Application of quantum statistics in psychological studies of decision processes*

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Abstract

We present a new approach to the old problem of how to incorporate the role of the observer in statistics. We show classical probability theory to be inadequate for this task and take refuge in the epsilon-model, which is the only model known to us capable of handling situations between quantum and classical statistics. An example is worked out and some problems are discussed as to the new viewpoint that emanates from our approach.

1 Introduction

The quantum probability theory is fundamentally different from the classical probability theory. The core of the difference lies in the fact that in quantum probability we are dealing with the *actualization* of a certain property *during the measurement process*, while in classical probability all properties are assumed to have a definite value *before* measurement, and that this value is the outcome of the measurement.

The content of quantum probability is the calculation of the probability of the actualization of one among different potentialities as the *result* of the measurement. Such an effect cannot, in general, be incorporated in a classical statistical framework. A lot has been said about the role of the observer in the human and social sciences. Indeed, let us consider the most simple example of a psychological questionnaire where the participant is allowed only two possible answers: *yes* and *no*. For certain questions people will have predefined opinion (e.g. Are you male/female?, Are you older than 20?). But it is not difficult to imagine a question for which the person who is being questioned has no opinion ready.

For such a situation one answer will be actualized among the two possible answers, at the moment the question is being asked. We are no longer in a

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classical statistical situation, because the reason why a particular answer is chosen is not so much dependent on the state of the participant or that of the questioner but is in fact highly contextual, that is, the answer is formed at the very moment the interaction between the participant and the examiner is taking place.

Because of the analogue with the quantum mechanical situation, one is very much inclined to investigate whether it is possible to describe such a situation more accurately within a quantum probabilistic framework. Clearly most questions will not be entirely quantum, nor entirely classical in nature, so what we need is a model that fills the now existing gap between the quantum probabilistic approach and the classical one. Such a model is provided by the epsilon-model that will be given in the next section.

It will be clear that if the introduction of quantum probability theory in psychology is successful, a fundamental problem of psychology, namely the incorporation of the observer in the statistics, can be examined from an entirely new point of view. Similar problems have plagued sociology (in fact, the example we give in section three constitutes a set of sociological problems) and anthropology and we like to think that investigating these areas with the techniques proposed here may lead to important new insights and methodologies.

2 The epsilon-model

There are several different ways of introducing the epsilon-model and we have chosen the most simple one for the present purpose (Aerts at al. 1993a,b; Aerts and Durt, 1994). Consider a particle whose possible states q can be represented by its position on the unit sphere. As a mathematical representation of this state we use a unit vector that we call v. Thus we can describe the set Σ of all possible states q_v as follows:

$$\Sigma = \{q_v \mid v \textit{ lies on the sphere}\} \tag{1}$$

Since we want to consider different measurements, we must have some way to identify a certain measurement. We shall do this by introducing another unit vector u. The set $\mathcal{M}$ of all measurements E_u with outcome O is then characterized as follows:

$$\mathcal{M} = \{E_u \mid u \textit{ lies on the sphere}\} \tag{2}$$

The outcome O of the measurement E_u is obtained as follows:

We have a piece of elastic L of length 2. One end of the elastic is attached to the point denoted by the vector u, while the other end is fixed to the diametrically opposite point $-u$. Once the elastic is placed, the particle falls from its original place (denoted by v) onto the elastic, and takes the shortest path when falling, that is, orthogonal to the elastic, and finally sticks to the elastic in a point a.

Now the elastic breaks somewhere. If we consider the two parts of the elastic, the part L_1 from a to $+u$, and the part L_2 from $-u$ to a, then it is obvious that the elastic must break either somewhere in L_1, or else somewhere in L_2.

If it breaks in L_1, the particle in a drawn towards the point $-u$. Similarly, if the elastic breaks in L_2, the particle in a is drawn towards the point $+u$. In the first case we shall say that the outcome $O = -1$, while in the second case we shall give the experiment the outcome $O = +1$.

This completes the measurement.

We see clearly how the act of measurement transforms the state q_v into the state q_u or q_{-u}, so that our measurement defines a state-transition in an indeterministic way, because we have no knowledge of the point where the elastic will break. The transition probabilities $p(u \mid v)$ and $p(-u \mid v)$ connected to the two possible transitions (q_v to q_u and q_v to q_{-u}) depend on the mechanism by which the elastic breaks.

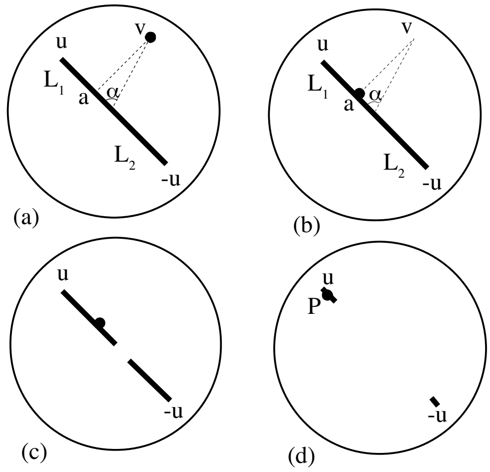

Fig. 1. A representation of the ϵ-model. A particle located in v, (a) falls orthogonally onto an elastic spanned between the two diametrical opposite points u and $-u$, (b) and sticks to it. Then the elastic breaks, (c), and the particle is pulled to one of the points u, (d), then we say that the measurement has outcome $+1$, or, $-u$, and then we say that it has outcome -1.

One can choose different models for this breaking that will lead to different probabilities. We will consider only the most simple case, where the probability of the elastic breaking in a certain segment is proportional to the length of the segment. Let us calculate the transition probabilities for this case.

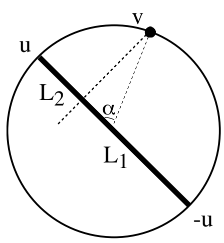

Fig. 2. If we make the hypothesis that the elastic breaks uniformly, the transition probability for the state v changing into u or into $-u$ is easily calculated.

We see that $p(u \mid v)$ is the probability that the elastic breaks in L_2, and hence equals the length of L_2 divided by 2, which is the length of the elastic. The length of L_2 (see Fig. 2) is $1 - a = 1 - u \cdot v = 1 - \cos\alpha$.

We find the transition probability to be:

$$p(u \mid v) = (1 - \cos\alpha)/2 = \sin^2(\alpha/2) \tag{3}$$

Similarly,

$$p(-u \mid v) = \cos^2(\alpha/2) \qquad (4)$$

These are precisely the quantum mechanical transition probabilities for the spin measurement of a spin 1/2 particle (e.g. the electron) in the direction u for a particle with a spin selected (or better prepared) in the direction v.

What we have here is a precise model for a two-dimensional quantum system.

How are we going to encompass the classical situation in our model?

Well, to be sure, it is the breaking of the elastic that invokes the measurement apparatus into the transition probability, ao one way to make the transition to a classical regime is simply to limit the length where the elastic can break.

We will call the length where the elastic may break $2 \cdot \epsilon$ with $\epsilon \in [0, 1]$, and we will center this length around the center of the sphere.

As we can see (Fig. 3) the transition probability will break up in three pieces according to the place where the particle falls. If the particle falls outside the part where the elastic can break, it will evolve deterministically towards the closest fix point of the elastic. If it falls inside the region where the elastic can break, the former reasoning applies: the length of the piece of elastic that counts is $\epsilon - \cos\alpha$ and we have to divide by $2 \cdot \epsilon$ instead of dividing by 2 (which was the old length of the elastic).

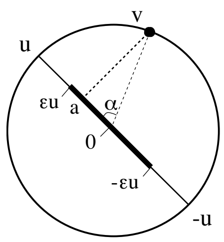

Fig. 3. A representation of the general situation. The elastic connecting the points $-u$ and u, is uniformly breakable between $-\epsilon u$ and ϵu, and unbreakable between $-u$ and $-\epsilon u$, and between ϵu and u.

We are now in a position to summarize the results:

1. if $\cos\alpha > \epsilon$, then $p(u \mid v) = +1$.
2. if $-\epsilon < \cos\alpha < \epsilon$, then $p(u \mid v) = (\epsilon - \cos\alpha)/2\epsilon$.
3. if $\cos\alpha < -\epsilon$, then $p(u \mid v) = 0$.

3 An example of a decision process

We will now try to show that the epsilon-model is capable of handling situations that are more general than a classical situation and, in fact even more general that a pure quantum statistical situation. In order to do so we must use a questionnaire of three questions. The necessity of using three questions to prove the inadequacy of classical probability theory is the content of the work of L. Accardi (Accardi and Fedullo, 1982) and I. Pitowsky (Pitowsky, 1989). Let us consider then a questionnaire consisting only of the following three questions:

- U: Are you a proponent of the use of nuclear energy? (*yes* or *no*)

- V: Do you think it would be a good idea to legalize soft-drugs? (*yes* or *no*)
- W: Do you think it is better for people to live in a capitalistic system? (*yes* or *no*)

The reason why we have chosen these questions is related to the fact that we want to isolate groups of people that are strong proponents and opponents for questions.

These groups will definitely not change their minds during a simple questionnaire. Yet the questions are also sufficiently complex, so that we can say that a large part of the total examined group (which is, of course, larger than the isolated groups of strong proponents and opponents) did not have an opinion before the question was posed. Although the questions are related, one can easily imagine that every possible combination of answers will be found if a sufficient amount of questionnaires are returned (e.g. 100 or more).

Let is make now the following somewhat arbitrary, but not a priori impossible assumptions about the probabilities:

We will say that in all cases 50% of the participants have answered question U with *yes*, and that only 15% of the total of persons where convinced of this choice before the question was actually posed. This means that 70% of the participants formed their answer at the moment the question was posed. For simplicity we shall make the same assumptions for questions V and W. We can picture this situation in the epsilon-model like this:

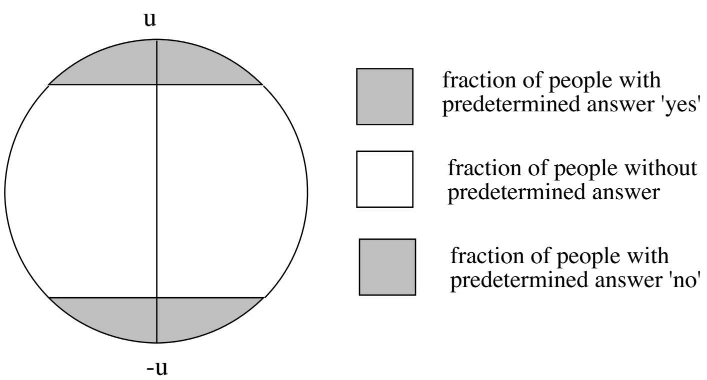

Fig. 4. A representation of the question U in the ϵ-model.

Next we will have to make some assumptions about the way the three questions are interrelated to one another. This too can be most easily visualized on the sphere of the epsilon-model. We have done this in Fig. 5 for questions U and V.

One can easily see how a person can be a strong proponent of the use of nuclear energy, while having no significant opinion about the legalization of soft-drugs (this corresponds to the area indicated with 1 in Fig. 5). Other persons have strong opinions in favor or against for both questions (areas 2 and 6 in Fig. 5). Still others have no prefixed opinion (area 4 in Fig. 5).

One may ask rightly to what extend these specific assumptions about the relation between the questions represent a real limitation to the use of the epsilon-model in a more general case. We shall postpone this discussion until the next section where we will show how some aspects (e.g. the apparent symmetry) are only presented here for the sake of simplicity, while others represent problems that need further investigation.

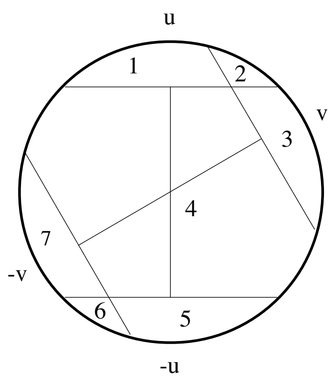

Fig. 5. A representation of the two questions U and V by means of the ϵ-model. We have numbered the 7 different regions. For example: (1) corresponds to the sample of persons that have predetermined opinion in favor of nuclear energy, but have no predetermined opinion for the other question: (2) corresponds to the sample of persons that have predetermined opinion in favor of nuclear energy and in favor of the legalization of soft drugs: (6) corresponds to a sample of persons that have predetermined opinion against the legalization of soft drugs and against nuclear energy: (4) corresponds to the sample of persons that have no predetermined opinion about none of the two questions, etc ...

Having said this we will assume the three questions to be related in the following manner:

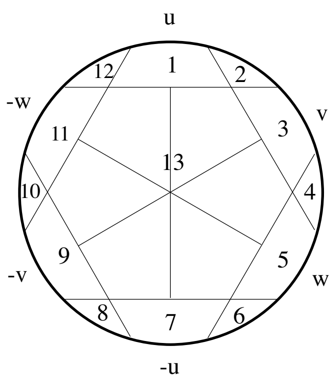

Fig. 6. A representation of the three questions U, V and W by means of the ϵ-model. We have numbered the 13 different regions. For example: (1) corresponds to the sample of persons that have predetermined opinion in favor of nuclear energy, but have no predetermined opinion for the other question: (4) corresponds to the sample of persons that have predetermined opinion in favor of nuclear energy and in favor of capitalism: (10) corresponds to a sample of persons that have predetermined opinion against the legalization of soft drugs and against capitalism: (13) corresponds to the sample of persons that have no predetermined opinion about none of the three questions, etc ...

So we could make three figures like Fig. 5: one for the relation between questions U and V, one for questions V and W and finally one for questions U and W.

We can see that questions V and W are related in much the same way as questions U and V; the only change is a rotation of 60 degrees around the center of the sphere.

The relation between questions U and W however, is slightly different, as can be seen in the next figure:

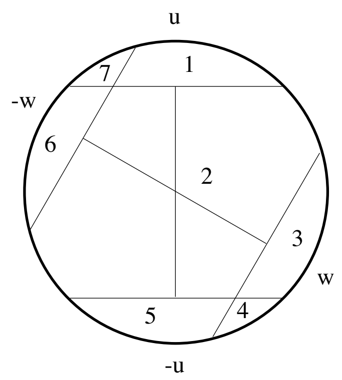

Fig. 7. A representation of the two questions U and W by means of the ϵ-model. We have numbered the 7 different regions. For example: (1) corresponds to the sample of persons that have predetermined opinion in favor of nuclear energy, but have no predetermined opinion for the other question: (2) corresponds to the sample of persons that have no predetermined opinion about none of the two questions: (4) corresponds to the sample of persons that have predetermined opinion against nuclear energy and in favor of capitalism: (7) corresponds to a sample of persons that have predetermined opinion in favor of nuclear energy and in favor of capitalism, etc ...

It will be clear that the questionnaire will have a more classical character if the fraction of people that had no opinion before the question (including those people that had an opinion, but changed it because of the questionnaire) is smaller. This fraction is modelled in the epsilon-model by means of epsilon. As indicated in section 2, *epsilon = 0* means a classical statistical situation, that is a situation where nothing changes as a result of the measurement, while *epsilon = 1* models a quantum statistical situation (the fraction of non-changing states have measure zero in this case).

In order to make a bridge between these two extremes, we need a concept of probability that involves both notions, *i.e.* a bridge between a classical conditional probability and a quantum transition probability. A suitable definition for such a probability is the following:

> $p(U = yes \mid V = yes)$ = the probability of somebody answering *yes* to question U, if we can predict that he will answer *yes* to question V

The important part of this new definition is the word "predict". Hereby we mean the group of strong proponents and opponents previously mentioned. So we have probability that is partly dependent on a reality that exists independent of the observer (... we can predict ...), and partly on the reality that is co-created by the observer.

It is easy to see that in the case of epsilon=1 this definition equals the transition probability, while in the case epsilon=0 the definition is the same as the old definition of the conditional probability. For a more thorough discussion, see Aerts (1995), Aerts (1996) and Aerts (1998).

The calculation of this conditional probability is of considerable length and complexity which is the main reason we have chosen not to include it in this article. For the interested reader we refer to Aerts (1995), Aerts (1996) and Aerts (1998).

The results of these calculations however are:

$p(U = yes \mid V = yes) = 0.78$
$p(U = yes \mid W = yes) = 0.22$
$p(V = yes \mid W = yes) = 0.78$

The compliment of these probabilities are, of course,

$p(U = no \mid V = yes) = 0.22$
$p(U = no \mid W = yes) = 0.78$
$p(V = no \mid W = yes) = 0.22$

We will now prove the following theorem:

Theorem: *The probabilities of the epsilon-model for this epsilon and this choice of U, V and W do not fit in a classical statistical framework.*

Phrased in this way it may seem as if it is only for this particular choice of parameters that the data do not fit a classical statistical model. In fact, most of the values of the epsilon-model do not fit in a classical framework. Mathematically we can extend the following proof which depends on the particular choices made here to a domain of values, by using the continuity of the calculated conditional probability. Let us now give the proof.

Proof: First of all we have:

$$P(U = yes) = P(U = no) = P(V = yes) = P(V = no) \quad (5)$$

$$= P(W = yes) = P(W = no) = \frac{1}{2} \quad (6)$$

and,

$$\mu(U_+) = \mu(U_-) = \mu(V_+) = \mu(V_-) = \mu(W_+) = \mu(W_-) = \frac{1}{2} \quad (7)$$

where $\mu(U_+)$ denotes the classical probability measure related to finding a *yes* to question U. If there exists a classical model, then we may use Bayes-axiom to calculate the conditional probability:

$$P(A \mid B) = \mu(A \cap B)/\mu(B) \quad (8)$$

Hence we have:

$$\frac{1}{2} \cdot P(V = yes \mid W = yes) = \mu(V_+ \cap W_+) \quad (9)$$

$$= \mu(U_+ \cap V_+ \cap W_+) + \mu(U_- \cap V_+ \cap W_+) \quad (10)$$

The right-hand side of the last two equations are found by decomposition over mutually exclusive events.

The above assumptions about the relations between the questions are equivalent to the following choices in the epsilon-model:

$$\epsilon = \sqrt{2}/2 \quad (11)$$

and

$$\alpha_U = 0, \alpha_V = \pi/3, \alpha_W = 2\pi/3 \tag{12}$$

We already mentioned that with these choices we have:

$$P(V = yes \mid W = yes) = 0.78 \tag{13}$$
$$P(U = yes \mid W = yes) = 0.22 \tag{14}$$

Thus we have:

$$\mu(U_+ \cap V_+ \cap W_+) + \mu(U_- \cap V_+ \cap W_+) = 0.39 \tag{15}$$

and,

$$\mu(U_+ \cap V_+ \cap W_+) + \mu(U_+ \cap V_- \cap W_+) = 0.11 \tag{16}$$

If we subtract the two equations , we find:

$$\mu(U_- \cap V_+ \cap W_+) = 0.28 + \mu(U_+ \cap V_- \cap W_+) \tag{17}$$

Because of the positive definite character of the probability measure:

$$\mu(U_- \cap V_+ \cap W_+) \geq 0.28 \tag{18}$$

On the other hand we have:

$$\frac{1}{2} \cdot P(U = no \mid V = yes) = \mu(U_+ \cap V_+) \tag{19}$$
$$= \mu(U_- \cap V_+ \cap W_+) + \mu(U_- \cap V_+ \cap W_-) \tag{20}$$

Because we also had:

$$P(U = no \mid V = yes) = 0.22 \tag{21}$$

We find:

$$\mu(U_- \cap V_+ \cap W_+) \leq 0.11 \tag{22}$$

Comparison of equations (18) and (22) gives us the required contradiction. □

We have also proven that this set of data cannot be incorporated in a purely quantum mechanical framework, that is, into a two dimensional Hilbert space. For the interested reader we refer again to Aerts (1995), Aerts (1996) and Aerts (1998). These two theorems could have been proven with the set of inequalities that Accardi has provided us with (Accardi and Fedullo, 1982) but we thought it would be interesting to present a specific proof.

4 Some problems related to the modelization

In this section we want to investigate the two following questions:

1. To what extent can the epsilon-model be enlarged as to encompass some more realistic situations?
2. What does all this mean for future inquiries? Considering the first question, we can point out the following serious restrictions in our example in section 3.

First of all, all data was symmetrical in what we considered here. This is only a superficial restriction. Indeed, the epsilon-model has been worked out for the non-symmetrical case as well. The conditional probability for the non symmetrical case was calculated by O. Lévêque.

Secondly, the questions we considered were two dimensional in nature, that is, only *yes* and *no* answers are possible. One could for example say: is not all this too far-fetched; suppose we add a third category saying "maybe", is not all resolved?

The answer is negative. The point is that a person may still change his mind, and whether this is from *"yes"* to *"no"* to *"maybe"* or from nothing to *"maybe"* does not alter the fact that there is a transition as a result of the measurement.

In addition we can reply that the epsilon-model can cope with more dimensional situations just as well, although we have to resort to a more mathematical equivalent so that the possibility of visualizing these situations on a three dimensional sphere is lost.

So, where does the epsilon-model show its weakness?

The weakness lies, of course, in the complexity of the human psychology. People may change their minds in a more or less irreversible way. For some people conviction is something you carry around the rest of your life, for others it is mere a game that can change from day to day.

This too could be incorporated in the model (in a statistical way of course) by means of another parameter that parametrizes the amount of induced change. We have called this parameter η, and although some preliminary work has been done, a whole area of investigation remains unexplored.

What does this all mean for future questionnaires?

Well, first of all we need to be fully aware of these facts. We can try to strategies: Firstly we may try to make these transitions relatively unimportant. We can do this by asking unimportant easy answers, but this is not a very satisfactory way of resolving the problem. Secondly we can try to create an atmosphere wherein a person is less likely to change mind: an impersonal questionnaire by a machine instead of an interview with a nice looking lady.

All this has already been tried.

Apart from this, we suggest another approach: investigation of the changing. Indeed, our mind was once a tabula rasa, and our very ideas and convictions are all formed as the result of these interactive changes. In order to understand better these interactive processes we will have to see humanity as a whole and not just an ensemble of individuals. We hope that this model even in its present rather limited form, can help in this kind of investigation.

5 Conclusions

Although much work needs to be done we have shown that the interactive transitions which take place in human interactions, are unlikely to be understood by means of classical probability theory. What we need is an interactive form of statistics, somewhere between quantum and classical probability. We have shown a first example of such a model and discussed some future routes to be followed.

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