perp \nu_1(s(p_1)) \Leftrightarrow q_1 \perp_1 p_1 \Leftrightarrow q_1 \in \{p_1\}^{\perp_1}$. Consider now an arbitrary outcome $x_{\mathcal{E}_2} \in O_{\mathcal{E}_2}$, and a state p_2 such that when S_2 is in p_2 the outcome $x_{\mathcal{E}_2}$ is a possible outcome for the experiment $e_{\mathcal{E}_1 \times \mathcal{E}_2}$. We have: $e_{\mathcal{E}}$ gives with certainty an outcome in A iff the state $q_1 \in \{p_1\}^{\perp_1}$. Hence we can always consider S_2 to be in the state p_2, and then $x_{\mathcal{E}_2}$ is a possible outcome. This shows that $A = A_1 \times O_{\mathcal{E}_2}$. And then $q_1 \in eig_{\mathcal{E}_1}(A_1)$ iff $q_1 \in \{p_1\}^{\perp_1}$, which shows that $\{p_1\}^{\perp_1} \in \mathcal{F}_1$. In an analogous way we can show that S_2 satisfies axiom ortho 2.

3.4 Quantum mechanics and separated entities.

We shall show now that the property lattice of the entity consisting of two separated entities does not satisfy the axiom of weak modularity and does not satisfy the axiom of the covering law if both entities are non-classical entities. To proof this theorem we use the characterization of classical entities of 2.5 of section 1, and introduce the concept of superselection rule. The idea is to call two elements of the closure structure separated by a superselection rule if their union is an element of the closure structure. This indeed means that the property corresponding to this union is actual iff one of the two properties corresponding to the elements is actual. To say it in quantum mechanical language: between states of elements that are separated by a superselection rule there do not exist 'superpositions'. Let us put forward this definition and then prove an important property in weakly modular and covering law satisfying structures.

Definition 14 *If For two elements $F, G \in \mathcal{F}$ we say that F and G are separated by a superselection rule iff $F \cup G \in \mathcal{F}$.*

Theorem 34 *Suppose that the axiom of state determination, the axiom of atomicity, axiom ortho 1 and axiom ortho 2 are satisfied. If the axiom of weak modularity 'or' the axiom of the covering law is satisfied, and p, q are two states of S that are separated by a superselection rule, then $p \perp q$.*

Proof: Suppose that $p, q \in \Sigma$ such that $cl_{eig}(\{p, q\}) = \{p, q\}$.
1) Suppose that the axiom of weak modularity is satisfied. Consider $cl[(\{p,q\} \cap \{q\}^\perp) \cup \{q\}]$. From weak modularity follows that $cl[(\{p,q\} \cap \{q\}^\perp) \cup \{q\}] = cl(\{p,q\})$. Hence $cl[(\{p,q\} \cap \{q\}^\perp) \cup \{q\}] = \{p,q\}$. Suppose that $p \not\perp q$, then we also have $\{p.q\} \cap \{q\}^\perp = \emptyset$, and hence $cl_{eig}[(\{p,q\} \cap \{q\}^\perp) \cup \{q\}] = \{q\}$. This means that $\{q\} = \{p,q\}$, and hence $q = p$. This shows that when the axiom of weak modularity is satisfied we have $p \perp q$ of $p = q$.
2) Suppose now that the axiom of the covering law is satisfied. If $p \not\perp q$ we have $(\{p,q\} \cup \{q\}) \cap \{q\}^\perp = \emptyset$ which gives that $\Sigma = [(\{p,q\} \cup \{q\}) \cap \{q\}^\perp]^\perp = cl[(\{p\}^\perp \cap \{q\}^\perp) \cup \{q\}]$. Hence Σ covers $\{p\}^\perp \cap \{q\}^\perp$. But $\{p\}^\perp \cap \{q\}^\perp \subset \{p\}^\perp \subset \Sigma$. And then follows from the covering law that $\{p\}^\perp \cap \{q\}^\perp = \{p\}^\perp$, which implies that $\{p\}^\perp = \{q\}^\perp$ and hence $p = q$. Again we have shown that when the covering law is satisfied then the states p and q are orthogonal or equal.

This last theorem shows that for entities where the axiom of weak modularity or the axiom of the covering law are satisfied, only orthogonal states can be separated by a superselection rule. We'll show in the next theorem that for the case of an entity S consisting of two separated entities S_1 and S_2 it is easy to find states that are separated by a superselection rule and are not orthogonal. This is what makes it impossible to describe this situation by quantum mechanics.

Theorem 35 *Suppose that S is an entity consisting of two separated entities S_1 and S_2, and suppose that the axiom of state determination, the axiom of atomicity, axiom ortho 1 and axiom ortho 2 are satisfied. If p_1, q_1 are two different states of S_1 and p_2, q_2 are two different states of S_2, then p and q, where $p = \nu_1(\{p_1\}) \cap \nu_2(\{p_2\})$ and $q = \nu_1(\{q_1\}) \cap \nu_2(\{q_2\})$ which are states of S, are separated by a superselection rule.*

Proof: We have $\{p_1\} = eig_{\mathcal{E}_1}(A_1)$, $\{q_1\} = eig_{\mathcal{E}_1}(B_1)$, $\{p_2\} = eig_{\mathcal{E}_2}(A_1)$ and $\{q_2\} = eig_{\mathcal{E}_2}(B_2)$. Then we have $\{p\} = eig_{\mathcal{E}}(A_1 \times A_2)$ and $\{q\} = eig_{\mathcal{E}}(B_1 \times B_2)$. From theorem 26 we know that $eig_{\mathcal{E}}(A_1 \times A_2 \cup B_1 \times B_2) = eig_{\mathcal{E}}(A_1 \cap B_1 \times A_2 \cup B_2) \cup eig_{\mathcal{E}}(A_1 \times A_2) \cup eig_{\mathcal{E}}(B_1 \times B_2) \cup eig_{\mathcal{E}}(A_1 \cup B_1 \times A_2 \cap B_2)$. Since p_1 and q_1 are different states, and p_2 and q_2 are different states, we have $eig_{\mathcal{E}_1}(A_1 \cap B_1) = \emptyset$ and $eig_{\mathcal{E}_2}(A_2 \cap B_2) = \emptyset$. From this follows that $eig_{\mathcal{E}}(A_1 \cap B_1 \times A_2 \cup B_2) = \emptyset$ and $eig_{\mathcal{E}}(A_1 \cup B_1 \times A_2 \cap B_2) = \emptyset$. Hence $eig_{\mathcal{E}}(A_1 \times A_2 \cup B_1 \times B_2) = eig\mathcal{E}(A_1 \times A_2) \cup eig_{\mathcal{E}}(B_1 \times B_2) = \{p, q\}$. This shows that $\{p, q\} \in \mathcal{F}$ and p and q are separated by a superselection rules.

We can now finally prove our main theorem.

Theorem 36 *Suppose that S is an entity consisting of two separated entities S_1 and S_2, and the axiom of state determination, the axiom of atomicity, axiom ortho 1, axiom ortho 2, are satisfied. If the axiom of weak modularity or the axiom of the covering law is satisfied for S, then one of the two entities S_1 or S_2 is a classical entity.*

Proof: Suppose that S_2 is not a classical entity. From theorem 20 follows that in this case there exists two different states p_2 and q_2 of S_2 that are not orthogonal. Suppose that p_1 and q_1 are arbitrary different states of S_1. From theorem 35 follows that, if $\{p\} = \nu_1(\{p_1\}) \cap \nu_2(\{p_2\})$ and $\{q\} = \nu_1(\{q_1\}) \cap \nu_2(\{q_2\})$, then p and q are separated by a superselection rule. If now the axiom of modularity, or the axiom of the covering law (one of the two is sufficient) is satisfied, then follows from theorem 34 that $p \perp q$. From theorem 30 follows then that $p_1 \perp_1 q_1$. Hence we have shown that two arbitrary different states of S_1 are orthogonal. Applying again theorem 20 we conclude that S_1 is a classical entity.

From this theorem follows that whenever two entities S_1 and S_2 have at least one non classical experiment (as in the case for entities described by quantum mechanics), the property lattice of the entity S consisting of the two separated entities S_1 and S_2 cannot satisfy the axiom of weak modularity and cannot satisfy the axiom of the covering law. As a consequence such an entity cannot be described by quantum mechanics, since quantum mechanics satisfies these axioms. And this is the reason that one encounters paradoxical situations if one does describe such a situation by quantum mechanics. The essential shortcoming that makes it impossible for quantum theory to describe separated entities, is due to the fact that a theory satisfying the axioms of quantum mechanics cannot describe non orthogonal states that are separated by a superselection rule. And an entity consisting of separated quantum entities has such states.

Most of the paradoxes of quantum mechanics are due to inevitable evolution in superposition states, by means of the Schrödinger equation, and this is also the case for the Einstein Podolsky Rosen paradox. Quantum mechanics predicts that even an entity S consisting of two separated entities S_1 and S_2 evolves to a superposition of product states, whenever there is an interaction between the two entities S_1 and S_2. This means that in all situations of separated quantum entities S_1 and S_2, with interaction between S_1 and S_2 (we remark again that separated does not mean without interaction), that constitute in macroscopic reality almost all interesting cases of two-body problems, cannot be described in quantum mechanics. We have analyzed in detail in which way the paradoxes disappear, when we decide to describe the entity S in the more general theory, that does not satisfy the two 'bad' axioms of weak modularity and the covering law (see ref [10,11,12,13]).

4. The origin of probability.

No attempts have ever been made to explain 'classical probability', because when it was introduced by Simon Laplace in the foregoing century, it was just considered as the description of a specific situation of reality. Only with the advent of quantum mechanics it seemed that a type of 'unexplainable' probability, now commonly called quantum probability, was appearing in reality. We have presented in earlier work a possible explanation for the quantum probability [14,15,16]. The idea is that the quantum probability is due to a lack of knowledge about the interaction between the experiment and the entity under study. Each experiment e is in fact a mixture of classical deterministic experiments e_λ, which we have named 'hidden measurements'. The probability is due to the fact that when we execute the experiment e, in reality one of the classical experiments e_λ is taking place, but since we lack knowledge about which one, repeated experiments e give different outcomes, giving rise to the presence of the quantum probability.

In this paper we show that the way in which we explained the quantum probability can be used to explain an arbitrary probability appearing in nature with an arbitrary experimental situation.

Definition 16 *Suppose that we have an entity S with a set of states Σ and a set of experiments $\mathcal{E}$. The probability that an experiment $e \in \mathcal{E}$ gives an outcome $x_e^i \in O_e$ when the entity is in a state $p \in \Sigma$ is denoted by $P(e = x_e^i \mid p)$.*

Definition 17 *Suppose that $E \subset \mathcal{E}$, and we consider the union experiment e_E that consists of performing one of the $e \in E$, then we shall say that e_E is a 'mixture' of the experiments $e \in E$ iff there exists a probability measure $\mu : \mathcal{B}(E) \rightarrow [0,1]$, where $\mathcal{B}(E)$ is a σ-algebra, such that $\mu(K)$ is the probability that an experiment in K is performed. We'll denote the 'mixture' by $e_{E,\mu}$.*

To be able to proof the next theorem, we need to introduce first some definitions.

Definition 18 *If $y_1, y_2, ..., y_n$ are vectors of $\Re^n$, we shall denote by $M(y_1, y_2, ..., y_n)$ the $n \times n$ matrix, where $M_{i,j} = (y_i)_j$. We denote by $det(y_1, ..., y_n)$ the determinant of this matrix $M(y_1, ..., y_n)$, and by $Par(y_1, ..., y_n)$ the parallelepiped spanned by the n vectors.*

Theorem 37 *Consider an entity S with set of states Σ and set of experiments $\mathcal{E}$, and set of probabilities $\{P_e(e = x_e^i \mid p) \mid e \in \mathcal{E}, p \in \Sigma, x_e^i \in O_e\}$. For each experiment $e \in \mathcal{E}$ it is possible to define a collection of classical experiments $C(\Lambda) = \{e_\lambda \mid \lambda \in \Lambda\}$, such that e is the 'mixture' of the collection $C(\Lambda)$, hence $e = e_{C(\Lambda),\mu} = \cup_{\lambda \in \Lambda} e_\lambda$, and $\mu : \mathcal{B}(\Lambda) \to [0,1]$ is a probability measure such that for $K \subset \Lambda$ we have that $\mu(K)$ is the probability that performing e, a classical experiment e_λ occurs, such that $\lambda \in K$. Then we have that $P_e(e = x_e^i \mid p) = \mu(A_p^i)$ where $A_p^i = \{\lambda \mid e_\lambda(p) = x_e^i\}$ is the set of all states where the classical experiment e_λ gives outcome x_e^i.*

Proof: Let us consider one specific experiment $e \in \mathcal{E}$, and suppose that the outcome set O_e is given by $\{x_e^1, x_e^2, ..., x_e^i, ..., x_e^n\}$. In relation with this one experiment e we have a collection of probabilities $\{P_e(e = x_e^i \mid p) \mid x_e^i \in O_e, p \in \Sigma\}$. We have to define a collection of classical experiments $e_\lambda \in \Lambda$, having all outcome sets O_e, such that $e = \cup_{\lambda \in \Lambda} e_\lambda$ is the mixture of the e_λ, and a probability measure $\mu : \mathcal{B}(\Lambda) \to [0,1]$, such that $P_e(e = x_e^i \mid p) = \mu(A_p^i)$, where $A_p^i = \{\lambda \mid e_\lambda(p) = x_e^i\}$. Let us denote $r^i(p) = P_e(e = x_e^i \mid p)$, then clearly $\sum_{i=1}^n r^i(p) = 1$ and $0 \leq r^i(p) \leq 1$. If we consider $r(p) = (r^1(p), r^2(p), ..., r^i(p), ..., r^n(p))$, then $r(p)$ for each p is a point of a simplex $S_n \subset \Re^n$, which is the convex closure of the set of canonical base vectors $h_1 = (1, ..., 0, ...0)$, $h_2 = (0, 1, 0, ..., 0)$, ...,$h_i = (0, ..., 1, ..., 0)$,..., $h_n = (0, ..., 1)$. For every state p of the entity S we have $s(p) = \sum_{i=1}^n r^i(p) h_i$. Let us now define the classical experiments.
1. Each experiment e_λ has the same set O_e of possible outcomes $\{x_e^1, x_e^2, ..., x_e^n\}$.
2. We take the parameter λ, that labels the 'classical experiments' e_λ, to be also points of the simplex S_n, hence $\lambda = (\lambda^1, \lambda^2, ..., \lambda^n)$ such that $0 \leq \lambda^i \leq 1$, and $\sum_{i=1}^n \lambda^i = 1$. For a given state p represented by the vector $r(p) = (r^1(p), r^2(p), ..., r^n(p))$, we call A_p^i the convex closure of the vectors h_1, h_2,..., $h_{i-1}, r(p), h_{i+1}$,..., h_n. Then clearly $S_n = \cup_{i=1}^n A_p^i$.
3. We define the experiment e_λ as follows: if $\lambda \in A_p^i$ then the experiment e_λ gives the outcome x_e^i if the entity is in state p. If λ is a point of the boundary of A_p^i and hence also a point of A_p^{i-1} or A_p^{i+1}, then the outcome of the experiment e_λ is indeterminate, but indeterminate in the classical sense (as for example in the case of a classical unstable equilibrium).
So we see that the classical experiments e_λ are deterministic in the classical sense, namely that they contain only a 'classical indeterminism' for those states that are border-points of the A_p^i. When the experiment e is performed, one of the e_λ occurs. Let us calculate the probability $P(e = x_e^i \mid p)$ for the experiment e to get the outcome x_e^i if the entity is in state p represented by $r(p)$. This probability is given by:

$$P(e = x_e^i \mid p) = \frac{m^n(A_p^i)}{m^n(S_n)} \tag{49}$$

with m^n the Lebesgue measure trace on $\Re^n$. We have $|det(y_1, y_2, ..., y_n)| = m^n(Par(y_1, y_2, ..., y_n))$ and $m^n(S_n) = c(n) m^n(Par(h_1, ..., h_n))$ and $m^n(A_{n,i}) = c(n) m^n(Par(h_1, ..., h_{i-1}, r(p), h_{i+1}, ..., h_n))$ which shows that :

$$\begin{aligned} P(e = x_e^i \mid p) &= \frac{m^n(Par(h_1, ..., h_{i-1}, r(p), h_{i+1}, ..., h_n))}{m^n(Par(h_1, ..., h_n))} \\ &= \frac{|det(h_1, ..., h_{i-1}, r(p), h_{i+1}, ..., h_n)|}{|det(h_1, ..., h_n)|} \\ &= |det(h_1, ..., h_{i-1}, r(p), h_{i+1}, ..., h_n)| \\ &= r^i(p) \end{aligned} \tag{50}$$

This theorem shows that we can build a model with classical experiments that gives the original probabilities. These probabilities are the consequence of our incomplete knowledge of how the measuring apparatus corresponding to the experiment e acts to produce an outcome. So the probabilities are due to a lack of knowledge about the experimental situations. In ref [17] we have investigated in which way this construction can be used for the quantum mechanical situation in general.

5. Experiments of the first kind.

If we are convinced that the quantum mechanical paradoxes are due to a structural shortcoming of the orthodox formalism, we should try to built a more general formalism. We can start by the theory that we have introduced here, and try to put forward axioms that do not lead to structural shortcomings, and hence can deliver finally a new quantum-like theory, which will be structurally not isomorphic to orthodox quantum theory. We would like to indicate how one could proceed, by introducing some concepts.

Definition 19 *Suppose that we have an entity S with a set of states Σ and a set of experiments $\mathcal{E}$. We say that an experiment $e \in \mathcal{E}$ is of the 'first kind' iff when an outcome $x_e^i \in O_e$ occurs, the state of the entity S is changed in a specific state p_e^i, that is an eigenstate of e with eigen outcome x_e^i.*

Let us remark that an experiment on a quantum entity represented by a self-adjoint operator is always an experiment of the first kind.

Theorem 38 *Consider a collection of experiments $E \in \mathcal{E}$ and the union experiment e_E that consists of choosing one of the experiments from E and performing the chosen experiment. We know that $O_E = \cup_{e \in E} O_e$ is the outcome set of the union experiment e_E. Then e_E is of the first kind iff e is of the first kind for each $e \in E$ and $O_e = O_f$ for all $e, f \in E$, such that $O_E = O_e$.*

Proof: Suppose that e_E is of the first kind. Consider an arbitrary $e \in E$ and suppose that we perform e and get an outcome x_e^i. Then, since e_E is of the first kind, the state of S is changed into a state p_E^i, which is an eigenstate of e_E with eigen outcome x_e^i, and hence also an eigenstate of e, with eigen outcome x_e^i. This shows that e is of the first kind. So we have shown already that if e_E is of the first kind, then each $e \in E$ is of the first kind. Consider now $e, f \in E$ and x_e^i an arbitrary outcome of e. Suppose that we perform e_E and get an outcome x_e^i. Since e_E is of the first kind we know that this has transformed the state of the entity into a state p_E^i, which is an eigenstate of e_E with eigen outcome x_e^i. Suppose that we perform the experiment e_E when S is in the state p_E^i and choose to perform $f \in E$. Then the outcome x_e^i occurs, which shows that $x_e^i \in O_f$. This shows that $O_e \subset O_f$. In an analogous way we show that $O_f \subset O_e$, which proves that $O_e = O_f$. Clearly then $O_E = O_e$. The inverse implication, suppose that all the $e \in E$ are experiments of the first kind, and $O_e = O_f$ for $e, f \in E$ implies that e_E is of the first kind, can easily be proved along the same lines.

When we consider a situation of an entity S, with a set of states Σ and a set of experiments $\mathcal{E}$, which are all experiments of the first kind, and a set of probabilities $\{P_e(e = x_e^i|p) \mid e \in \mathcal{E}, x_e^i \in O_e, p \in \Sigma\}$, then applying theorem 37, we know that we can define for each experiment e a collection of classical experiments e_λ, such that e is a mixture of these classical experiments, and the lack of knowledge about which classical experiment occurs, when the experiment e is executed, gives rise to the probabilities. From theorem 37 follows that if e is of the first kind, then all the e_λ are of the first kind, and have the same outcome set O_e. The construction that we have proposed in proof of theorem 38, can again be applied for this situation. We just have to define the action of a specific classical experiment on the state. From theorem 38 follows that we have to propose the following: an experiment e_λ, such that $\lambda \in A_p^i$, changes the state p to the eigenstate p_e^i of the experiment e with eigenoutcome x_e^i.

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