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Quantum, Classical and Intermediate II : the Vanishing Vector Space Structure.

Diederik Aerts*, Bob Coecke†, Thomas Durt‡ and Frank Valckenborgh

TENA⋆, Free University of Brussels,

Pleinlaan 2, B-1050 Brussels, Belgium.

Abstract. We put forward an approach where physical entities are described by the set of their states, and the set of their relevant experiments. In this framework we will study a general entity that is neither quantum nor classical. We show that the collection of eigenstate sets forms a closure structure on the set of states. We also illustrate this framework on a concrete physical example, the ϵ-example. this leads us to a model for a continuous evolution from the linear closure in vector space to the standard topological closure.

1. Introduction.

It is well known that classical theories and quantum theories are structurally completely different (Boolean versus non-Boolean, commutative versus non-commutative, Kolmogorovian versus non-Kolmogorovian). In the field of the more general approaches (lattice theories, *algebra's, probability models), the classical situation can be considered to be a 'special' case of the general situation. A classical entity is described by a Boolean lattice, a commutative *algebra and a Kolmogorovian probability model, while for a general entity the lattice is not necessarily Boolean, the *algebra not always commutative, and the probability model not necessarily Kolmogorovian. Hence these general approaches can study the quantum as well

* Senior Research Associate of the National Fund for Scientific Research

† Research Assistant of the National Fund for Scientific Research

‡ Research Assistant of the Inter-University Institute for Nuclear Sciences

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as the classical and also a mixture of both cases. If we go to a concrete quantum entity, and even when we represent this concrete quantum entity in one of these general approaches, the situation is somewhat more complicated. In the lattice theoretic approach, a concrete quantum entity, described in a separable complex Hilbert space, is represented by the complete, orthocomplemented, weakly modular atomic lattice satisfying the covering law [1][6][10] that corresponds to this Hilbert space. This lattice is irreducible and hence does not contain any classical parts (the center of the lattice is given by $\{0, I\}$). This shows that a concrete quantum entity is described by a non-Boolean lattice, but the lattice is more than just non-Boolean, it is strictly non-Boolean, because it is irreducible. The fact that the quantum is non-Boolean and irreducible, while the classical is Boolean, explains why on this level the classical and the quantum cannot be put together into one general description. So, in the category of lattice theory (and also the *algebra and the probability model approach) already a fundamental generalization, as compared to the orthodox quantum theory in a Hilbert space, is present. It is the description of physical entities by lattices that are non-Boolean but not irreducible, and hence contain at the same time quantum and classical properties. These type of general situations have been studied intensively in all the categories that we have mentioned [2][3][6](chapter 21.6-7)[10]. For a complete orthocomplemented weakly modular atomic lattice there exists a theorem that shows that the lattice is isomorphic to the direct union of a set of irreducible lattices [2][3][4][6][10][11], such that the classical properties of the entity are represented by subsets of the index set, while the quantum properties are represented by elements of the irreducible components.

In this paper we put forward an approach to study what we will call the "intermediate" situation. In this intermediate situations we encounter properties that are neither classical nor quantum, but in between. These properties must not be confused with the mixture of classical and quantum properties that can be described in the traditional lattice approach. Indeed, in the traditional approach the mixture contains only classical properties (described by subsets of the index set of the direct union) or pure quantum properties (described by elements of the irreducible components). The mathematical structure that we put forward in this paper to describe the intermediate situations is the closure structure. An immediate consequence is that the lattice corresponding to the closure structure does not satisfy the 'quantum' axioms: it is not a complete, orthocomplemented, weakly modular atomic lattice, satisfying the covering law. In a forthcoming paper we shall investigate in detail which of the axioms are not satisfied, and we mention the result: the lattice is complete and atomic, but not orthocomplemented, not weakly modular and it does not satisfy the covering law. This result explains why the intermediate situations considered in this paper could not be described by the earlier approaches, and it shows that the in this paper introduced closure structure is more fundamental concerning the description of the set of properties of a physical entity than the lattice structure. Also this aspect shall be analyzed more in detail in this forthcoming paper.

As already mentioned, the aim of this paper is the introduction of one framework to study quantum, classical and intermediate situations. We first study the collection of eigenstate-sets of a general entity. This we do in parallel with the study of the ϵ-example, a model introduced in [5] that covers (true a parameter ϵ) a classical (ϵ=0), a family of intermediate ($\epsilon \in]0,1[$) and a quantum situation (ϵ=1). We will show that the collection of eigenstate-sets of an arbitrary entity always corresponds to a closure structure on the set of states. We also study this closure structure for the specific case of the sphere model. In this way we obtain a continuous evolution from the linear closure in vector space to the standard topological closure true the parameter ϵ.

2. Preliminaries.

In [5] we introduced a model on the Poincaré sphere (that also corresponds with a concrete physical entity) that describes a class of intermediate situations in the sense that the encountered structure is neither quantum nor classical. In the same paper we also started to develop a mathematical framework with experiments with more than two outcomes. This we did to incorporate the critique of [12] on our usual approach. Since this paper should be seen as part II of [5], we refer to [5] for the description of the model introduced in it. Also for a more detailed analyses concerning what we mean by state and measurement, we refer to that same paper.

For the reader who is only interested in the in this paper introduced formalism, and not in the model introduced in [5] (and here used as an illustration of the framework), we will repeat all necessary notations that will be used in the formalism.

The set of possible states that we consider for the physical entity S shall be denoted by Σ. The state can change under influence of a measurement e, and we call this a measurement process. The set of measurements that we consider in relation with the physical entity S

shall be denoted by $\mathcal{E}$. With each measurement e corresponds an outcome-set O_e, and when O_e is a finite set we denote it by $\{o_1, ..., o_n\}$. The probability that the state p is changed by the measurement e into the state q shall be denoted by $P_e(q \mid p)$ and we call it the transition probability of p to q by e :

$$P_e : \Sigma \times \Sigma \to [0, 1] \tag{1}$$

3. The Lattice of Eigenray sets.

In quantum mechanics, a state p of the quantum entity S is represented by a ray (one dimensional subspace) of a Hilbert space $\mathcal{H}$ and we denote by ψ_p a unit vector in this ray. A measurement e is represented by a self-adjoint operator H_e on the Hilbert space $\mathcal{H}$, and the outcome-set O_e of the measurement e is given by the spectrum σ_e of this self-adjoint operator H_e. By the spectral theorem, every such self-adjoint operator H_e is completely determined by its spectral measure : $Proj : \mathcal{B}(\sigma_e) \to \mathcal{P}(\mathcal{H})$, where with $A \in \mathcal{B}(\sigma_e)$, corresponds an orthogonal projection $Proj_A$, element of the set of all orthogonal projections $\mathcal{P}(\mathcal{H})$, and $\mathcal{B}(\sigma_e)$ is the collection of Borel-sets of the set of real numbers σ_e, the spectrum of the operator H_e. Suppose that we consider the two elements, $A \in \mathcal{B}(\sigma_e)$ and $A^C = \sigma_e \backslash A$, for an arbitrary self-adjoint operator H_e representing a measurement e. If the measurement e is performed, the entity S being in a state p, the outcome is with certainty in one of the two subsets A or A^C. In this way we make correspond with each element A of $\mathcal{B}(\sigma_e)$ and its complement A^C a 'yes-no'-experiment to the measurement e. These 'yes-no'-experiments determine the measurement e, and are represented by the spectral projections $Proj_A$ and $Proj_{A^C}$ of the spectral decomposition of the operator H_e.

Instead of considering the structure of these spectral projections, and hence the structure of $\mathcal{P}(\mathcal{H})$, as the principal one, as is often the case in quantum logic approaches, we want to identify this same structure by considering the subsets of the set of states $\Sigma_{\mathcal{H}}$ that correspond to these spectral projections. Hence, if $A \in \mathcal{B}(\sigma_e)$, we denote by $eig(A)$ the set of all rays $\bar{\psi}$ such that ψ is an eigenvector of $Proj_A$, with eigenvalue 1, and call $eig(A)$ the 'eigenray-set' corresponding to the measurement e, with 'eigenoutcome-set' A. Let us denote by $\mathcal{F}(\Sigma_{\mathcal{H}})$ the collection of all eigenray-sets. Hence, if we define for an arbitrary orthogonal projection $Proj \in \mathcal{P}(\mathcal{H})$, the subset $F_{Proj} = \{\psi \mid Proj(\psi) = \psi\}$ of $\Sigma_{\mathcal{H}}$, then :

$$\mathcal{F}(\Sigma_{\mathcal{H}}) = \{F_{Proj} \mid Proj \in \mathcal{P}(\mathcal{H})\} \tag{2}$$

It is well known that $\mathcal{P}(\mathcal{H}), \subset, \cap, \perp$ is a complete, atomic, orthomodular lattice, satisfying the covering law [6]. But also $\mathcal{F}(\Sigma_\mathcal{H})$ has this structure, because it is lattice-theoretically isomorphic to $\mathcal{P}(\mathcal{H})$.

Theorem 1. *We have :*

$$\emptyset, \Sigma_\mathcal{H} \in \mathcal{F}(\Sigma_\mathcal{H}) \tag{3}$$

$$F_i \in \mathcal{F}(\Sigma_\mathcal{H}) \Rightarrow \cap_i F_i \in \mathcal{F}(\Sigma_\mathcal{H}) \tag{4}$$

Proof : $\emptyset$ is the eigenray-set corresponding to the orthogonal projection $0 \in \mathcal{P}(\mathcal{H})$, and $\Sigma_\mathcal{H}$ is the eigenray-set corresponding to the orthogonal projection $1 \in \mathcal{P}(\mathcal{H})$. Take $F_i \in \mathcal{F}(\Sigma_\mathcal{H})$ such that $Proj_{F_i}$ are the corresponding orthogonal projections. If we consider $\cap_i Proj_{F_i}(\mathcal{H})$, then this is a closed subspace of the Hilbert space $\mathcal{H}$. Let us call $Proj$ the orthogonal projection on this closed subspace, then $Proj \in \mathcal{P}(\mathcal{H})$, and the eigenray-set corresponding to $Proj$ is $\cap_i F_i$.

We can now show that $\mathcal{F}(\Sigma_\mathcal{H})$ has the structure of a complete lattice :

Theorem 2. *If we define for two elements $F, G \in \mathcal{F}(\Sigma_\mathcal{H})$, $F < G \iff F \subset G$, and for a family of elements $F_i \in \mathcal{F}(\Sigma_\mathcal{H})$, $\wedge_i F_i = \cap_i F_i$ and $\vee_i F_i = \cap_{F_i \subset F, F \in \mathcal{F}(\Sigma_\mathcal{H})} F$, then $\mathcal{F}(\Sigma_\mathcal{H}), <, \wedge, \vee$ is a complete lattice, with partial order relation $<$, infimum $\wedge$ and supremum $\vee$.*

Proof : Clearly $<$ is a partial order relation, and $\wedge$ is an infimum for this partial order relation. Since $\Sigma_\mathcal{H} \in \mathcal{F}(\Sigma_\mathcal{H})$, it makes sense to define for a family of elements $F_i \in \mathcal{F}(\Sigma_\mathcal{H})$ the element $\vee_i F_i = \cap_{F_i \subset F, F \in \mathcal{F}(\Sigma_\mathcal{H})} F$. Clearly $F_k < \vee_i F_i \ \forall\, k$. Suppose that $F_i < F \ \forall\, i$, then $\cap_{F_i \subset G, F \in \mathcal{F}(\Sigma_\mathcal{H})} F < G$, which shows that $\vee_i F_i < G$. This proves that $\vee_i F_i$ is the supremum of the family F_i.

Instead of concentrating on the additional lattice-theoretical structure of $\mathcal{F}(\Sigma_\mathcal{H})$, namely atomicity, orthomodularity, and the covering law, we want to elaborate another, less investigated structure of $\mathcal{F}(\Sigma_\mathcal{H})$. Let us remark that (3) and (4) of theorem 1 are also satisfied for the collection of closed subsets of a topology on the set $\Sigma_\mathcal{H}$. If $\mathcal{F}$ would be the collection of closed subsets of a topology on $\Sigma_\mathcal{H}$, then one additional requirement would be satisfied, namely :

$$F, G \in \mathcal{F} \Rightarrow F \cup G \in \mathcal{F} \tag{5}$$

and this requirement (5) is not satisfied in $\mathcal{F}(\Sigma_{\mathcal{H}})$. Although for a topology we need (3), (4) and (5), a structure that only satisfies (3) and (4) still entails many properties of topological nature. Such structures are called closure-structures [6][8][9], and we shall see that they form the objects of the natural category of the eigenstate-sets for a general entity.

4. The Eigenstate-sets.

We now want to introduce the structure of the eigenstate-sets for a general physical entity, and we shall see that they generate a closure structure on the set of states. We repeat again, to avoid all confusion, the elements of the ϵ-situation that we want to study :

Definition 1. *Our physical entity S is the point particle P, that is in one of the points of the surface of the sphere, hence $\Sigma = \{p_v \mid v \in surf\}$. The measurements that are available to us for the study of this entity are the measurements $e^{\epsilon}_{u,d}$, for some $\epsilon \in [0,1]$. Hence $\mathcal{E}^{\epsilon} = \{e^{\epsilon}_{u,d} \mid u \in surf, d \in [-1+\epsilon, 1-\epsilon]\}$ for $\epsilon \in [0,1]$. The transition probabilities are given by : $P^{\epsilon}(p_u \mid p_v) = \frac{1}{2\epsilon}(v \cdot u - d + \epsilon)$ and $P^{\epsilon}(p_{-u} \mid p_v) = \frac{1}{2\epsilon}(d + \epsilon - v \cdot u)$.*

Let us now define a general eigenstate-set.

Definition 2 : *We consider the situation where we have an entity S with a set of states Σ and a set of measurements $\mathcal{E}$ that can be performed with the entity. We consider a particular measurement e with a set of possible outcomes O_e. When $A \subset O_e$, we define $eig(A) \subset \Sigma$ such that when $p \in eig(A)$, and the entity S is in the state p, the outcome of the measurement e is with certainty (probability equal to 1) contained in A. We say that $eig(A)$ is an eigenstate-set of the measurement e with eigenoutcome-set A.*

The connection between the eigenstate set $eig_e(A)$ of an entity and a property of the entity, as introduced in [1][10], is easy to point out. Therefore we shortly explain the concept of 'property' and 'test' (experimental project or question) as used in [1][10].

A test is an experiment that can be performed on the entity and where one has agreed in advance what is the positive result. In this way, each test α defines a particular property a of the entity. We say that a property a is 'actual' for a given entity S if in the event of the test α the positive result would be certainly obtained. Other properties which are not actual are said to be potential. We suppose known the set Q of all tests that can be performed on the entity. A test $\alpha \in Q$ is said to be 'true' if the corresponding property a is actual, which means that if we would perform α the positive outcome would come out with certainty. A test α is said to be stronger than a test β (we denote $\alpha < \beta$), if whenever α is true also β is true. A test α and a test β are said to be equivalent if $\alpha < \beta$ and $\beta < \alpha$. We identify the equivalence classes containing α with its corresponding property a. The set $\mathcal{L}$ of all properties is a complete lattice with the inherited order relation. We can represent each property a by the set $\mu(a)$ of all states of the entity S that make this property actual, which defines a map $\mu : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$, which is called the Cartan map.

We can see that $eig_e(A)$ is the Cartan image $\mu(a_e^A)$ of the property a_e^A, defined by means of the test α_e^A, consisting of performing the experiment e and giving the positive answer if the outcome is an element of A. Since the Cartan map is an isomorphism on its image, the property lattice $\mathcal{L}$ of the entity S is isomorphic to the collection of eigenstate sets. The statement 'the property a_e^A is actual' is equivalent to the statement 'the state of S is in $eig_e(A)$' and to the statement 'the experiment e gives with certainty an outcome in A'.

Since we want to apply this definition to the situation of the sphere-elastic-example, where the considered measurements have only two possible outcomes, we repeat the foregoing general definition for the case of a measurement with a finite outcome-set.

Definition 3 : *Suppose that we have an entity S with a set of states Σ and a measurement e with outcome set $\{o_1, ..., o_n\}$. The state p of the entity S is an eigenstate of the measurement e with eigenoutcome o_k if we can predict with certainty (probability equal to 1) that if the measurement e would be performed, the outcome o_k will occur. The set of eigenstates corresponding to an outcome o_k will be denoted by $eig(o_k)$.*

For our sphere-elastic-model it is easy to see that p_u and p_{-u} are eigenstates of the measurement $e_{u,d}^\epsilon$ for all d and ϵ. But whenever ϵ is different from 1 the measurement $e_{u,d}^\epsilon$ has other eigenstates. Indeed, for a measurement $e_{u,d}^\epsilon$ there is an eigenstate-set $eig(o_{u,d,1}^\epsilon)$ of eigenstates with eigenoutcome $o_{u,d,1}^\epsilon$ and an eigenstate-set $eig(o_{u,d,2}^\epsilon)$ of eigenstates with eigenoutcome $o_{u,d,2}^\epsilon$. We can easily identify these eigenstate-sets for a given measurement $e_{u,d}^\epsilon$, taking into account the results of [5], and we have represented them in figures 1a and 1b.

$$eig(o_{u,d,1}^\epsilon) = \{p_v \mid d + \epsilon \leq v \cdot u\} \tag{6}$$

$$eig(o^{\epsilon}_{u,d,2}) = \{p_v \mid v \cdot u \leq d - \epsilon\} \tag{7}$$

Definition 4 : *Suppose that we have an entity S with a set of states Σ and a measurement e with outcome set $\{o_1, ..., o_n\}$. The state p of the entity S is a superposition-state of the measurement e if it is not an eigenstate of the measurement e.*

We can easily identify the superposition-states of the measurement $e^{\epsilon}_{u,d}$, and shall denote the set of all these superposition-states by $sup^{\epsilon}(u,d)$:

$$sup^{\epsilon}(u,d) = \{p_v \mid d - \epsilon < v \cdot u < d + \epsilon\} \tag{8}$$

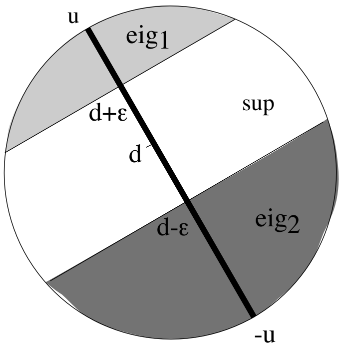

Fig. 1a : A representation of the measurement $e^{\epsilon}_{u,d}$, for a choice of d=0.2, and ϵ=0.4. We have represented the set of eigenstates $eig(o^{\epsilon}_{u,d,1})$ of $e^{\epsilon}_{u,d}$ (the light-gray area on the drawing), and the set of eigenstates $eig(o^{\epsilon}_{u,d,2})$ (the dark gray area on the figure). The set of superposition states $sup^{\epsilon}(u,d)$ corresponds to the white area on the drawing.

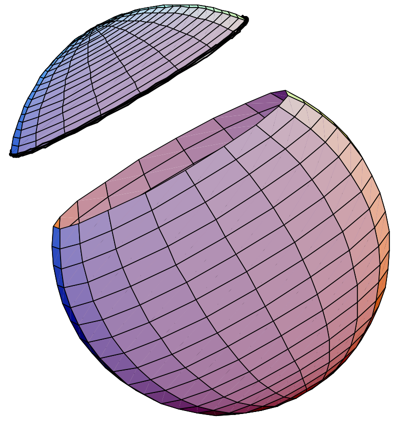

Fig. 1b : The top spherical circle is a three dimensional representation of the set of eigenstates corresponding to $eig(o^{\epsilon}_{u,d,1})$, as shown in figure 1a. Its complement, the bottom spherical sector, represents the set of states $sup^{\epsilon}(u,d) \cup eig(o^{\epsilon}_{u,d,2})$. We have drawn a black circle under the top spherical sector to indicate that this is a closed set.

4.1 The quantum situation.

For $\epsilon = 1$ we always have $d = 0$. Hence we find :

$$eig(o^{1}_{u,0,1}) = \{p_v \mid +1 \leq v \cdot u\} = \{p_u\}$$
$$sup^{1}(u,0) = \{p_v \mid -1 < v \cdot u < +1\}$$
$$eig(o^{1}_{u,0,2}) = \{p_v \mid v \cdot u \leq -1\} = \{p_{-u}\} \tag{9}$$

which shows that for the quantum situation, and hence measurements $e^1_{u,0} \in \mathcal{E}^1$, the eigenstates are the states p_u and p_{-u}, and all the other states are superposition states (see figures 2a and 2b).

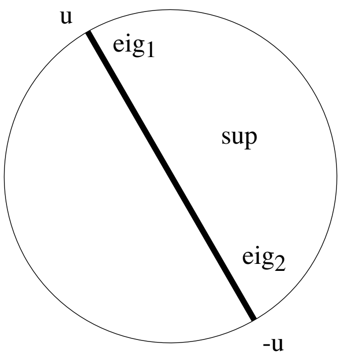

Fig. 2a : In the quantum situation, for ϵ=1, we have: $eig(o^{\epsilon}_{u,d,1})$ =$\{p_u\}$, and $eig(o^{\epsilon}_{u,d,2})$=$\{p_{-u}\}$.

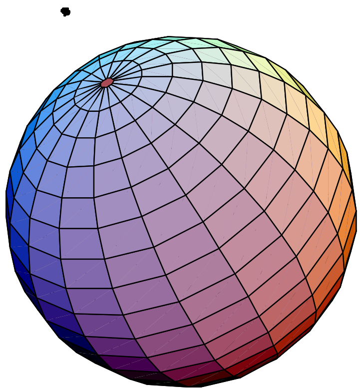

Fig. 2b : A three-dimensional representation of the quantum situation.

4.2 The classical situation.

The classical situation is the situation without fluctuations. If $\epsilon = 0$, then d can take any value in the interval $[-1, +1]$, and we have:

$$
\begin{aligned}
eig(o^0_{u,d,1}) &= \{p_v \mid d < v \cdot u\} \\
sup^0(u,d) &= \{p_v \mid d = v \cdot u\} \\
eig(o^0_{u,d,2}) &= \{p_v \mid v \cdot u < d\}
\end{aligned} \tag{10}
$$

which shows that for the classical situation, the only superposition states are the states p_v such that $v \cdot u = d$, and all the other states are eigenstates (see figures 3a and 3b).

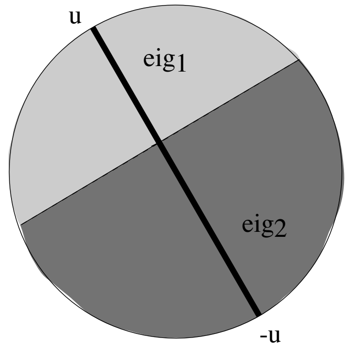

Fig. 3a : A representation of the classical situation for $d = 0.2$.

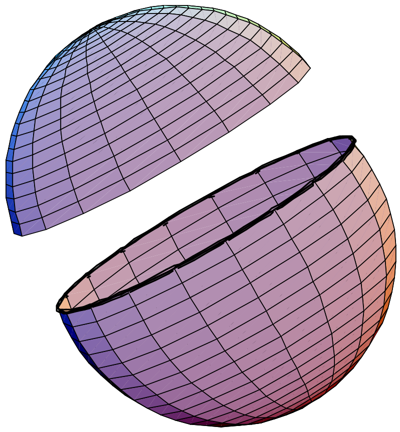

Fig. 3b : A three-dimensional representation of the classical situation for $d = 0.2$.

4.3 The general situation.

To be able to give a clear picture of the intermediate situations, we shall introduce some additional concepts. First we remark that the regions of eigenstates $eig(o^\epsilon_{u,d,1})$ and $eig(o^\epsilon_{u,d,2})$ are determined by the points of spherical sectors of $surf$ centered around the points u and $-u$ (see figures 1a and 1b).

We shall denote a closed spherical sector centered around the point $u \in surf$ with angle θ by $sec(u,\theta)$. With 'closed' we mean that also the circle on $surf$ with center u and angle θ, that we shall denote by $circ(u,\theta)$, is contained in $sec(u,\theta)$ (see fig. 1b). We remark that in the classical situation, for $\epsilon = 0$, $eig^0_1(u,d)$ and $eig^0_2(u,d)$ are given by open spherical sectors centered around u and $-u$. Let us denote such an open spherical sector centered around u and with angle θ by $sec^o(u,\theta)$.

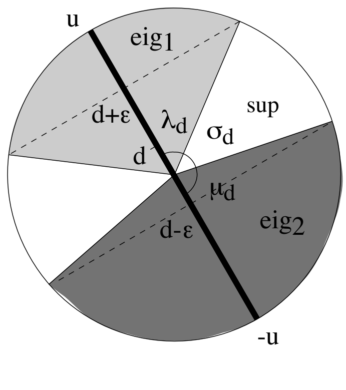

Fig. 4 : A representation of the general situation. The set of eigenstates of the measurement $e^\epsilon_{u,d}$ with eigenoutcome $o^\epsilon_{1,u}$, denoted by $eig(o^\epsilon_{u,d,1})$, is determined by the points of a spherical sector $sec(u,\lambda^\epsilon_d)$. The set of eigenstates with eigenoutcome $o^\epsilon_{2,u}$, denoted by $eig(o^\epsilon_{u,d,2})$, is determined by the points of the spherical sector $sec^\epsilon(-u,\mu^\epsilon_d)$. The set of superposition states is determined by the spherical sector centered around u and $-u$ with characteristic angle σ^ϵ_d.

Definition 5 : (see figure 4) *Let us call* λ_d^ϵ *the angle of the spherical sectors corresponding to* $eig(o_{u,d,1}^\epsilon)$ *for all* u*, hence for* $0 < \epsilon$ *we have* $eig(o_{u,d,1}^\epsilon) = \{p_v \mid v \in sec(u, \lambda_d^\epsilon)\}$*, and* $eig(o_{u,d,1}^0) = \{p_v \mid v \in sec^o(u, \lambda_d^0)\}$*. We can verify easily that* $eig(o_{u,d,2}^\epsilon)$ *is determined by a spherical sector centered around the point* $-u$*. We call* μ_d^ϵ *the angle of this spherical sector, hence for* $0 < \epsilon$ *we have* $eig(o_{u,d,2}^\epsilon) = \{p_v \mid v \in sec(-u, \mu_d^\epsilon)\}$ *and* $eig(o_{u,d,2}^0) = \{p_v \mid v \in sec^o(-u, \mu_d^\epsilon)\}$*. Let us call* σ_d^ϵ *the angle of the open spherical sector determined by* $sup^\epsilon(u, d)$*.*

We have :

$$cos\lambda_d^\epsilon = \epsilon + d \tag{11}$$

$$cos\mu_d^\epsilon = \epsilon - d \tag{12}$$

$$\sigma_d^\epsilon = \pi - \lambda_d^\epsilon - \mu_d^\epsilon \tag{13}$$

$$\lambda_{-d}^\epsilon = \mu_d^\epsilon \text{ and } \sigma_{-d}^\epsilon = \sigma_d^\epsilon \tag{14}$$

The eigenstate-sets corresponding to one particular measurement e satisfy in general the following properties :

Theorem 3. *Let us introduce the following notations :* $\mathcal{P}(O_e)$ *is the collection of all subsets of* O_e*, and* $\mathcal{P}(\Sigma)$ *is the collection of all subsets of* Σ*. Then the map*

$$eig : \mathcal{P}(O_e) \to \mathcal{P}(\Sigma) \tag{15}$$

satisfies the following properties :

$$eig(\emptyset) = \emptyset \tag{16}$$

$$eig(O_e) = \Sigma \tag{17}$$

$$eig(\cap_i A_i) = \cap_i eig(A_i) \tag{18}$$

Proof : $p \in eig(\cap_i A_i) \iff$ S being in state p, an execution of e gives with certainty an outcome in $\cap_i A_i \iff$ S being in state p, an execution of e gives with certainty an outcome in A_i for each $i \iff p \in eig(A_i)$ for each $i \iff p \in \cap_i eig(A_i)$.

We can easily see that $eig(A \cup B)$ in general is not equal to $eig(A) \cup eig(B)$ for $A, B \in \mathcal{P}(O_e)$. For example in the case of the sphere-elastic-example, we have $eig(\{o_{1,u}^\epsilon\} \cup \{o_{2,u}^\epsilon\})$

$= eig(O_e) = \Sigma$, while $eig(o^{\epsilon}_{1,u}) \cup eig(o^{\epsilon}_{2,u})$ is different from Σ, because of the existence of the superposition states (see figures 1a and 1b).

Before we study more in detail the structure of the map eig and its image, we want to investigate the general structure of the set of measurements itself.

5. The structure of the set of measurements

We consider the general situation of an entity S, where Σ is its set of states, and $\mathcal{E}$ its set of measurements. Each measurement $e \in \mathcal{E}$ has an outcome-set O_e.

It is important to remark that if we have two arbitrary measurements $e, f \in \mathcal{E}$ we can define the measurement $e \cup f$ that consists of performing the measurement e **or** performing the measurement f. This measurement $e \cup f$ is a new measurement with a set of possible outcomes given by $O_e \cup O_f$. Clearly we have for $e, f, g \in \mathcal{E}$:

$$e \cup e = e \tag{19}$$

$$(e \cup f) \cup g = e \cup (f \cup g) \tag{20}$$

$$e \cup f = f \cup e \tag{21}$$

We can now put forward a general definition :

Definition 6 : *Let us consider an arbitrary subset $E \subset \mathcal{E}$, and introduce the measurement, corresponding to this subset E, that we shall denote by e_E, in the following way : the measurement e_E consists of choosing one of the measurements $e \in E$, and perform the chosen measurement.*

The measurement e_E has an outcome-set $O_E = \cup_{e \in E} O_e$. The enlarged set of measurements that can be performed on the entity S will be represented by $cl(\mathcal{E})$, the collection of all subsets of the set $\mathcal{E}$, where each subset defines an measurement in the way we have explained in definition 6. The reason for this notation will get clear in the next section. If we introduce $O = \cup_{e \in \mathcal{E}} O_e$, then O is the outcome-set of the measurement $e_{\mathcal{E}}$, and the outcome-set of any measurement $e_E \in cl(\mathcal{E})$ is contained in O. To distinguish we shall call the original measurements $e \in \mathcal{E}$ 'primitive' measurements, and the newly introduced measurements 'union' measurements.

For the case of our example, we choose $\mathcal{E}^\epsilon = \{e^\epsilon_{d,u} \mid d \in [-1+\epsilon, +1-\epsilon], u \in surf\}$, and the enlarged set of measurements is given by $cl(\mathcal{E}^\epsilon)$.

We have now all the material to explain the fundamental point of the 'hidden measurement' approach. The entity S is defined by its set of states Σ and a set of 'primitive measurements' $\mathcal{E}$, and the complete set of measurements, giving rise to the structures that we shall put forward in the following, is given by the set of 'union measurements' or the closure $cl(\mathcal{E})$ of the original set of primitive measurements $\mathcal{E}$. The hidden measurement approach, and the possibility to built the quantum mechanical models by this approach, shows that for a quantum entity S it is always possible to find a set of 'classical' measurements that by the operation of the 'union' generates all the measurements of $cl(\mathcal{E})$. With other words: all of the probability structure for a quantum entity comes from the fact that measurements are unions of classical deterministic measurements. To illustrate this situation, let us make explicit this set of 'generating classical measurements' for the case of the sphere model.

Consider the sphere model for a given ϵ, with a set of primitive measurements $\mathcal{E}^\epsilon = \{e^\epsilon_{u,d} | \ u \in surf, d \in [-1+\epsilon, +1-\epsilon]\}$. Taking into account the way in which the measurement takes place we see that $e^\epsilon_{u,d} = \cup_{x \in [d-\epsilon, d+\epsilon]} e^0_{u,x}$, where $e^0_{u,x}$ is the 'classical' measurement, as it is defined in 3.2 for the case of zero fluctuations ($\epsilon = 0$). This means that the set:

$$\mathcal{E}^0 = \{e^0_{u,d} | \ u \in surf, d \in [0,1]\} \tag{22}$$

is a generating set for the set of all measurements $cl(\mathcal{E}^\epsilon)$.

Theorem 4. *For the sphere model in situation 2, with set of states $\Sigma = \{p_v | \ v \in surf\}$, and set of measurements $\mathcal{E}^\epsilon = \{e^\epsilon_{u,d} \mid u \in surf, d \in [-1+\epsilon, +1-\epsilon]\}$ we have :*

$$cl(\mathcal{E}^0) = cl(\mathcal{E}^\epsilon) \tag{23}$$

where $\mathcal{E}^0 = \{e^0_{u,d} | \ u \in surf, d \in [-1, +1]\}$ is the set of 'classical' measurements corresponding to the sphere model with $\epsilon = 0$.

Definition 7. *In the same way as we have introduced the eigenstate-set corresponding to a set of outcomes for a primitive measurement (definition 3) we can introduce the eigenstate-set corresponding to a set of outcomes of a general union measurement. Suppose that e_E is* *a union measurement and* O_E *its outcome set. For* $A \subset O_E$ *we define eig(A) to be the set of all states, such that when* S *is in one of these states, the measurement* e_E *gives with certainty an outcome in* O_E*. We denote the collection of eigenstate-sets* $eig(\mathcal{P}(O_e))$*, corresponding to a primitive measurement* e*, by* $\mathcal{F}_e$*, and the collection of all eigenstate-sets by* $\mathcal{F}(\Sigma)$*. Hence :*

$$\mathcal{F}(\Sigma) = \{F \mid F \subset \Sigma, \exists \text{ measurement } e_E \in cl(\mathcal{E}), \text{ and } A \subset \cup_{e \in E} O_e, \text{ such that } F = eig(A)\} \tag{24}$$

Theorem 5. *Suppose that* e_E *is a measurement,* O_E *its outcome-set, and* $A \subset O_E$*, and* $eig(A)$ *the eigenstate-set corresponding to* A*. Then we can consider the sets* $A \cap O_e$*, and the corresponding eigenstate-sets* $eig(A \cap O_e)$*. We have :*

$$eig(A) = \cap_{e \in E} eig(A \cap O_e) \tag{25}$$

Proof : $p \in eig(A) \iff S$ being in state p, the measurement e_E gives with certainty an outcome inside $A \iff$ each primitive measurement $e \in E$, S being in state p, gives with certainty an outcome in $A \cap O_e \iff p \in eig(A \cap O_e)$ for each $e \in E \iff p \in \cap_{e \in E} eig(A \cap O_e)$.

This theorem shows that each eigenstate-set is the set theoretical intersection of eigenstate-sets corresponding to primitive measurements.

The structure of the property lattice as constructed in [1][10] is that of a complete lattice. Since the collection of eigenstate sets is connected to the property lattice by the Cartan map, this complete lattice structure should be present in some way or another in the collection of eigenstate sets. By means of the introduction of the 'union' experiment, we can identify what is at the origin of the completeness of the property lattice. To do this we have to reflect about the relation between the outcome sets O_e and O_f of different experiments e and f.

Definition 8. *We call two experiments* $e, f \in \mathcal{E}$ *'distinguishable' if we can define* O_e *and* O_f *in such a way that* $O_e \cap O_f = \emptyset$*.*

Two experiments e and f are distinguishable if they can be distinguished from each other by means of their outcomes. Experiments that we 'consciously' perform on the entity S are always distinguishable, because we can 'name' the outcome corresponding to the given experiment (when a certain outcome occurs with an experiment e, even if physically equivalent with the outcome of another experiment, we can distinguish it if we know that we were performing the experiment e and not this other experiment).

Suppose that we consider a test α_e^A, consisting of performing the experiment e and giving the positive answer 'yes' if the outcome is in A, and a test α_f^B, consisting of performing the experiment f and giving a positive answer 'yes' if the outcome is in B. Piron has introduced in [10] the concept of 'product test', and if α_e^A tests whether the property a_e^A is actual and α_f^B tests whether the property a_f^B is actual, then $\alpha_e^A \cdot \alpha_f^B$ tests whether both properties a_e^A and a_f^B are actual. It is by requiring that the collection of tests of the entity S contains all the product questions, that the lattice of properties becomes a complete lattice. The product test is defined by means of the experiment $e \cup f$, and is given by $\alpha_{e \cup f}^{A \cup B}$, consisting of performing the experiment $e \cup f$ and giving a positive answer 'yes' if the outcome is in $A \cup B$. We remark that, although the product test can always be defined, it only tests whether the two properties a_e^A and a_f^B are actual, if e and f are distinguishable experiments. Indeed, suppose that e and f are not distinguishable, then $O_e \cap O_f \neq \emptyset$, which means that there is at least one outcome $x_e^i = x_f^j \in O_e \cap O_f$. Suppose that A does not contain this outcome while B does, then it is possible that the entity S is in a state p such that e has as possible outcomes the set $A \cup \{x_e^i\}$, which is a state where a_e^A is not actual, and where f has as possible outcomes B. Then $e \cup f$ has as possible outcomes $A \cup B$, which means that in this state p the test $\alpha_{e \cup f}^{A \cup B}$ gives with certainty a positive outcome. This shows that in this case of non distinguishable experiments, $\alpha_{e \cup f}^{A \cup B}$ does not test the actuality of both properties.

Theorem 6 *Let us denote by $\mathcal{F}$ the collection of eigenstate sets of the experiment $e_\mathcal{E}$. If all the experiments are distinguishable then $\mathcal{F}_E \subset \mathcal{F}$ for $E \subset \mathcal{E}$.*

Proof: Consider an arbitrary element $F \in \mathcal{F}_E$. Then there exists $A \subset O_E$ such that $F = eig_E(A)$. Consider $A' = A \cup (\cup_{e \in E^C} O_e)$, then we have $eig_\mathcal{E}(A') = eig_E(A)$, which shows that $F \in \mathcal{F}$.

Corollary 1 *When all the experiments are distinguishable then the collection of all eigenstate sets is given by $\mathcal{F}$.*

From now on we shall consider the experiments $e \in \mathcal{E}$ to be 'distinguishable' experiments, which correspond to the situation where we consciously can choose between the different experiments, and in this way distinguish between the outcomes. Hence we have $O_e \cap O_f = \emptyset$ for $e, f \in \mathcal{E}$.

The eigenstate-sets corresponding to the primitive measurements, hence $\cup_{e \in \mathcal{E}} \mathcal{F}_e$, form a generating set for the collection of all eigenstate-sets. Our aim is to characterize the collection of all eigenstate-sets $\mathcal{F}(\Sigma)$.

Theorem 7. *If $\mathcal{F}(\Sigma)$ is the collection of all eigenstate-sets corresponding to an entity S, with a set of primitive measurements $\mathcal{E}$, we have :*

$$\emptyset, \Sigma \in \mathcal{F}(\Sigma) \tag{26}$$

$$F_i \in \mathcal{F}(\Sigma) \Rightarrow \cap_i F_i \in \mathcal{F}(\Sigma) \tag{27}$$

Proof : Suppose that $F_i \in \mathcal{F}(\Sigma)\ \forall\ i$, then there exists a A_i such that $F_i = eig(A_i)\ \forall\ i$. We also have that $\forall\ i\ A_i \subset O$, and hence $eig(A_i) = \cap_{e \in \mathcal{E}} eig(A_i \cap O_e)$. From this follows that $\cap_i F_i = \cap_i \cap_{e \in \mathcal{E}} eig(A_i \cap O_e) = \cap_{e \in \mathcal{E}} \cap_i eig(A_i \cap O_e) = \cap_{e \in \mathcal{E}} eig(\cap_i A_i \cap O_e) = eig(\cap_i A_i)$.

If we remember theorem 1, then we see that we find a similar 'generalized topological structure' for the situation of an arbitrary entity S with set of states Σ and collection of measurements $\mathcal{E}$. Let us now see how this structure fits in the category of closure structures.

6. The eigen-closure structure.

In this section we shall introduce the closure structure that enable us to characterize the collection of eigenstate-sets for an arbitrary entity S. First we introduce some definitions, and proof some general theorems :

Definition 9. *Consider a set X. We say that cl is a closure on this set X if, for $K, L \subset X$, we have :*

$$K \subset cl(L) \tag{28}$$

$$K \subset L \Rightarrow cl(K) \subset cl(L) \tag{29}$$

$$cl(cl(K)) = cl(K) \tag{30}$$

$$cl(\emptyset) = \emptyset \tag{31}$$

A set X with a closure cl satisfying (28),(29),(30) and (31) shall be called a 'closure-structure'.

Theorem 8. *If X is a set equipped with a closure cl, and we define a subset $F \subset X$ to be closed if $cl(F) = F$, then the family $\mathcal{F}$ of closed subsets of X satisfies :*

$$\emptyset \in \mathcal{F}, X \in \mathcal{F} \tag{32}$$

$$F_i \in \mathcal{F} \Rightarrow \cap_i F_i \in \mathcal{F} \tag{33}$$

This theorem shows that a closure structure generates a collection of closed subsets $\mathcal{F}$ that satisfy (32) and (33). Since these properties are satisfied in the collection of eigenstate-sets corresponding to a general entity S (see theorem 7), we can wonder whether this collection of eigenstate-sets can be characterized by a closure structure. The following theorem shows that this is indeed the case.

Theorem 9. *Suppose we have a set X and a family $\mathcal{F}$ of subsets of X that satisfy (32) and (33), and for an arbitrary subset $K \subset X$ we define $cl(K) = \cap_{K \subset F, F \in \mathcal{F}} F$, then cl is a closure and $\mathcal{F}$ is the set of closed subsets of X defined by this closure.*

Proof : Let $K, L \subset X$. Clearly $K \subset cl(K)$. If $K \subset L$ then $cl(K) \subset cl(L)$. If $F \in \mathcal{F}$ then $cl(F) = F$. Since $cl(K) \in \mathcal{F}$ we have $cl(cl(K)) = cl(K)$. This shows that cl is a closure. Consider a set K such that $cl(K) = K$, then $K = \cap_{K \subset F, F \in \mathcal{F}} F$, and hence $K \in \mathcal{F}$.

Theorem 7 and theorem 9 show that for an entity S, with a set of possible states Σ, the collection of eigenstate-sets $\mathcal{F}(\Sigma)$ is the collection of closed subsets corresponding to a closure structure, and theorem 9 shows us how to define this closure structure.

Definition 10. *Let us consider an entity S, with a set of possible states Σ, and a set of measurements $\mathcal{E}$. For an arbitrary set of states $K \subset \Sigma$ we define :*

$$cl_{eig}(K) = \cap_{K \subset F, F \in \mathcal{F}(\Sigma)} F \tag{34}$$

where $\mathcal{F}(\Sigma)$ is the collection of eigenstate-sets, then cl_{eig} is the closure that defines the closure structure corresponding to this collection of eigenstate-sets. We shall call it the eigen-closure structure of S.

To characterize more easily this eigen-closure structure cl_{eig} of S, also in the case of the sphere-model, we introduce the following definition :

Definition 11. *Suppose we have a set X and $\mathcal{F}$ is the set of closed subsets corresponding to a closure cl on X. We say that the collection $\mathcal{B}$ of subsets of X is a 'generating set' for $\mathcal{F}$ if for each subset $F \in \mathcal{F}$ we have a family $B_i \in \mathcal{B}$ such that $F = \cap_i B_i$.*

Theorem 10. *Suppose we have a set X equipped with a closure cl and $\mathcal{B}$ is a generating set for the set of closed subsets $\mathcal{F}$. Then for an arbitrary subset $K \subset X$ we have :*

$$cl(K) = \cap_{K \subset B, B \in \mathcal{B}} B \tag{35}$$

Proof : We know that $cl(K) = \cap_{K \subset F, F \in \mathcal{F}} F$. Because $\mathcal{B}$ is a generating set for $\mathcal{F}$ we have $F = \cap_{F \subset B, B \in \mathcal{B}} B$. Hence $cl(K) = \cap_{K \subset F}(\cap_{F \subset B} B) = \cap_{K \subset B, B \in \mathcal{B}} B$.

Theorem 11. *The collection of eigenstate-sets corresponding to primitive experiments , hence $\cup_{e \in \mathcal{E}} \mathcal{F}_e$, is a generating set for the eigen-closure structure of the entity S.*

Theorem 8 also shows that a closure structure is a generalization of an ordinary topology, and knowing this we can understand why the concepts of closure that are well known in physics ; the topological closure of subsets of a phase-space in classical mechanics, and the linear closure of subsets of a vector space in quantum mechanics, are both examples of closures.

7. The eigen-closure structure of the sphere-model.

In [6] we have characterized the eigenstate-sets corresponding to the primitive measurements of the sphere-model. These are, for a given fixed value of ϵ, the subsets $eig(o^{\epsilon}_{u,d,1})$ and $eig(o^{\epsilon}_{u,d,2})$, for $u \in surf$ and $d \in [-1+\epsilon, +1-\epsilon]$. This will be the working-object to construct the eigen-closure structure, namely the collection of all eigenstate-sets. Let us construct a closure-procedure for the sphere-model.

Definition 12. *Let us consider the surface of the sphere, denoted by $surf$, of the sphere-model and an arbitrary subset Q, $Q \subset surf$. Let us define the set*

$$H(\alpha) = \{sec(u, \alpha) \mid u \in surf\} \cup \{surf\} \tag{36}$$

We introduce for $Q \subset surf$ the following :

$$cl^{\alpha}(Q) = \cap_{Q \subset R, R \in H(\alpha)} R \tag{37}$$

Theorem 12. *cl^{α} is a closure for every α on the set $surf$.*

Theorem 13. *For every subset $Q \subset surf$ we have $cl^{\alpha}(Q) \subset cl^{\beta}(Q)$ if $\beta \leq \alpha$.*

Proof : Suppose that $\beta \leq \alpha$. We remark that in this case $sec(u, \beta) = cl^{\alpha}(sec(u, \beta))$. Let us consider $cl^{\beta}(Q) = \cap_{Q \subset R, R \in H(\beta)} R$. We can substitute each spherical sector $sec(u, \beta)$ of this intersection by $cl^{\alpha}(sec(u, \beta)$, and then we have that $cl^{\beta}(Q)$ is given by an intersection of elements of $H(\alpha)$. This shows that $cl^{\alpha}(Q) \subset cl^{\beta}(Q)$.

This theorem will allow us to construct in a more explicit way the eigen-closed subsets of our model. Indeed, we remark again that for a fixed ϵ the set of generating subsets of the eigen-closure structure on Σ defined by the collection of eigenstate-sets consists of the set $H(\lambda_d^{\epsilon})$. The foregoing theorem shows that we only have to take into account the $H(\lambda_d^{\epsilon})$ for a maximum of the λ_d^{ϵ}. A maximum of λ_d^{ϵ} for a given ϵ, means a minimum of $cos(\lambda_d^{\epsilon})$, hence a minimum of $\epsilon + d$. Let us denote the minimum of d by d_m, then clearly $d_m = -1 + \epsilon$.

Theorem 14. *Consider an arbitrary subset $A \subset \Sigma$, and let $Q = \{u \mid p_u \in A\}$. Then :*

$$cl_{eig}(A) = \{p_w \mid w \in \cap_{Q \subset R, R \in H(\lambda^{\epsilon}_{d_m})} R,\ d_m = -1 + \epsilon\} \tag{38}$$

Proof : Immediate consequence of theorem 13.

This theorem allows to construct explicitly the property-closure of arbitrary subsets of the set of states. Let us now investigate in detail the eigen-closure structure that we have constructed for the ϵ-sphere-model (the sphere-model equipped with the ρ_{ϵ} elastic).

8. Different situations and examples.

8.1 The quantum case.

We choose $\epsilon = 1$. Then $d_m = -1 + \epsilon = 0$.

We have $cos(\lambda_0^1) = 1$, hence $\lambda_0^1 = 0$. We have $H(0) = \{\{p_u\} \mid u \in surf\} \cup \{surf\}$.

Consider a arbitrary $A \subset \Sigma$, and $Q = \{u \mid p_u \in A\}$. Then $cl_{eig}(A) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(0)} R\}$.

Let us consider some examples of subsets $A \subset \Sigma$.

If we consider an arbitrary state p_v of the sphere-example, we will that $cl_{eig}(\{p_v\}) = \{p_v\}$. Consider an arbitrary value of ϵ, and an arbitrary state p_w different from p_v. Since v is different from w, the angle $\theta(v,w)$ between the vectors v and w is strictly greater then zero. As we show on figure 5, we can construct an experiment $e^{\epsilon}_{u,d}$ such that p_v is an eigenstate of $e^{\epsilon}_{u,d}$, and p_w is not. This shows that $p_w \notin cl_{eig}(\{p_v\})$. Since p_w is an arbitrary state different from p_v, we conclude that $cl_{eig}(\{p_v\}) = \{p_v\}$.

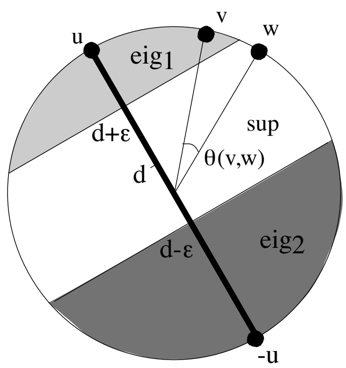

Fig. 5 : For a given ϵ, and two different states p_v and p_w, it is easy to construct a measurement $u^{\epsilon}_{u,d}$, such that p_v is an eigenstate of $e^{\epsilon}_{u,d}$, and p_w is not.

Thus we have that each singleton $\{p_u\}$ is a closed subset of Σ, and hence $cl_{eig}(\{p_u\}) = \{p_u\}$. So we start our investigation with subsets containing at least two states.

a. $A = \{p_x, p_y\}$, for $x \neq y$., hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(0)} R\} = \Sigma$. The closure of the set consisting of two different states is Σ, which is an expression of the superposition principle in quantum mechanics.

b. A is an arbitrary subset of Σ different from $\emptyset$ and different from a singleton.

Since in this case there exist $x \neq y$ such that $\{p_x, p_y\} \subset A$, we have $\Sigma \subset cl_{eig}(\{p_x, p_y\}) \subset cl_{eig}(A) \subset \Sigma$, which shows that $cl_{eig}(A) = \Sigma$. The only closed subset different from $\emptyset$ or from a singleton is Σ. Hence the collection of eigenstate-sets $\mathcal{F}(\Sigma) = \{\emptyset, \{p_u\}, \Sigma\}$, a situation that corresponds to what we know in quantum mechanics.

8.2 An intermediate near-to-quantum case.

We choose $\epsilon = \frac{3}{4}$. Then $d_m = -\frac{1}{4}$.

We have $cos(\lambda^{\frac{3}{4}}_{-\frac{1}{4}}) = +\frac{1}{2}$. Hence $\lambda^{\frac{3}{4}}_{-\frac{1}{4}} = \frac{\pi}{3}$. So we have $H(\frac{\pi}{3}) = \{sec(u, \frac{\pi}{3}) \mid u \in surf\} \cup \{surf\}$.

a. $A = \{p_x, p_y\}$, for $x \neq y$ hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\frac{\pi}{3})} R\}$. Let construct $cl_{eig}(A)$.

1. $\frac{2\pi}{3} < \theta(x, y)$.

Then there exists no $sec(u, \frac{\pi}{3})$ such that $\{p_x, p_y\} \subset sec(u, \frac{\pi}{3})$. Hence $cl^p(\{p_x, p_y\}) = \Sigma$.

2. $\theta(x, y) \leq \frac{2\pi}{3}$.

Consider the two points v and w, such that $\theta(x, w) = \theta(y, w) = \theta(x, v) = \theta(y, v) = \frac{\pi}{3}$, and the spherical sectors $sec(v, \frac{\pi}{3})$ and $sec(w, \frac{\pi}{3})$. Our claim is that $cl_{eig}(\{p_x, p_y\}) = \{p_u \mid u \in sec(w, \frac{\pi}{3}) \cap sec(v, \frac{\pi}{3})\}$. Indeed if $sec(u, \frac{\pi}{3})$ is an arbitrary spherical sector such that $x \in sec(u, \frac{\pi}{3})$ and $y \in sec(u, \frac{\pi}{3})$, then clearly $sec(w, \frac{\pi}{3}) \cap sec(v, \frac{\pi}{3}) \subset sec(u, \frac{\pi}{3})$ (see figure 6a).

b. $A = \{p_x, p_y, p_z\}$, for $x \neq y$, $x \neq z$ and $y \neq z$, hence $Q = \{x, y, z\}$.

1. $\nexists\ sec(u, \frac{\pi}{3})$ such that $Q \subset sec(u, \frac{\pi}{3})$.

In this case $cl_{eig}(\{p_x, p_y, p_z\}) = \Sigma$.

2. $\exists\ sec(u, \frac{\pi}{3})$ such that $Q \subset sec(u\frac{\pi}{3})$.

Consider three points u, v and w such that $\theta(u, x) = \theta(u, y) = \theta(v, x) = \theta(v, z) = \theta(w, z) = \theta(w, y) = \frac{\pi}{3}$, and the three spherical sectors $sec(u, \frac{\pi}{3})$, $sec(v, \frac{\pi}{3})$ and $sec(w, \frac{\pi}{3})$.
$cl_{eig}(\{p_x, p_y, p_z\}) = \{p_v \mid v \in sec(u, \frac{\pi}{3}) \cap sec(v, \frac{\pi}{3}) \cap sec(w, \frac{\pi}{3})\}$ (see figure 6c).
c. A is an arbitrary subset of Σ.

1. $\nexists\ sec(u, \frac{\pi}{3})$ such that $A \subset sec(u, \frac{\pi}{3})$.

In this case $cl_{eig}(A) = \Sigma$.

2. $\exists\ sec(u, \frac{\pi}{3})$ such that $A \subset sec(u, \frac{\pi}{3})$.

Then $cl_{eig}(Q) = \cap_{Q \subset sec(v, \frac{\pi}{3}), v \in surf} sec(v, \frac{\pi}{3})$. We have to construct this set for each given Q apart.

8.3 The in-between case.

We choose $\epsilon = \frac{1}{2}$. Then $d_m = -\frac{1}{2}$.

We have $cos(\lambda_{-\frac{1}{2}}^{\frac{1}{2}}) = 0$. Hence $\lambda_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{2}$. So we have $H(\frac{\pi}{2}) = \{sec(u, \frac{\pi}{2}) \mid u \in surf\} \cup \{surf\}$.

a. $A = \{p_x, p_y\}$, for $x \neq y$ hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\frac{\pi}{2})} R\}$. Let construct $cl_{eig}(A)$.

We remark that there always exists a spherical sector $sec(u, \frac{\pi}{2})$ such that $Q \subset sec(u, \frac{\pi}{2})$. Indeed we only have to consider the big circle through x and y, and take u the North Pole of this big circle. This shows that for $\epsilon \leq \frac{1}{2}$ the property-closure of two states cannot be equal anymore to Σ as it is in the quantum case, and as it can be for $\frac{1}{2} \leq \epsilon \leq 1$.

The construction of v and w, and $sec(v, \frac{\pi}{2})$ and $sec(w, \frac{\pi}{2})$, that we explained in detail for the case $\epsilon = \frac{3}{4}$ also works in this case. But since the contouring circles $circ(u, \frac{\pi}{2})$ of spherical sectors with angle $\frac{\pi}{2}$ are big circles of $surf$, we have that $sec(v, \frac{\pi}{2}) \cap sec(w, \frac{\pi}{2})$ is the big circle through x and y. Hence $cl_{eig}(\{x, y\})$ is the big circle from x to y (see figure 6b).

b. $A = \{p_x, p_y, p_z\}$, for $x \neq y$, $x \neq z$ and $y \neq z$, hence $Q = \{x, y, z\}$.

Again we remark that always $\exists\ sec(u, \frac{\pi}{2})$ such that $Q \subset sec(u, \frac{\pi}{2})$, hence even the property-closure of three different states cannot result in Σ if $\epsilon \leq \frac{1}{2}$. An analogous reasoning than the one we have made for the two states shows that $cl_{eig}(\{x, y, z\})$ is given by the sphere-triangle of $surf$ formed by x, y and z (see figure 6d).

c. A is an arbitrary subset of Σ.

1. $\nexists\ sec(u, \frac{\pi}{2})$ such that $A \subset sec(u, \frac{\pi}{2})$.

In this case $cl_{eig}(A) = \Sigma$.

2. $\exists\ sec(u, \frac{\pi}{2})$ such that $A \subset sec(u, \frac{\pi}{2})$.

Then $cl_{eig}(Q) = \cap_{Q \subset sec(v, \frac{\pi}{2}), v \in surf} sec(v, \frac{\pi}{2})$.

8.4 An intermediate near-to-classical case.

We choose $\epsilon = \frac{1}{4}$. Then $d_m = -\frac{3}{4}$.

We have $cos(\lambda^{\frac{1}{4}}_{-\frac{3}{4}}) = -\frac{1}{2}$. Hence $\lambda^{\frac{1}{2}}_{-\frac{3}{4}} = \frac{2\pi}{3}$. So we have $H(\frac{2\pi}{3}) = \{sec(u, \frac{2\pi}{3}) \mid u \in surf\} \cup \{surf\}$.

a. $A = \{p_x, p_y\}$, for $x \neq y$ hence $Q = \{x, y\}$.

$cl_{eig}(\{p_x, p_y\}) = \{p_v \mid v \in \cap_{Q \subset R, R \in H(\frac{2\pi}{3})} R\}$. Let construct $cl_{eig}(A)$.

Again there always exists a spherical sector $sec(u, \frac{2\pi}{3})$ such that $Q \subset sec(u, \frac{2\pi}{3})$. Let us call $cl_{eig}(Q)^C$ the set theoretical complement of $cl_{eig}(Q)$. We remark that $sec(u, \frac{2\pi}{3})^C = sec^o(-u, \frac{\pi}{3})$, which is the open spherical sector centered around $-u$ with angle $\frac{\pi}{3}$. It can be shown that $cl_{eig}(Q)^C = \cup_{sec^o(u, \frac{\pi}{3}) \subset Q^C, u \in surf} sec^o(u, \frac{\pi}{3})$ and Q^C is the complement of $\{x, y\}$. It is easy to see that we can overlap the whole of Q^C with open spherical sectors $sec^o(u, \frac{\pi}{3})$, which means that $cl_{eig}(Q)^C = Q^C$ and hence $cl_{eig}(Q) = Q$. This shows that in this case of $\epsilon = \frac{1}{4}$ for two different states there is no superposition, in the sense that $cl_{eig}(\{p_x, p_y\}) = \{p_x, p_y\}$, and this is generally true for $\epsilon < \frac{1}{2}$. This however does not mean that we have already reached a classical situation once $\epsilon < \frac{1}{2}$. This becomes clear in the next case.

b. $A = \{p_x, p_y, p_z\}$, for $x \neq y$, $x \neq z$ and $y \neq z$, hence $Q = \{x, y, z\}$.

Again we remark that always $\exists$ $sec(u, \frac{2\pi}{3})$ such that $Q \subset sec(u, \frac{2\pi}{3})$, hence even the property-closure of three different states cannot result in Σ if $\epsilon \leq \frac{1}{2}$. But in this case we have again additional states coming in in the closure. Indeed, if the three points are 'close enough to each other', $cl_{eig}(Q)$ is the triangle with points x, y and z and sides pieces of circles with angle $\frac{\pi}{3}$ (see figure 6e).

c. A is an arbitrary subset of Σ.

1. $\nexists$ $sec(u, \frac{2\pi}{3})$ such that $A \subset sec(u, \frac{2\pi}{3})$.

In this case $cl_{eig}(A) = \Sigma$.

2. $\exists$ $sec(u, \frac{2\pi}{3})$ such that $A \subset sec(u, \frac{2\pi}{3})$.

Then $cl_{eig}(Q) = \cap_{Q \subset sec(u, \frac{2\pi}{3}), v \in surf} sec(u, \frac{2\pi}{3})$.

8.5 The classical case.

We choose $\epsilon = 0$. Then $d_m = -1$.

We have $cos(\lambda^0_{-1}) = -1$. Hence $\lambda^0_{-1} = \pi$. So we have $H(\pi) = \{sec^o(u, \pi) \mid u \in surf\} \cup \{surf\}$.

As we shall show, we only have to consider the case where A is an arbitrary subset of Σ. Let us construct $cl_{eig}(A)$.

We remark that $sec^o(u, \pi) = \{u\}^C$, hence the open spherical sectors in this case are just the complements of the points of $surf$. Clearly there always exists a spherical sector $sec^o(u, \pi)$ such that $Q \subset sec^o(u, \pi)$. If we introduce again $cl_{eig}(Q)^C$ as the set theoretical complement of $cl_{eig}(Q)$, and remark that any set equals the union of its points, we have $cl_{eig}(Q)^C = \cup_{u \in Q^C, u \in surf}\{u\} = Q^C$. Hence $cl_{eig}(Q) = Q$ for any Q. This shows that in this case of $\epsilon = 0$ there is never a superposition since the closure of any subset of the set of states equals this subset. With other words, if $\epsilon = 0$ we have $\mathcal{A}(\Sigma) = \mathcal{P}(\Sigma)$, and the collection of the eigenstate-sets is isomorphic to $\mathcal{P}(\Sigma)$.

It is interesting to remark that for all the cases $\epsilon \neq 0$ the closure $cl_{eig}(Q)$ of an arbitrary subset $Q \subset surf$ is a topologically closed subset, since it is the intersection of closed sets. Only for the special case of $\epsilon = 0$ the closure $cl_{eig}(Q)$ of an arbitrary subset $Q \subset surf$ is equal to Q, and this is possible since it is the intersection of topologically open sets, and an infinite intersection of open sets can be open or closed.

Fig. 6 : Construction of the closure set $cl_{eig}(A)$ with A=$\{p_x,p_y\}$ and A=$\{p_x,p_y,p_z\}$ for different values of ϵ. We use the representation introduced in figures 1b, 2b and 3b.

6a : ϵ=3/4, A=$\{p_x,p_y\}$
6b : ϵ=1/2, A=$\{p_x,p_y\}$
6c : ϵ=3/4, A=$\{p_x,p_y,p_z\}$
6d : ϵ=1/2, A=$\{p_x,p_y,p_z\}$
6e : ϵ=1/4, A=$\{p_x,p_y,p_z\}$

Fig. 6a

Fig. 6b

Fig. 6c

Fig. 6d

Fig. 6e

9. The occurrence of the standard topology in the ϵ-model.

We would like to return now to the physical meaning of the ϵ-example. The probabilities in the model find their origin in the presence of fluctuations on the measurement apparatuses. The quantum situation appears if the magnitude of the fluctuations on the measurement apparatuses is maximal. In this case the elastic can break uniformly in all of its points. We have proposed to characterize the classical situation as being the situations with zero fluctuations, where the elastic breaks in one predetermined point. This leads us to a collection of eigenstate-sets $\mathcal{F}(\Sigma)$ that is Boolean and isomorphic to $\mathcal{P}(\Sigma)$. We have two remarks in relation with this finding.

1) A situation of zero fluctuations on the measurement apparatuses is a very strong idealization. It would be much more realistic to describe the classical situation as being the situation where the fluctuations can be minimized as much as one wants. Let us see whether we can construct the eigen-closure structure corresponding to this slightly more realistic situation.

2) We have remarked already that a closure structure is a generalization of an ordinary topology. For $\epsilon = 0$ each subset K of Σ is closed for cl_{eig} and hence $\mathcal{F}(\Sigma) = \mathcal{P}(\Sigma)$, which shows that closure structure for the case $\epsilon = 0$ corresponds to the discrete topology on Σ. This result is in some sense not very satisfactory as to the possibility of characterizing the eigenstate-sets of an entity by means of the closure structure. We would rather have liked the classical situation to correspond to a closure structure corresponding with the ordinary classical topology on the sphere.

To answer both questions we want to investigate the classical situation in more detail.

Definition 13. *We define $\mathcal{E}_{stand}$ as the set of all $e^{\epsilon}_{u,d}$ such that $0 < \epsilon$, and $\mathcal{F}_{stand}$ as the collection of eigenstate-sets generated by the measurements $\mathcal{E}_{stand}$*

We know that $\mathcal{B}^{\epsilon} = H(\lambda^{\epsilon}_{d})$ is a generating set for the collection of closed subspaces $\mathcal{F}^{\epsilon}$. From this follows that $\mathcal{B}_{stand} = \cup_{0<\epsilon}\mathcal{B}^{\epsilon}$ is a generating set for $\mathcal{F}_{stand}$, and if we consider an arbitrary subset $K \subset \Sigma$, and $Q = \{u \mid p_u \in K\}$, we can find the closure structure corresponding to this collection of eigenstate-sets. Let us denote the closure generating this closure structure by cl_{stand}

Theorem 15. *Consider an arbitrary subset $K \subset \Sigma$, and let $Q = \{u \mid p_u \in K\}$. Then :*

$$cl_{stand}(K) = \{p_w \mid w \in \cup_{Q \subset R, R \in \mathcal{B}_{stand}} R\} \qquad (39)$$

Theorem 16. *The eigen-closure structure introduced by cl_{stand}, with collection of eigen-closed subsets $\mathcal{F}_{stand}$ is the ordinary topology on the surface of the sphere $surf$, and as a consequence we have for $K, L \subset \Sigma$:*

$$cl_{stand}(K \cup L) = cl_{stand}(K) \cup cl_{stand}(L) \tag{40}$$

and from this follows that $\mathcal{F}_{stand}$ is a Boolean lattice.

Proof : Consider $\mathcal{B}_{stand}$, and $R \in \mathcal{B}_{stand}$, R different from $surf$. Then R is a closed spherical sector $sec(u, \lambda_d^\epsilon)$. If we consider R^C, then it is an open spherical sector $sec^o(u, \pi - \lambda_d^\epsilon)$. If we choose an arbitrary $0 < \delta$ we can always find an $0 < \epsilon$ such that $\pi - \lambda_d^\epsilon < \delta$. This proves that $Z = \{R^C \mid R \in \mathcal{B}_{stand}\}$ is a base for the ordinary topology on the surface of the sphere, and as a consequence $\mathcal{F}_{stand}$ corresponds to the ordinary topology on the sphere. For $Q \subset surf$ we denote by $\bar{Q}$ the closure of Q in this ordinary topology of the sphere. Then follows that for $K \subset \Sigma$ we have $cl_{stand}(K) = \{p_v \mid v \in \bar{Q}\}$, if $K = \{p_v \mid v \in Q\}$. Since for the ordinary topology on the surface of the sphere we have that the closure of the union of two subsets is the union of the closures of the subsets, we have $cl_{stand}(K \cup L) = cl_{stand}(K) \cup cl_{stand}(L)$. Let us show now that this is sufficient to make the closure structure Boolean. Suppose that we consider $K, L, M \in \mathcal{F}_{stand}$, then $cl_{stand}(K \cup L) \cap M = (cl_{stand}(K) \cup cl_{stand}(L)) \cap M = (K \cup L) \cap M = (K \cap M) \cup (L \cap M) \subset cl_{stand}((K \cap M) \cup (L \cap M))$. On the other hand we have $(K \cap M) \cup (L \cap M) \subset (K \cup L) \cap M$, which implies that $(K \cap M) \cup (L \cap M) \subset cl_{stand}(K \cup L) \cap M$ and because $cl_{stand}(K \cup L) \cap M$ is closed we have $cl_{stand}((K \cap M) \cup (L \cap M)) \subset cl_{stand}(K \cup L) \cap M$. This shows that $cl_{stand}(K \cup L) \cap M = cl_{stand}((K \cap M) \cup (L \cap M))$. We also have $cl_{stand}(K \cup M) \cap cl_{stand}(L \cup M) = (K \cup M) \cap (L \cup M) = (K \cap L) \cup M \subset cl_{stand}((K \cap L) \cup M)$. In an analogous way we show that $cl_{stand}((K \cap L) \cup M) \subset cl_{stand}(K \cup M) \cap cl_{stand}(L \cup M)$ which shows that $cl_{stand}((K \cap L) \cup M) = cl_{stand}(K \cup M) \cap cl_{stand}(L \cup M)$. From these equalities follows that $\mathcal{F}_{stand}$ is a Boolean closure structure.

If we define, in a more realistic way, the classical situation as being the situation where the fluctuations on the measurement apparatuses can be made arbitrary small (but not necessarily zero), the collection of eigenstate-sets of this situation is isomorphic to the set of all closed subsets of the surface of the sphere. It is a Boolean lattice, however not isomorphic to the set of all subsets of Σ. Theorem 16 shows that it is interesting to introduce the concept of closure structure for the collection of eigenstate-sets of an entity, and that it is a generalization of the ordinary concept of topology. Such a closure structure is able to formalize the ordinary topological closures as well as the linear closures and in this sense captures the classical and the quantum, including also the intermediate.

10. Conclusion.

In this paper we introduced a framework to study quantum, classical and intermediate situations. The main ingredient of this framework is the closure structure, which we encountered while we studied the eigenstate-sets. We also studied this closure structure for the specific case of the ϵ-model. In this way we obtained a one parameter family that contains the linear closure in vector space, and gave rise to the standard topological closure.

As to the physical relevance of the intermediate structure that we have introduced, we think of the physics in the mesoscopic region (between micro and macro) of reality. It is well known that orthodox quantum mechanics causes great difficulties in this region, and that physicists and theoretical chemists use a kind of mixture of classical and quantum theories in an almost heuristic way. We could imagine that the sphere model, for a value of $\epsilon < 1$ could deliver a description of the spin 1/2 of very large molecules. Unfortunately there exists no real experimental data that could allow the testing of such a hypothesis. In our group we are however reflecting about specific experiments that could give some information on the behavior of physical entities that are between the micro and macro level of reality.

We want to conclude coming back to the classical limit case of the sphere model, and remark that this this classical situation is not completely deterministic. Indeed, for each measurement $e^0_{u,d} \in \mathcal{E}^0$, we have classical states of unstable equilibrium, namely the states for which the point particle falls exactly on the point where the elastic breaks. Following our general definition of superposition states these states of unstable equilibrium are superposition states. But they have measure zero compared to the eigenstates and hence don't contribute in the same way to the structures of the entity. That is the reason why even in the presence of such a probability due to unstable equilibrium states, the classical situation gives rise to the well known classical structures. For the collection of eigenstate-sets it gives rise to a Boolean algebra isomorphic to $\mathcal{P}(\Sigma)$. With growing fluctuations on the measurement apparatuses this classical unstable equilibrium explodes and gives rise to quantum like structures. It grows to the full quantum structure when the fluctuations are maximal.

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