Published as: Aerts, D., Coecke, B., Durt, T. and Valckenborgh, F., 1997, "Quantum, classical and intermediate I: a model on the Poincaré sphere", Tatra Mt. Math. Publ., **10**, 225.

Quantum, Classical and Intermediate I : a Model on the Poincaré Sphere.

Diederik Aerts*, Bob Coecke†, Thomas Durt‡ and Frank Valckenborgh

TENA⋆, Free University of Brussels,

Pleinlaan 2, B-1050 Brussels, Belgium.

Abstract. Following an approach, that we have called the hidden-measurement approach, where the probability structure of quantum mechanics is explained as being due to the presence of fluctuations on the measurement situations, we introduce explicitly a variation of these fluctuations, with the aim of defining a procedure for the classical limit. We study a concrete physical entity and show that for maximal fluctuations the entity is described by a quantum model, isomorphic to the model of the spin of a spin 1/2 quantum entity. For zero fluctuations we find a classical structure, and for intermediate fluctuations we find a structure that is neither quantum nor classical, to which we shall refer as the 'intermediate' situation.

1. Introduction.

Quantum theory is different from classical theories in many aspects. It entails a non-Kolmogorovian probability calculus (quantum probability), a non-Boolean propositional calculus (quantum logic), and a non-commutative measurement calculus (*-algebra's). Many aspects of these structural differences between quantum theory and classical theories have been investigated, and we refer to [13] for an overview. These different approaches all help to understand some of the still mysterious aspects of quantum theory, but even now, after so many years of intensive study, many parts of the theory remain obscure.

* Senior Research Associate of the National Fund for Scientific Research

† Research Assistant of the National Fund for Scientific Research

‡ Research Assistant of the Inter-University Institute for Nuclear Sciences

⋆ Supported by Flanders' Federale Diensten voor Wetenschappelijke, Technische en Culturele Aangelegenheden in the framework of the IUAP-III n°9

In our group in Brussels we are developing an approach where the quantum structures arise as a consequence of the presence of fluctuations on the measurement situations. In our approach the indeterminism of the measurement process is due to the fact that a measurement e consists of different possible sub-measurements $e(x)$ that are not distinguished macroscopically. We have named these $e(x)$ 'hidden measurements' in analogy with the 'hidden states' appearing in ordinary hidden variable theories. During a measurement process corresponding to the measurement e, one of these hidden measurements $e(x)$ actually takes place, and each one of these $e(x)$ is deterministic. Therefore the probabilistic aspect of the measurement e finds its origin in the 'lack of knowledge' about which one of the hidden measurements $e(x)$ is actually performed. This x can also be seen as a hidden variable of the measurement apparatus corresponding to e, and the hidden variable theory that results in this way, not describing hidden states but hidden measurements, is 'contextual' by definition. That is the reason why the quantum mechanical models that we have been able to build in our approach do not conflict with the well-known no-go-theorems about hidden variable descriptions. Another, equivalent, way of expressing this situation consists of saying that there are fluctuations present on the measuring apparatuses connected to the measurement e, and these fluctuations are described by the random variable x.

Earlier we have shown that the introduction of fluctuations on the measuring apparatuses allows us to build a hidden-measurement-model for an arbitrary finite-dimensional Hilbert space quantum entity [4, 5], and we have proposed a mechanistic physical entity that delivers a model for a two-dimensional Hilbert space quantum entity, namely a spin 1/2 model [4, 5]. The quantum probability model of this example has been studied in detail [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], and apart from being a hidden-measurement representation that can be realized in the laboratory, in also entails a representation of the two dimensional complex Hilbert space in a three dimensional real Euclidean space. That similar hidden-measurements representations in real Euclidean spaces can be constructed for more than two dimensional complex Hilbert spaces is shown in [15]. In [7] it is shown that also more general probability structures then the ones encountered in quantum mechanics can be realized by explicit hidden-measurement models, and this result is generalized in [16] for situations with experiments with an infinite numer of outcomes. It is also shown that these hidden-measurement-models give rise to a quantum logical propositional structure [9, 10].

If the quantum structure can be explained by the presence of fluctuations on the measuring apparatuses, we can go a step further, and wonder what types of structure arise when we consider the original models, with fluctuations on the measuring apparatuses, and introduce a variation of the magnitude of these fluctuations. In this paper we shall study the 'sphere-model' under varying fluctuations, parameterizing these variations by a number $\epsilon \in [0,1]$, such that $\epsilon = 1$ corresponds to the situation of maximal fluctuations, giving rise to a quantum structure, and $\epsilon = 0$ corresponds to the situation of zero fluctuations, generating a classical structure[1], and other values of ϵ correspond to intermediate situations, giving rise to a probability structure that is neither quantum nor classical.

Since these intermediate situations generate a structure that is neither quantum nor classical, we want to study them by using a mathematical framework capable of representing the quantum structures as well as the classical structures. The quantum logic approaches to quantum mechanics [13] are well fitted for such a program. In [12] we will study the collection of eigenstate-sets of our sphere model in the intermediate situation, in parallel with the study of a general entity. We shall show that the collection of eigenstate-sets of an arbitrary entity always corresponds to a closure structure on the set of states. We also study this closure structure for the specific case of the sphere model. We show that ϵ defines a continuous evolution from the linear closure in vector space to the standard topological closure.

The aim of this paper is to study in detail the sphere-model (to which corresponds a real physical entity), that we have introduced in the earlier papers for explaining the probabilities in quantum mechanics, and to see that the newly introduced intermediate situations give rise to certain structures that are neither quantum nor classical. In this way we have built a macroscopical physical entity with a non-classical, and non-quantum structure.

2. Conceptual basis.

The basic concepts of our approach will be 'measurements' to be performed on a physical entity and 'states' of this physical entity. A measurement is a physical procedure that can be carried out with the physical entity under consideration, and that leads to a recognizable and identifiable 'outcome'. To state clearly what we mean by the concept of 'state of a physical entity' is somewhat more difficult, because different concepts of state are used in physics.

[1] We shall show in [12] that the characterization of the classical situation is somewhat more complex.

Even in the quantum formalism itself two mathematical concepts of state are used, and several different interpretations are given. There is the so called 'pure state' of a quantum entity, represented in the quantum formalism by the unit vector (or the ray corresponding to this vector) of the complex Hilbert space, and there is the 'mixed state' of the quantum entity, represented by a density operator on the Hilbert space. When we use the 'state of a physical entity' we have in mind the 'pure state', because we consider the 'mixed state' to be a derived physical concept, namely a statistical mixture of pure states. As is well known, also for the 'pure state' of a quantum entity does not exist a consensus about its interpretation. We adopt a realistic interpretation where the state of a physical entity at a certain time represents the 'reality' of this physical entity at this time[2].

The set of possible states that we consider for the physical entity S shall be denoted by Σ. The state can change under influence of a measurement e, and we call this a measurement process. The set of measurements that we consider in relation with the physical entity S shall be denoted by $\mathcal{E}$. With each measurement e corresponds an outcome-set O_e, and when O_e is a finite set we denote it by $\{o_1, ..., o_n\}$. If the entity S is in a state p, and the measurement e is performed, the state of the entity can change in different ways, and after the measurement the entity can be encountered in different states. The probability that the state p is changed by the measurement e into the state q shall be denoted by $P_e(q \mid p)$ and we call it the transition probability of p to q by e.

Thus, in this paper and in particularly in [12], we develop a mathematical framework with experiments with more than two outcomes. This we do to incorporate the critique of

[2] This is how effectively the concept of 'state of an entity' is used in ordinary language. When we speak of the 'state' of the economy today, then we refer to 'how' the economy 'is' today. In earlier times, before the birth of quantum mechanics, physicists had no problem to manipulate this 'realistic' concept of 'state'. Because of the many interpretative problems related to quantum mechanics, and the widespread belief that a realistic interpretation was impossible, many different definitions of 'state' have been put forward. Meanwhile however, it has been shown that a realistic interpretation is possible [1][2][3][8][11][18][19][20][21], and that the pure state of a quantum entity at a certain time can be interpreted as representing part of 'what is' at this time. Since the example that we shall introduce is a physical example, we shall use without any possible confusion this realistic interpretation for the concept of state.

[14]. Hence in our approach the entity S is described by two sets:

$$\Sigma = \{p \mid p \text{ is a possible state of the entity } S\} \tag{1}$$

$$\mathcal{E} = \{e \mid e \text{ is a measurement on the entity } S\} \tag{2}$$

and with each measurement e corresponds a transition probability:

$$P_e : \Sigma \times \Sigma \to [0, 1] \tag{3}$$

where $P_e(q \mid p)$ is the probability that, the entity S being in state p, a performance of the measurement e results in a state transition to the state q[3].

Let us now indicate how this structure is encountered in quantum mechanics. In quantum mechanics the states are described by the use of a complex Hilbert space $\mathcal{H}$, such that a state p of the quantum entity S is represented by a ray (one dimensional subspace), denoted by $\bar{\psi}_p$, where ψ_p is the unit vector in this ray. The set of all rays is denoted by $\Sigma_{\mathcal{H}}$. A measurement e is represented by a self-adjoint operator H_e on this Hilbert space $\mathcal{H}$, and the outcome-set O_e of the measurement e is given by the spectrum σ_e of this self-adjoint operator H_e. Let us denote the collection of all self-adjoint operators by $\mathcal{E}_{\mathcal{H}}$. To find the way in which the measurement e generates a state transition we remark that by the spectral theorem, every self-adjoint operator H_e is completely determined by its spectral measure : $Proj : \mathcal{B}(\sigma_e) \to \mathcal{P}(\mathcal{H})$, where with $A \in \mathcal{B}(\sigma_e)$, corresponds an orthogonal projection $Proj_A$, element of the set of all orthogonal projections $\mathcal{P}(\mathcal{H})$, and $\mathcal{B}(\sigma_e)$ is the collection of Borel-sets of the set of real numbers σ_e, the spectrum of the operator H_e. If the entity S is in a state $\bar{\psi}_p$ before the measurement, and a performance of the measurement e gives an outcome in a set $A \in \mathcal{B}(\sigma_e)$, then the state $\bar{\psi}_p$ is changed into the state which is represented by the ray generated by the vector $Proj_A(\psi_p)$, let us denote it by $\bar{\psi}_q$. The transition probability $P_e(\bar{\psi}_q \mid \bar{\psi}_p)$ corresponding to this change of state is $|< Proj_A(\psi_p) \mid \psi_p >|=|< \psi_q \mid \psi_p >|^2$ In this way we have identified for the quantum entity S the representation of the set of states by $\Sigma_{\mathcal{H}}$, the set of measurements by $\mathcal{E}_{\mathcal{H}}$, and the transition probability by $|< \psi_q \mid \psi_p >|^2$.

3. The sphere-model.

[3] See also [1][2][18][22]. We could give a description in the general formalism of Foulis and Randall, and shall point out the way to do this in a forthcoming paper. Since our main aim in this paper is to show the intermediate, non-quantum and non-classical character of the model, we have limited the formalism to fulfill these specific needs.

Let us now introduce our model and the physical example representing it. We first introduce the model, and then indicate which specific physical example can be shown to correspond to it.

The physical entity that we consider is a point particle P that can move on the surface of a sphere with center O and radius 1, and we shall denote this surface by $surf$. This particle P is our physical entity S (see Fig. 1). In our model of the point particle we consider the unit-vector v where the particle is located on $surf$ at a certain instant of time t as representing the state of this particle at time t, its place on the surface of the sphere, that we shall denote by p_v. Hence $\Sigma = \{p_v \mid v \in surf\}$.

Let us now introduce the measurements. We consider two diametrically opposite points u and $-u$ on the surface of the sphere. We shall systematically denote by $[-u, u]$ the 'interval' of real numbers $[-1, 1]_u$, coordinating the points of the line between u and $-u$ in such a way that -1 coordinates $-u$ and 1 coordinates u this to alleviate the notations. The measurement e_u consists of the following happening: the particle P falls from its original place v orthogonally onto the line between u and $-u$, and arrives in a point a (see Fig 1), coordinated in the interval $[-u, u]$ by the real number $v \cdot u$. In the interval $[-u, u]$ we consider a random variable x, and the measurement proceeds as follows. If $x \in [-u\ ,\ v \cdot u[$, the particle P, being in the point a, moves to the point u, and we say that the measurement e_u gives outcome $o_{u,1}$. If $x \in]v \cdot u\ ,\ u]$, the particle P moves to the point $-u$, and we say that the measurement e_u gives outcome $o_{u,2}$. If $x = v \cdot u$ the particle P moves with probability $\frac{1}{2}$ to the point $-u$, and then the measurement e_u gives outcome $o_{u,1}$, and it moves with probability $\frac{1}{2}$ to the point u, and then the measurement e_u gives outcome $o_{u,2}$. This completes the description of the measurement e_u.

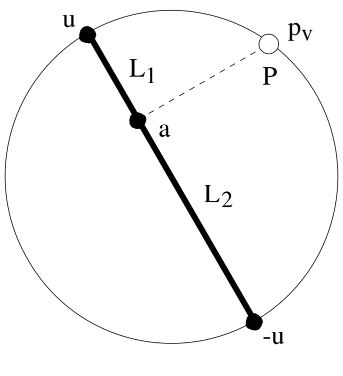


Fig. 1 : A point particle P is in a state p_v at the point v of the surface of the sphere. The measurement e_u consists of considering the line L joining the point u of the surface of the sphere with the point $-u$. When the measurement starts the particle P falls from v orthogonally onto this line in a point a, and then moves on to the point u or to the point $-u$ depending on the value of a random variable x. This random variable is defined in an interval $[-1,+1]$, of which the numbers coordinate the points of the line L between u and $-u$, such that -1 coordinates $-u$ and $+1$ coordinates u. If x coordinates a point between a and $-u$ then the particle P moves to the point u, and the measurement e_u gives outcome $o_{u,1}$, while if x coordinates a point between a and $-u$, then the particle P moves to the point $-u$, and the measurement e_u gives the outcome $o_{u,2}$.

We can see that the measurement e_u transforms the state p_v into a new state p_u if outcome $o_{u,1}$ occurs, or a state p_{-u} if outcome $o_{u,2}$ occurs. This shows that our measurement e_u indeed defines a state-transition, but clearly this change of state is not deterministic, in the sense that the original state p_v can be changed into two different states p_u or p_{-u}. The transition probabilities $P_{e_u}(p_u \mid p_v)$ and $P_{e_u}(p_{-u} \mid p_v)$ connected with either of these two possible changes by the measurement e_u (p_v into p_u, or p_v into p_{-u}) depend on the distribution of the random variable $x \in [-u, u]$.

Before we explore the model in more detail, let us show that it can be realized in the laboratory by means of a very simple physical entity and very simple measurements to be performed on this entity. Therefore we suppose that the particle P is a material point particle, located on the sphere. The measurement e_u happens by means of a piece of elastic of length 2. The piece of elastic is fixed, with one of its end-points in the point u of the surface and the other end-point in the diametrically opposite point $-u$ (Fig 1). Once the elastic is placed, the material particle P falls from its original place v orthogonally onto the elastic, and sticks on it in the point a. Then the piece of elastic breaks. If we consider the two parts of the elastic, the part L_1 from a to u, and the part L_2 from a to $-u$, it must break in a point of one of these two parts. If it breaks in L_1, the particle P will be drawn to the point $-u$ by the elastic still connected to it, and we will say that the measurement e_u gives outcome $o_{u,2}$. If it breaks in L_2, the particle P will be drawn to the point u by the elastic still connected to it, and we will say that the measurement e_u gives outcome $o_{u,1}$. If it breaks exactly in the point a itself, the particle P can go both ways. We suppose that in this case the particle P goes to the point u with probability $\frac{1}{2}$, and to the point $-u$ with probability $\frac{1}{2}$. It is clear that this 'sphere-elastic-example' is a physical realization of the sphere model. It is important to know that such a realization exist, because it shows that the sphere model is not just a mathematical construction. Other physical realizations can be invented, such as the charge-example put forward in [4]. It is also clear by means of this example, that the hidden-variable x can be interpreted as describing the (hidden) internal construction of the measuring apparatus, in this case the mechanism of breaking of the piece of elastic. Different models can be proposed for this mechanism of breaking of the elastic, all corresponding in principle to different types of elastic, but we shall first of all consider the most simple type of elastic, and show that it makes the sphere-model a model for a two-dimensional Hilbert space quantum mechanical entity (e.g. the spin of a particle with spin 1/2).

3.1 The uniform sphere model.

Let us suppose that for an arbitrary measurement e_u, the hidden-variable $x \in [-u, u]$ is uniformly distributed in the interval $[-u, u]$. We can calculate easily the transition probabilities related to this situation.

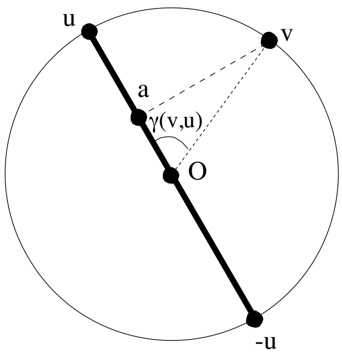


Fig. 2 : The four points u,$-u$, v and a are situated in a plane. If we denote by $\gamma(v,u)$ the angle between the two vectors v and u, then we can easily see that $a{=}cos\gamma(v,u){\cdot}u$. In this way we can calculate the lengths of L_1 and L_2, and also the transition probabilities connected with the possible changes of states under influence of the measurement e_u.

We see that the three points v, u and $-u$ are situated in a plane through the diameter of the sphere (see Fig. 2). Also the point a is in this plane, which means that the point P moves in this plane. Since x is uniformly distributed in the interval $[-u, u]$, we have that $P_{e_u}(p_u \mid p_v)$ is given by the length of $[-u\ ,\ v \cdot u]$ divided by the length of the total interval $[-u, u]$, which equals 2. Before we calculate the length of $[-u\ ,\ v \cdot u]$, we introduce the angle $\gamma(v, u)$ between two space-vectors v and u, such that,

$$cos\gamma(v, u) = v \cdot u \tag{4}$$

Let us now proceed the calculation of the probabilities. We have $a = cos\gamma(v, u) \cdot u$, and hence $|[-u\ ,\ v \cdot u]| = |u + a| = |u + cos\gamma(v, u) \cdot u| = (1 + cos\gamma(v, u))$. From this follows that :

$$P_{e_u}(p_u \mid p_v) = \frac{1}{2}(1 + cos\gamma(v, u)) \tag{5}$$

In an analogous way we find :

$$P_{e_u}(p_{-u} \mid p_v) = \frac{1}{2}(1 - cos\gamma(v, u)) \tag{6}$$

These transition probabilities are the same as the ones related to the outcomes of a Stern-Gerlach spin measurement on a spin 1/2 quantum particle, of which the quantum-spin-state in direction $v = (cos\phi sin\theta,\ sin\phi sin\theta, cos\theta)$, denoted by $\bar{\psi_v}$, is represented by the vector

$$\psi_v = (e^{-i\phi/2}cos\theta/2, e^{i\phi/2}sin\theta/2) \tag{7}$$

of the two-dimensional complex Hilbert space, and the measurement e_u corresponding to the spin measurement in direction $u = (cos\beta sin\alpha,\ sin\beta sin\alpha, cos\alpha)$ by the self adjoint operator

$$H_u = \frac{1}{2}\begin{pmatrix} cos\alpha & e^{-i\beta}sin\alpha \\ e^{i\beta}sin\alpha & -cos\alpha \end{pmatrix} \tag{8}$$

on this Hilbert space. Indeed, let us calculate for example the quantum transition probability $P(\bar{\psi_u} \mid \bar{\psi_v})$ of the quantum state $\bar{\psi_v}$ into the quantum state $\bar{\psi_u}$, where ψ_u is the eigenstate of the self-adjoint operator H_u with eigenvalue $+\frac{1}{2}$. Following the rules of quantum mechanics, this transition probability is given by the square of the absolute value of the in-product in this two-dimensional complex Hilbert space, and hence :

$$\begin{aligned}
P_{e_u}(\bar{\psi_u} \mid \bar{\psi_v}) &= \mid <\psi_v \mid \psi_u> \mid^2 \\
&= \mid < e^{-i\phi/2}cos\theta/2, e^{i\phi/2}sin\theta/2 \mid e^{-i\beta/2}cos\alpha/2, e^{i\beta/2}sin\alpha/2 > \mid^2 \\
&= |e^{i\frac{\phi-\beta}{2}} \cdot cos\frac{\theta}{2} \cdot cos\frac{\alpha}{2} + e^{-i\frac{\phi-\beta}{2}} \cdot sin\frac{\theta}{2} \cdot sin\frac{\alpha}{2}|^2 \\
&= |cos\frac{\phi-\beta}{2} \cdot cos\frac{\theta-\alpha}{2} + isin\frac{\phi-\beta}{2} \cdot cos\frac{\theta+\alpha}{2}|^2 \\
&= cos^2\frac{\phi-\beta}{2} \cdot cos^2\frac{\theta-\alpha}{2} + sin^2\frac{\phi-\beta}{2} \cdot cos^2\frac{\theta+\alpha}{2} \\
&= \frac{1}{4}((1+cos(\phi-\beta)) \cdot (1+cos(\theta-\alpha)) + (1-cos(\phi-\beta)) \cdot (1+cos(\theta+\alpha))) \\
&= \frac{1}{2}(1+cos(\phi-\beta) \cdot sin\theta \cdot sin\alpha + cos\theta \cdot cos\alpha) \\
&= \frac{1}{2}(1+v \cdot u) \\
&= \frac{1}{2}(1+cos\gamma(v,u))
\end{aligned} \tag{9}$$

In an analogous way we can calculate the transition probability $P_{e_u}(\bar{\psi_{-u}} \mid \bar{\psi_v})$ of the quantum state $\bar{\psi_v}$ to the quantum state $\bar{\psi_{-u}}$. We get :

$$\begin{aligned}
P_{e_u}(\bar{\psi_{-u}} \mid \bar{\psi_v}) &= \mid <\psi_v \mid \psi_{-u}> \mid^2 \\
&= \mid < e^{-i\phi/2}cos\theta/2, e^{i\phi/2}sin\theta/2 \mid -ie^{-i\beta/2}sin\alpha/2, ie^{i\beta/2}cos\alpha/2 > \mid^2 \\
&= \frac{1}{2}(1-cos\gamma(v,u))
\end{aligned} \tag{10}$$

Hence the quantum probabilities $P(\bar{\psi_u} \mid \bar{\psi_v})$ and $P(\bar{\psi_{-u}} \mid \bar{\psi_v})$ are the same as the transition probabilities $P_{e_u}(p_u \mid p_v)$ and $P_{e_u}(p_{-u} \mid p_v)$ calculated in our sphere-elastic-model. This shows the 'equivalence' of the sphere-elastic-model with the quantum model of the spin of a spin $\frac{1}{2}$ particle.

Definition 1. *A sphere model where the hidden variable is distributed uniformly shall be called a 'uniform' sphere model.*

Theorem 1. *The uniform sphere model is a model for a quantum entity described in a two dimensional Hilbert space.*

If we consider again the physical elastic-sphere-example, that constitutes a physical realization of the sphere model, then it is easy to see the uniform sphere model is realized if we use elastics where the probability of breaking in a certain segment of the elastic is proportional to the length of this segment. From theorem 1 follows that the physical example with this types of elastics is a macroscopic physical model for a quantum entity described in a two dimensional Hilbert space.

3.2 The general sphere model.

We can easily imagine elastics that break in different ways depending on their physical construction or on other environmental happenings. These different elastics shall correspond to the situation where the hidden variable $x \in [-u, u]$ is distributed in a non-uniform way. We can describe such a general situation by a distribution ρ, (that for the physical elastic-sphere-example describes the probability distribution of 'breaking' of the elastic), in the following way:

$$\rho : [-u, u] \to [0, +\infty[\tag{11}$$

such that

$$\int_{\Omega} \rho(x)dx \tag{12}$$

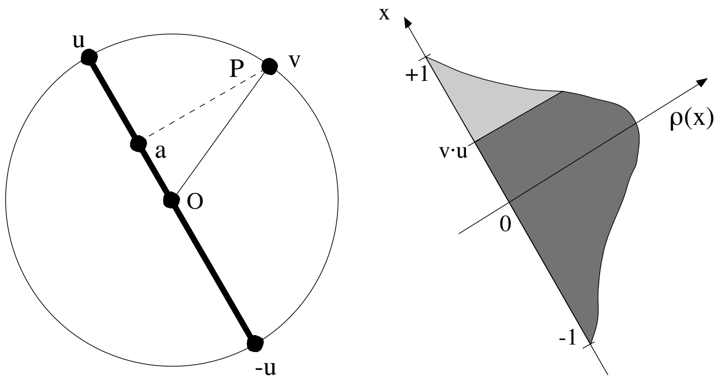
is the probability that the hidden variable $x \in \Omega \subset [-u, u]$ (in case of the physical elastic-sphere-example this is the probability that the elastic breaks in Ω). We also have :

$$\int_{[-u,u]} \rho(x)dx = 1 \tag{13}$$

(which in the case of the physical elastic-sphere-example expresses the fact that the elastic always breaks during a measurement).

Definition 2. *A measurement e_u where the hidden variable $x \in [-u, u]$ is distributed as described by ρ shall be called a ρ-measurement and denoted by e_u^ρ.*

In figure 3 we have represented such a measurement e_u^ρ, and drawn the probability distribution function $\rho(x)$ such that the definition interval of this function coincides with the line joining the points u and $-u$. The transition probability that the particle P arrives at point u under the influence of the measurement e_u^ρ, that we denote by $P_\rho(p_u \mid p_v)$, is given by the integral over ρ from -1 to $v \cdot u$, which is represented by the dark-gray area on figure 3. In an analogous way the light-gray area on figure 3 represents the transition probability, $P_\rho(p_{-u} \mid p_v)$, that the particle P arrives at the point $-u$ under the influence of the measurement e_u^ρ.



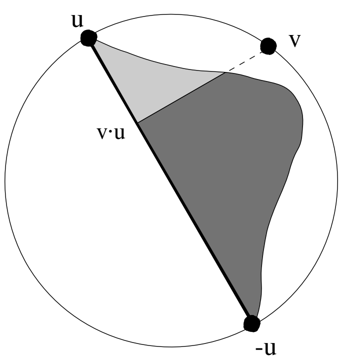


Fig. 3 : We have represented a measurement e_u^ρ, and drawn the probability distribution $\rho(x)$ of the hidden variable x, such that the definition interval of this distribution coincides with the line joining the two points u and $-u$. The probability that the particle P arrives in the point u is given by the area contained under the function ρ from -1 to $v \cdot u$, the dark-gray area on the figure, and the probability that the particle P arrives at the point $-u$ is given by the area from $v \cdot u$ to $+1$, the light-grey area on the figure.

$$P_\rho(p_u \mid p_v) = \int_{-1}^{v \cdot u} \rho(x) dx \tag{14}$$

$$P_\rho(p_{-u} \mid p_v) = \int_{v \cdot u}^{+1} \rho(x) dx \tag{15}$$

A uniform sphere-model is a special case of a general ρ-sphere-model, where ρ is the constant function $\frac{1}{2}$, and, as we have shown, this type of sphere-model generates the quantum situation (fig. 4).

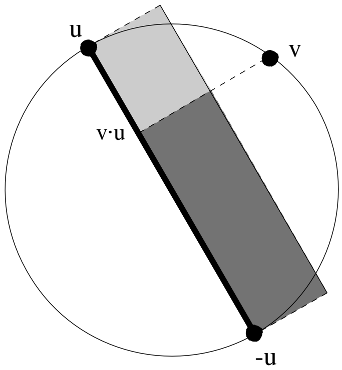


Fig. 4 : A representation of the uniform probability distribution corresponding to the uniform sphere model. It generates a model for a two-dimensional Hilbert space quantum entity.

3.3 The classical-deterministic sphere model.

We have considered the general sphere model, where the transition probabilities are described by a probability distribution ρ, and seen how this general sphere model becomes a two dimensional quantum model when the probability distribution is a constant function, and this situation we have indicated by the uniform sphere model. The other extreme is given by the situation where the hidden variable $x \in [-u, u]$ is fixed in a point $d \in [-u, u]$, and hence described by the probability distribution $\rho(x) = \delta(x - d)$. In this situation only in the case where P falls exactly onto d the measurement process is indeterministic, and all other cases give rise to a determined outcome (for the physical elastic sphere example this corresponds to the situation where the elastic can only break in the point d). This is the situation that we shall consider as the classical-limit case, where all fluctuations on the measurement apparatuses have disappeared.

Now that we have identified the quantum situation (uniform measurements), and the classical situation (deterministic measurements), we can see easily that the general situation of a ρ-sphere model represents an intermediate situation. We shall study in the following these intermediate situations and see that they are neither classical nor quantum.

3.4 Introduction of the parameters ϵ and d.

The intermediate situation that we want to study in this paper corresponds with a situation where the hidden variable x is distributed uniformly in an interval $[d - \epsilon, d + \epsilon] \subset [-u, u]$.

Hence the probability distribution describing this situation is given by:

$$\rho^\epsilon = \frac{1}{2\epsilon} X_{[d-\epsilon, d+\epsilon]} \tag{16}$$

where $X_{[a,b]}$ is the characteristic function of the interval $[a, b] \subset [-u, u]$, and

$$d \in [-1 + \epsilon, 1 - \epsilon] \tag{17}$$

We have represented, along the lines of the other figures, this measurement in figure 5, and chosen a case where $d = 0.2$ and $\epsilon = \frac{1}{2}$. If we consider again the elastic sphere model, then this situation corresponds to an elastic that breaks in a uniform way in the interval $[d - \epsilon, d + \epsilon]$, while it is unbreakable in the intervals $[-1, d - \epsilon]$ and $[d + \epsilon, 1]$. We have represented in figure 5 two cases. If v is such that $v \cdot u \in [d - \epsilon, d + \epsilon]$, the measurement e_u is still indeterministic, in the sense that after the measurement the point P being in v can end up in u, and then the state p_v has been changed into the state p_u, or it can end up in $-u$, and then the state p_v has been changed into the state p_{-u}. If v is such that $v \cdot u \notin [d - \epsilon, d + \epsilon]$, then the measurement e_u is deterministic, and the point particle P located in v ends up with certainty in u, or with certainty in $-u$.

Definition 3. *A measurement e_u with a hidden variable x distributed uniformly in the interval $[d - \epsilon, d + \epsilon] \subset [-u, +u]$ shall be called an ϵ-measurement, and denoted by $e^\epsilon_{u,d}$.*

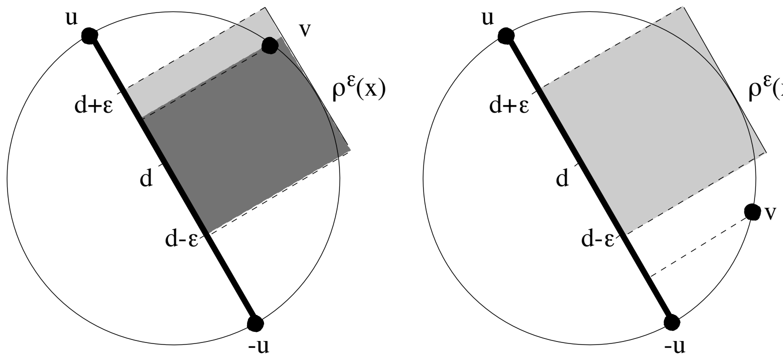


Fig. 5 : A representation of the measurement $e^\epsilon_{u,d}$ and a drawing of the probability distribution function connected to it. We have chosen a case where d=0.2, and ϵ=$\frac{1}{2}$. If we consider the sphere-example realizing this situation, then the elastic breaks uniformly inside the interval $[d-\epsilon, d+\epsilon]$, and is unbreakable outside this interval, in the points of the set $[-1, d-\epsilon] \cup [d+\epsilon, 1]$.

We can analyze the change of states by a measurement e^ϵ_u immediately from figure 5, but we can also calculate the probabilities, using (18) and (19). We then have :

$$P_\epsilon(p_u \mid p_v) = \int_{-1}^{v \cdot u} \rho^\epsilon(x) dx \tag{18}$$

$$P_\epsilon(p_{-u} \mid p_v) = \int_{v\cdot u}^{+1} \rho^\epsilon(x)dx \tag{19}$$

and can consider different cases :

1. $d + \epsilon \leq v \cdot u$.

Then $P_\epsilon(p_u \mid p_v) = 1$ and $P_\epsilon(p_{-u} \mid p_v) = 0$.

2. $d - \epsilon < v \cdot u < d + \epsilon$

We have :

$$\begin{aligned} P_\epsilon(p_u \mid p_v) &= \int_{d-\epsilon}^{v\cdot u} \frac{dx}{2\epsilon} \\ &= \frac{1}{2\epsilon}(v \cdot u - d + \epsilon) \end{aligned} \tag{20}$$

and

$$\begin{aligned} P_\epsilon(p_{-u} \mid p_v) &= \int_{v\cdot u}^{d+\epsilon} \frac{dx}{2\epsilon} \\ &= \frac{1}{2\epsilon}(d + \epsilon - v \cdot u) \end{aligned} \tag{21}$$

3. $v \cdot u \leq d - \epsilon$

Then $P_\epsilon(p_u \mid p_v) = 0$ and $P_\epsilon(p_{-u} \mid p_v) = 1$.

We have represented the transition probability $P_\epsilon(p_u \mid p_v)$ as a function of $\gamma(v, u)$ and ϵ for the case $d = 0$ in figure 6, and then easily see how the quantum-situation ($\epsilon = 1$) transforms into the classical situation ($\epsilon = 0$).

For $\epsilon = 1$ we are in the situation of a uniform sphere model, and $\rho^1(x) = \frac{1}{2}$, and the sphere model is isomorphic to a two-dimensional Hilbert space model. We can also see that $lim_{\epsilon\to 0}\rho^\epsilon(x) = \delta(d - x)$, such that for $\epsilon = 0$ we are in the situation of a classical sphere model, without fluctuations.

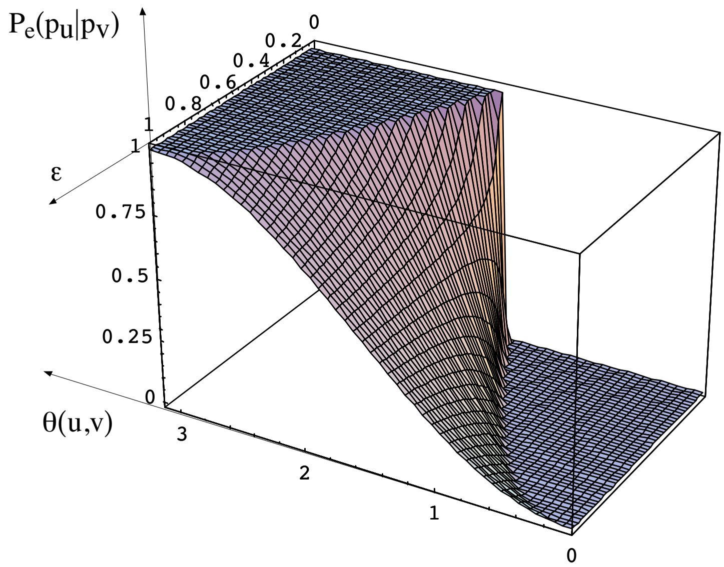


Fig. 6 : A graphical representation of the transition probability $P_e(u \mid v)$ for different angles $\theta(u,v) = \pi - \gamma(u,v)$, and for different values of ϵ. One can see on the drawing how the quantum situation evolves for vanishing fluctuations to the classical situation.

4. Conclusion.

We have investigated the ϵ-example in a very detailed way, and we have shown that for varying ϵ the amount of quantum behavior present in the example changes. For $\epsilon = 1$, corresponding to maximal fluctuations on the measurement apparatuses, we find a model for the spin of a spin $\frac{1}{2}$ quantum entity. For $\epsilon = 0$, corresponding to zero fluctuations, we find a classical situation. In between we have intermediate situations, neither classical nor quantum.

The ϵ-model that we have proposed in this paper describes a class of intermediate situations (for each value of ϵ) but is not yet the general study of the 'intermediate' situation. Indeed, we have chosen one parameter ϵ for all the measurements $e^{\epsilon}_{u,d}$, such that the amount of fluctuations (the amount of hidden measurements) for each measurement is determined by the value of this ϵ. In principle this amount of fluctuations could be determined by different parameters, and even in the extreme case by one parameter per measurement (there would be then a class of parameters ϵ_u for each point $u \in surf$). For such a more general description, there are intermediate situations of a different nature than the ones that we have described in this paper. In [12] we will outline an approach that is sufficiently general to study also these types of intermediate situations. The collection of eigen-state sets is then determined by a closure structure on the state space.

With the sphere model corresponds a realisable physical entity, e.g. the elastic-sphere example, whereon we have defined realisable physical measurements, that could eventually be executed in the laboratory, and will then generate experimental data fitting into a quantum mechanical structure. We are well aware of the strong existing paradigm stating that quantum-reality is intrinsically different from classical macroscopic reality, and in this way forbidding to imagine quantum structures existing inside this macroscopic reality. It seems that our elastic example confronts this paradigm, and indeed it does, and to avoid prejudices rising immediately we must explain exactly in which way it does.

It is difficult in principle to imagine the physical entity S, the point particle P, at any state being located in a point of the sphere[4], as a quantum entity. But this difficulty comes from the fact that we, in our imagination, in preparing and identifying the state of this physical entity S automatically use measurements other than the e_u^ρ that we have introduced. Usually we 'see' the point on the sphere and suppose unconsciously that the 'seeing' experiment is a measurement that is available to us for the study of the entity S that is the point particle P on the surface of the sphere. What we should do is to imagine the physical entity S, hence the point particle P on the sphere, as only knowable to us by means of the measurements e_u^ρ that we have introduced. This is indeed exactly the situation that we encounter when we make investigations about quantum entities in the micro-world. We cannot 'see' or 'touch' these entities, and have only knowledge about them by means of the measurements that we can carry out on them. This remark makes it possible for us to state exactly the philosophy of our approach. If we state that the entity S is described by the sets Σ and $\mathcal{E}$, then we also suppose that the elements of $\mathcal{E}$ are the only measurements available to us for the identification and preparation of the states Σ of the entity S. It is in this light that our classification scheme, quantum, classical and intermediate, has to be understood. And we believe that it is the correct way of approaching the knowledge that we have about the reality of the micro-world.

5. References.

[1] Aerts, D. (1981), *"The one and the many'*, Doctoral Dissertation, Free University of Brussels, Pleinlaan 2, 1050 Brussels

[4] Preparation and identification are different aspects of a measurement in quantum mechanics. It can be seen easily that for the elastic sphere example both aspects model a two dimensional quantum entity [6].

[2] Aerts, D. (1982), *'Description of separated physical entities without the paradoxes encountered in quantum mechanics'*, Found. Phys. **12**, 1131.

[3] Aerts, D. (1983), *'The description of one and many physical systems'*, in *Les Fondements de la Mécanique Quantique*, ed. C. Gruber et al, A.V.C.P., case postale 101, 1015 Lausanne, Suisse.

[4] Aerts, D. (1986), *'A possible explanation for the probabilities of quantum mechanics'*, J. Math. Phys. **27**, 203.

[5] Aerts, D. (1987), *'The origin of the non-classical character of the quantum probability model'*, in *"Information, Complexity and Control in Quantum Physics"*, ed. Blanquiere, A., et al., Springer Verlach.

[6] Aerts, D. (1991), *'A macroscopic classical laboratory situation with only macroscopic classical entities giving rise to a quantum mechanical description'*, in *"Quantum Probabilities, volume VI"* ed., Accardi, L., World Scientific.

[7] Aerts, D. (1994), *'Quantum structures, separated physical entities and probability',* Found. Phys. **24**, 1227.

[8] Aerts, D. (1995), *'Quantum structures: an attempt to explain the origin of their appearance in nature',* to be published in Int. J. Theor. Phys.

[9] Aerts, D. and Van Bogaert, B. (1992), *'A mechanical classical laboratory situation with a quantum logic structure'*, International Journal of Theoretical Physics **10** ,1839.

[10] Aerts, D., Durt, T., Grib, A.A., Van Bogaert, B. and Zapatrin, R.R. (1992), *'Quantum Structures in Macroscopic Reality'*, to be published in Int. J. of Theor. Phys. **32**, 489.

[11] Aerts, D., Durt, T. and Van Bogaert, B. (1993), *'Quantum Indeterminism, the Classical Limit and Non-Locality'*, in the proceedings of the Symposium on the Foundations of Modern Physics, Helsinki 1992, World Scientific Publishing Company, Signapore.

[12] Aerts, D., Coecke, B., Durt, T. and Valckenborgh, F. (1995), *'Quantum, Classical and Intermediate II : the Vanishing Vector Space Structure'*

[13] Beltrametti, E. and Cassinelli, G. (1981), *"The logic of quantum mechanics"*, Encyclopedia of Mathematics and its applications, Addison-Wesley Publishing Company.

[14] Cattaneo, G. and Nistico, G. (1990) *'A Note on Aerts' Description of Separated Entities'*, Found. Phys. **20**, 119.

[15] Coecke, B. (1995) *'A Hidden Measurement Model for Pure and Mixed States of Quantum Physics in Euclidean Space'*, to be published in Int. J. Theor. Phys.

[16] Coecke, B. (1995), *'New Examples of Hidden Measurement Systems and Outline of a General Scheme'*, submitted for the proceedings of the 5th Winterschool on measure theory, Liptovski Jan, Slovakia.

[17] Czachor, M. (1992), *'On Classical Models of Spin'*, Found. Phys. 5, **3**, 249.

[18] Foulis, D., Piron, C. and Randall, C. (1983), *'Realism, Operationalism and Quantum Mechanics'*, Found. Phys. **13**, 813.

[19] Piron, C. (1976), *"Foundations of Quantum Physics"*, W.A. Benjamin, Inc.

[20] Piron, C. (1989), *'Recent Developments in Quantum Mechanics'*,, Helv. Phys. Acta **62**, 82.

[21] Piron, C. (1990), *"Mécanique Quantique, Bases et Applications"*, Presses Polytechnique et Universitaire Romandes.

[22] Randall, C. and Foulis, D. (1983), *'A mathematical language for quantum physics'*, in *Les Fondements de la Mécanique Quantique*, ed. C. Gruberet al, A.V.C.P., case postale 101, 1015 Lausanne, Suisse.