A mechanistic macroscopic physical entity with a three-dimensional Hilbert space description*

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Abstract

It is sometimes stated that Gleason's theorem prevents the construction of hidden-variable models for quantum entities described in a more than two-dimensional Hilbert space. In this paper however we explicitly construct a classical (macroscopical) system that can be represented in a three-dimensional real Hilbert space, the probability structure appearing as the result of a lack of knowledge about the measurement context. We briefly discuss Gleason's theorem from this point of view.

1 Introduction

Even after more than 60 years there remain many problems on the 'understanding' of quantum mechanics. From the early days, a main concern of the majority of physicists reflecting on the foundations of the theory has been the question of understanding the nature of the quantum probability. At the other hand, it was a problem to understand the appearance of probabilities in classical theories, since we all agree that it finds its origin in a lack of knowledge about a deeper deterministic reality. The archetypic example is found in thermodynamics, where the probabilities associated with macroscopic observables such as pressure, volume, temperature, energy and entropy are due to the fact that the 'real' state of the entity is characterized deterministically by all the microscopic variables of positions and momenta of the constituting entities, the probabilities describing *our* lack of knowledge about the microscopic state of the entity. The variables of momenta and positions of the individual entities can be considered as 'hidden variables', present in the underlying reality. This example can stand for many of the attempts that have been undertaken to explain the notion of quantum probability, and the underlying theories are called 'hidden variable' theories. In general, for a hidden variable theory, one aims at constructing a theory of an underlying deterministic reality, in such a way that the quantum

observables appear as observables that do not reach this underlying 'hidden' reality and the quantum probabilities finding their origin in a lack of knowledge about this underlying reality.

Von Neumann [1] gave a first impossibility proof for hidden variable theories for quantum mechanics. It was remarked by Bell [2] that in the proof of his No-Go theorem, Von Neumann had made an assumption that was not necessarily justified, and Bell explicitly constructs a hidden variable model for the spin of a spin-$\frac{1}{2}$ quantum particle. Bell also criticizes the impossibility proof of Gleason [3], and he correctly points out the danger of demanding extra 'mathematical' assumptions without an exact knowledge on their physical meaning. Very specific attention was paid to this danger in the study of Kochen and Specker [4], and their impossibility proof is often considered as closing the debate. We can state that each of these impossibility proofs consists in showing that a hidden variable theory (under certain assumptions) gives rise to a certain mathematical structure for the set of observables of the physical system under consideration, while the set of observables of a quantum system does not have this mathematical structure. Therefore it is impossible to replace quantum mechanics by a hidden variable theory (satisfying the assumptions). To be more specific, if one works in the category of observables, then a hidden variable theory (under the given assumptions) gives rise to a commutative algebraic structure for the set of observables, while the set of observables of a quantum system is non-commutative. If one works in the category of properties (yes-no observables) then a hidden variable theory (satisfying the assumptions) has always a Boolean lattice structure for the set of properties while the lattice of properties of a quantum system is not Boolean. If one works in the category of probability models, then a hidden variable theory (satisfying the assumptions) has always a Kolmogorovian probability model for the set of properties while the quantum probability model is not Kolmogorovian. Most of the mathematically oriented physicists, once aware of these fundamental structural differences, gave up the hope that it would ever be possible to replace quantum mechanics by a hidden variable theory. However, it turned out that the state of affairs was even more complicated than the structural differences in the different mathematical categories would make us believe. We have already mentioned that the No-Go theorems for hidden variables, from Von Neumann to Kochen and Specker, depended on some assumptions about the nature of these hidden variable theories. We shall not go into details about the specific assumptions related to each specific No-Go theorem, because in the mean time it became clear that there is one central assumption that is at the core of each of these theorems: the hidden variables have to be hidden variables of the *state* of the physical entity under consideration and specify a deeper underlying reality of the physical entity itself, independent of the specific measurement that is performed. Therefore we shall call them state hidden variables. This assumption is of course inspired by the situation in thermodynamics, where statistical mechanics is the hidden variable theory, and indeed, the momenta and positions of the molecules of the thermodynamical entity specify a deeper underlying reality of this thermodynamical entity, independent of the macroscopic observable that is measured. It was already remarked that there exists always the mathematical possibility to construct so-called contextual hidden variable models for quantum particles, where one allows the hidden variables to depend on the measurement under consideration (e.g. the spin model proposed by Bell [2]). For the general case we refer to a theorem proved by Gudder [5]. However,

generally this kind of theories are only considered as a mathematical curiosum, but physically rather irrelevant. Indeed, it seems difficult to conceive from a physical point of view that the nature of the deeper underlying reality of the quantum entity would depend on the measurement to be performed. To conclude we can state that : (1) only state hidden variable theories were considered to be physically relevant for the solution of the hidden variable problem, (2) for non-contextual state hidden variable theories the No-Go theorem of Kochen and Specker concludes the situation; it is not possible to construct a hidden variable theory of the non-contextual state type that substitutes quantum mechanics.

What we want to point out is that, from a physical point of view, it is possible to imagine that not only the quantum system can have a deeper underlying reality, but also the physical measurement process for each particular measurement. If this is true, then the physical origin of the quantum probabilities could be connected with a lack of knowledge about a deeper underlying reality of the measurement process. In [6,7,8] this idea was explored and it has been shown that such a lack of knowledge gives indeed rise to a quantum structure (quantum probability model, non-commutative set of observables, non-Boolean lattice of properties). This uncertainty about the interaction between the measurement device and the physical entity can be eliminated by introducing hidden variables that describe the fluctuations in the measurement context. However, they are not state hidden variables, they rather describe an underlying reality for each measurement process, and therefore they have been called 'hidden measurements', and the corresponding theories 'hidden measurement theories'. Suppose that weperform a measurement e on a physical system S and that there is a lack of knowledge on the measuring process connected with e, in such a way that there exist 'hidden measurements' e_λ, where each e_λ has the same outcome set as e, and each e_λ is deterministic, which means that for a given state p of the system S, for each λ the hidden measurement e_λ has a determined outcome. Now the fundamental idea is that each time when the measurement e is performed, it is actually one of the e_λ, each with a certain probability, that takes place in the underlying hidden reality. In [6,7,8] it is shown that a hidden measurement model can be constructed for any arbitrary quantum mechanical system of finite dimension , and the possibility of constructing a hidden measurement model for an infinite dimensional quantum system can be found in [8]. Although the models presented in these papers illustrate our point about the possibility of explaining the quantum probabilities in this way, there is always the possibility to construct more concrete macroscopic models, only dealing with real macroscopic entities and real interactions between the measurement device and the entities, that give rise to quantum mechanical structures. It is our point of view that these realistic macroscopic models are important from a physical and philosophical point of view, because one can visually perceive how the quantum-like probability arises. One of the authors introduced such a real macroscopic model for the spin of a spin-$\frac{1}{2}$ quantum entity. When he presents this spin model for an audience, it was often raised that this kind of realistic macroscopic model can only be built for the case of a two-dimensional Hilbert space quantum entity, because of the theorem of Gleason and the paper of Kochen and Specker. Gleason's theorem is only valid for a Hilbert space with more than two dimensions and hence not for the two-dimensional complex Hilbert space that is used in quantum mechanics to describe the spin of a spin-$\frac{1}{2}$ quantum entity. In the paper of Kochen and Specker also a spin model for the

spin of a spin-$\frac{1}{2}$ quantum entity is constructed, and a real macroscopic realization of this spin model is proposed. They point out on different occasions that such a real model can only be constructed for a quantum entity with a Hilbert space of dimension not larger than two. The aim of this paper is to clarify this dimensional problem. Therefore we shall construct a real macroscopic physical entity and measurements on this entity that give rise to a quantum mechanical model for the case of a three-dimensional real Hilbert space, a situation where Gleason's theorem is already fully applicable. We remark that one of the authors [10] presented a model for a spin-1 quantum entity that allows in a rather straightforward way a hidden measurement representation. Nevertheless, since he only considered a set of coherent spin-1 states (i.e., a set of states that spans a three-dimensional Hilbert space, but that does not fill it) his model cannot be considered as a satisfactory counter argument against the No-Go theorems.

In the first two sections, we briefly give the two-dimensional examples of Aerts and Kochen-Specker and analyze their differences. In section 4 we investigate the dimensional problem related to the possible hidden variable models. Afterwards, we construct a hidden measurement model with a mathematical structure for its set of states and observables that can be represented in a three-dimensional real Hilbert space.

2   The two-dimensional model

The physical entity that we consider is a point particle P that can move on the surface of the unit sphere S^2. Every unit vector $\mathbf{v}$ represents a state p_v of the entity. For every point $\mathbf{u}$ of S^2 we define a measurement e_u as follows : a rubber string between $\mathbf{u}$ and its antipodal point $-\mathbf{u}$ catches the particle P that falls orthogonally and sticks to it. Next, the string breaks somewhere with a uniform probability density and the particle P moves to one of the points $\mathbf{u}$ or $-\mathbf{u}$, depending on the piece of elastic it was attached to. If it arrives in $\mathbf{u}$ we will give the outcome o_1^u to the experiment, in the other case we will say that the outcome o_2^u has occurred. After the measurement the entity will be in a new state: p_u in the case of outcome o_1^u and p_{-u} in the other case. Taking into account that the elastic breaks uniformly, it is easy to calculate the probabilities for the two results:

$$P(o_1^u|p_v) = \frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$$

$$P(o_2^u|p_v) = \frac{1-\cos\theta}{2} = sin^2\frac{\theta}{2}$$

with $\theta = \mathbf{u} \cdot \mathbf{v}$. We have the same results for the probabilities associated with the spin measurement of a quantum entity of spin-$\frac{1}{2}$ (see [2,8]), so we can describe our macroscopic example by the ordinary quantum formalism where the set of states is given by the points of a two-dimensional complex Hilbert space. Clearly, we can also interpret this macroscopic example as a hidden variable model of the spin measurement of a quantum entity of spin-$\frac{1}{2}$. Indeed, if the point λ where the string disintegrates is known, the measurement outcome is certain. The probabilities in this model appear because of our lack of knowledge of the precise interaction between the entity and the measurement device. Every spin measurement e_u can be considered as a class of classical spin measurements e_u^λ with determined outcomes, and the probabilities are the result of an averaging

process. In this example it is clear that the hidden variable λ is neither a variable of the entity under study nor a variable pertaining to the measurement apparatus. Rather, it is a variable belonging to the measurement process as a whole.

3   The Kochen-Specker example

In Kochen and Specker's model, again a point P on a sphere represents the quantum state of the spin-$\frac{1}{2}$ entity. However, at the same time the entity is in a hidden state which is represented by another point T_P of S_P^+, the upper half sphere with P as its north pole, determined in the following way. A disk D of the same radius as the sphere is placed perpendicular to the line OP which connects P with the center O of the sphere and centred directly above P. A particle is placed on the disk that is now shaken "randomly", i.e., in such a way that the probability that the particle will end up in a region U of the disk is proportional to the area of U. The point T is then the orthogonal projection of the particle. The probability density function $\mu(T)$ is

$$\mu(T) = \begin{cases} \frac{1}{\pi}\cos\theta & 0 \le \theta \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

where θ is the angle between T and P. If a measurement is made in the direction OQ the outcome "spin up" will be found in the case that $T \in S_Q^+$ and "spin down" otherwise. As a result of the measurement the new state of the entity will be Q in case of spin up and $-Q$ otherwise. The new hidden state T_Q is now determined as before, the disk being placed now at Q if the new state is Q and at $-Q$ if otherwise. It can be shown that the same probabilities as for the quantum spin-$\frac{1}{2}$ entity occur. It is important to remark that the hidden variable here pertains to the entity under study, as was made clear by using the expression "hidden state". But is this really the case? As we look closer we see that for every consecutive spin measurement to reveal the correct probabilities, we need each time a randomisation of the hidden state T. Thus every time a measurement occurs the hidden variable has to be reset again. In practice this means that for every measurement a new value of the variable will be needed. Thus we can make the philosophical important step to remove this "hidden state" from the entity and absorb it within the context of the measurement itself, indeed a reasonable thing to do. Once this is done, the analogy with the model of section 3 is obvious. But it is also clear that a new idea has been introduced, namely the shift of the hidden variable from the entity towards the measurement process. This is not only a new feature for a hidden variable theory, but also a natural way out of the traps of the No-Go theorems.

4   The Dimensional problems

As was pointed out by several authors (see [3,4,5,6,13,14]), it is possible to prove that "reasonable" hidden variable theories don't exist for Hilbert spaces with a dimension greater than two. Moreover, other arguments show the necessity for a proof of existence of a hidden variable model with a more than two-dimensional state space. There is for instance the theorem of Gleason which states that for

a propositional system corresponding to a three-dimensional real Hilbert space there exists a unique probability function over the set of propositions which satisfies some very plausible properties. This means that every hidden variable theory (satisfying these assumptions) can only reveal the same probabilities as the quantum probability function and this would render the hidden variable theory redundant, because no extra information can be gained. To prove that the No-Go theorems are too restrictive it is thus necessary (but also sufficient!) to give one "reasonable" example with a three-dimensional Hilbert state space and this is exactly what we will do now.

5 The 3-Dimensional model

In this section we introduce a mechanistic macroscopic physical entity with a three-dimensional Hilbert space quantum description. Probably there exist models that are much more elegant than the one we propose, because the explicit realization would be rather non-trivial, but for our purpose it is sufficient to prove that there exists at least one. Once again we remark that the system that we present is not a representation of a quantum mechanical entity, but a macroscopic physical entity that gives rise to the same probability structure as one encounters in quantum mechanics. First we propose the model and, for reasons of readability, we present a geometrical equivalent in $\mathbb{R}^3$. In this way we can easily prove the equivalence between the model and the quantum mechanical case. In section 5.3 we shall study the probability structure of the model.

5.1 The practical realization

The entity S that we consider is a rod of length 2 which is fixed in its center point c, both sides of which have to be identified. The set of states of the entity, i.e. the set of rays in Euclidean 3-space, possibly characterized by one of the two end points of the rod (denoted by x_p), will be denoted by Σ_S. The measurement apparatus consists of three mutual orthogonal rods, parallel with rays $\widehat{x}_e^1, \widehat{x}_e^2, \widehat{x}_e^3$, fixed in 3-space. The entity and the measurement device are coupled for a measurement in the following way : (see Fig. 1) :
• Connection in x_e^i: the rod floats in a slider which is fixed orthogonal to the rod of the measurement apparatus.
• Connection in x_p: the three interaction-rods are fixed to one slider, which floats on the "entity-rod".
• We also fix three rubber strings between the entity-rod and the three rods of the measurement apparatus.
The last ingredient that takes part in the interaction is something we call a "random gun". This is a gun, fixed on a slider that floats on and turns around the entity-rod in such a way that:
• The gun is shooting in a direction orthogonal to the entity-rod.
• The movement and the frequency of shooting are at random but such that the probability of shooting a bullet in a certain direction, and from a certain point of the entity-rod is uniformly distributed, i.e., the gun distributes the bullets uniformly in all directions and from all the points of the rod. If a bullet hits one of the connections, both the rod and string break, such that the entity can

start moving (there is one new degree of freedom), and it is clear that the two non broken strings will tear the entity into the plane of the measurement-rods to which it is still connected.
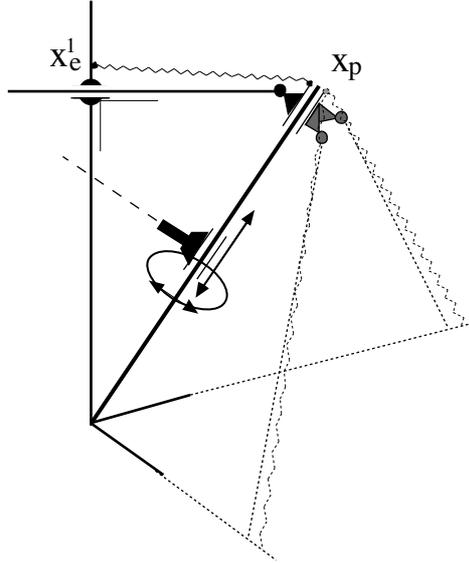

Fig. 1 : Practical realization of the model. With rods, sliders, strings and a "random gun" we construct a device with a mathematical structure equivalent to the one for a quantum entity with a three-dimensional real Hilbert state space.

5.2 A geometrical equivalent of the model

To facilitate the calculation of the probabilities we will describe what happens during the measurement from a geometrical point of view. We know that a state p of the entity is characterized by the angles $\theta_1, \theta_2, \theta_3$ between the rod and an arbitrary selected set of three orthonormal axis in Euclidean 3-space E^3. It is clear that this set of states corresponds in a one-to-one way with the states of an entity described in a three-dimensional real Hilbert space.

The set of measurements to be performed on this entity S is characterized as follows. Let $\widehat{x}_e^1, \widehat{x}_e^2, \widehat{x}_e^3$ be the three mutual orthogonal rays coinciding with the rods of the measurement apparatus. As a consequence, for a given state p, and a given experiment e, we have the three angles $\theta_1, \theta_2, \theta_3$ as representative parameters to characterize the state, relative to the measurement apparatus. We denote by x_e^1, x_e^2, x_e^3, the orthogonal projections of x_p on the three rays $\widehat{x}_e^1, \widehat{x}_e^2, \widehat{x}_e^3$, forming a set of points representative for the couple (p, e). The geometrical description of the measurement process goes as follows:

i) Every point x_e^i is connected with x_p by a segment denoted by $[x_e^i, x_p]$ with length $sin\theta_i$. Therefore the length of the projection of $[x_e^i, x_p]$ on the rod is $cos(\frac{\Pi}{2} - \theta_i).sin\theta_i = sin^2\theta_i$.

ii) Next, one of the connections $[x_e^i, x_p]$ breaks with a probability proportional to the length of the projection of $[x_e^i, x_p]$ on the rod (In Fig. 4 and Fig. 5 we suppose that $[x_e^1, x_p]$ breaks). The rod rotates into the plane of the two remaining points x_e^j, x_e^k, to which it is still connected, and such that the point x_p', the projection of x_p on the $\widehat{x_e^j x_e^k}$-plane, lies on the rod. As a consequence,

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the connections $[x_e^j, x_p']$ and $[x_e^k, x_p']$ are still orthogonal to the corresponding axes $\widehat{x}_e^j$ and $\widehat{x}_e^k$.
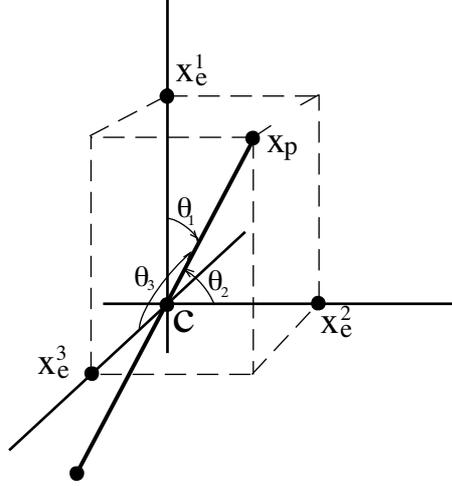

Fig. 2 : The states of the classical mechanistic entity, a rod in Euclidean 3-space, represented by x_p, one of the two end points of the rod. Thus, the different states p of the entity are represented by the angels $\theta_1, \theta_2, \theta_3$ between the rod and three mutual orthogonal rays $\widehat{x}_e^1, \widehat{x}_1^2, \widehat{x}_1^3$, representative for a measurement e. x_e^1, x_e^2, x_e^3, the orthogonal projections of x_p on the three rays are thus representative for the couple (p,e).
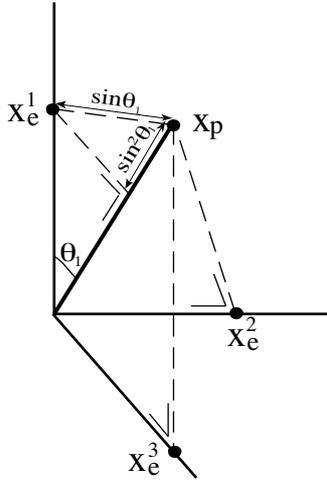

Fig. 3 : The first step of the measurement. Every point x_e^i is connected with x_p by a segment denoted by $[x_e^i, x_p]$. The length of the projection of $[x_e^1, x_p]$ on the rod is $sin^2\theta_1$.

iii) We proceed with this new, two-dimensional situation characterized by the elements $\{x_p', x_e^j, x_e^k\}$ as before, denoting the angle between x_p' and x_e^j as θ_j'. One of the segments, $[x_e^j, x_p']$ or $[x_e^k, x_p']$, seizes to exist, again with a probability proportional to the length of the projection of this segment on the rod, equal to $sin\theta_i.sin^2\theta_j'$. Ultimately, the rod rotates towards and stabilises at the third ray, to which it is still connected.

The global process can thus be seen as a measurement e, with three possible outcomes o_e^1, o_e^2, o_e^3, on an entity S in a state p.
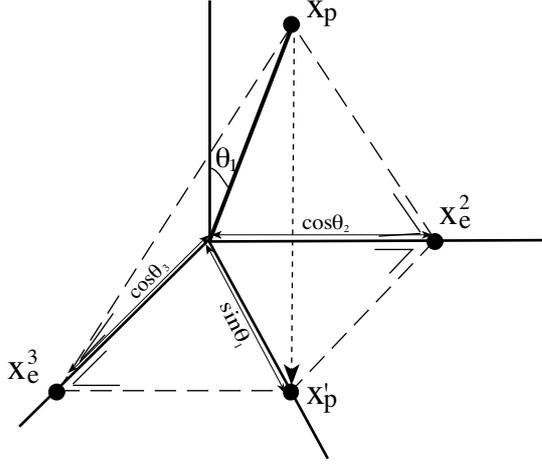

Fig. 4 : The second step of the measurement. One of the connections, $[x_e^1, x_p]$, breaks with probability proportional to the length of the projection of $[x_e^1, x_p]$ on the rod. The rod rotates into the plane of the two points x_e^2, x_e^3 in such a way that the connections $[x_e^2, x_p']$ and $[x_e^3, x_p']$ are still orthogonal to the corresponding axes $\widehat{x_e^2}$ and $\widehat{x_e^3}$.
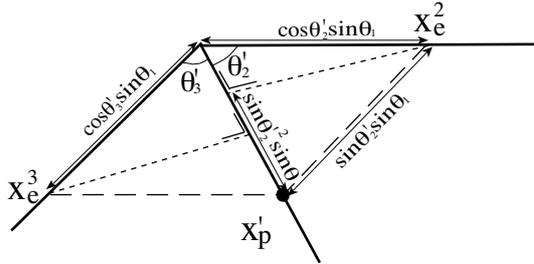

Fig. 5 : We proceed with $\{x_p', x_e^2, x_e^3\}$ as we did with $\{x_p, x_e^1, x_e^2, x_e^3\}$. One of the two existing connections breaks with probability proportional to the length of the projection of the corresponding segment on the rod, equal to $sin\theta_1.sin^2\theta_2'$.

5.3 The probability structure of the model

After this geometrical representation of our model it becomes very easy to calculate the probability to obtain an outcome o_e^k, equivalent with neither obtaining o_e^i nor o_e^j, and thus with the breaking of these two connections. Suppose that first $[x_e^i, x_p]$ breaks and then $[x_e^j, x_p']$. Since $cos^2\theta_i + cos^2\theta_j + cos^2\theta_k = 1$ we have $sin^2\theta_i + sin^2\theta_j + sin^2\theta_k = 2$. So we find $(sin^2\theta_i)/2$ for the probability for the breaking of $[x_e^i, x_p]$. Since $cos^2\theta_j' + cos^2\theta_k' = 1$ we have $sin^2\theta_j' + sin^2\theta_k' = 1$. Thus $sin^2\theta_j'$ is the conditional probability for the breaking of $[x_e^j, x_p']$ supposing that the connection between x_e^i and the rod broke first. This

yields $\frac{1}{2}sin^2\theta_i.sin^2\theta'_j = \frac{1}{2}sin^2\theta_i.cos^2\theta'_k = \frac{1}{2}sin^2\theta_i.(\frac{cos\theta_k}{sin\theta_i})^2 = \frac{1}{2}cos^2\theta_k$ for the requested probability. Analogously, we find the same result for the probability that first $[x^j_e, x_p]$ and then $[x^i_e, x'_p]$ breaks.

Therefore we find:

$$P(o^k_e|p) = \cos^2\theta_k$$

where θ_k is the angle between x_p and x^k_e, the eigenstate with eigen-outcome o^k_e of the measurement e on the entity S.

Now we are able to compare the probability structure associated with our model with the one encountered in quantum mechanics. For a three-dimensional real Hilbert space $\mathcal{H}_{\mathbb{R}^3}$ we can write a self-adjoint operator H_e with $\{\widehat{x}^1_e, \widehat{x}^2_e, \widehat{x}^3_e\}$ a set of mutual orthogonal eigen-rays and $\{o^1_e, o^2_e, o^3_e\}$ the corresponding eigenvalues (some of them may be equal), as $H_e = \sum_{i=1}^{i=3} o^i_e E_{\widehat{x}^i_e}$, where $E_{\widehat{x}^i_e}$ is the projector on the ray $\widehat{x}^i_e$. Therefore, we have for every o^i_e, eigen-outcome of a measurement e and associated with an eigenstate represented by a ray $\widehat{x}^i_e$, and for every state p of the system, represented by a ray $\widehat{x}_p$:

$$P(o^i_e|p) = | < x^i_e| \ x_p > |^2 = \cos^2\theta_i$$

where θ_i is the angle between the rays $\widehat{x}^i_e$ and $\widehat{x}_p$. It is therefore clear that the entity in our model corresponds in a one-to-one way with a quantum entity described in a three-dimensional real Hilbert space.

6   Discussion

In this paper we have presented a macroscopic device with a quantum-like probability structure and state space. Since one can interpret this model as a hidden variable description for a quantum entity, we can analyse the relationship with Gleason's theorem, which implies the existence of a unique probability measure for a physical entity if its state space is a more than two-dimensional separable Hilbert space $\mathcal{H}$ and if this probability measure satisfies some reasonably looking a priori assumptions. For pure states Gleason's theorem takes the following form: if $p : \mathcal{L}(\mathcal{H}) \to [0, 1]$ is a (generalised) probability measure, there exists a unit vector $\psi \in \mathcal{H}$ such that $\forall \ P \in \mathcal{L}(\mathcal{H}) : p(P) = \ < \psi \ | \ P\psi >$, with $\mathcal{L}(\mathcal{H})$ the lattice of closed subspaces of the Hilbert space. In our case it asserts that the probability to obtain say o^i_e necessarily takes the form that was given above in this paper. Therefore it is implicit in the assumptions of the theorem that the probabilities only depend on the initial and final state of the entity. However, referring to our model we see that it is easy to invent other probability measures that actually do depend on the intermediate states of the entity and therefore do not satisfy the assumptions of Gleason's theorem. For instance, one can imagine that the random gun is absent and the interaction rods break with a uniform probability density, resulting in the first probability being proportional to $sin\theta_i$ in stead of $sin^2\theta_i$. Since the hidden measurement approach is obviously a contextual theory that keeps the Hilbert space framework for its state space, but situates the origin of the quantum probability in the measurement environment, there is no need for the existence of dispersion-free probability measures on $\mathcal{L}(\mathcal{H})$ as in the conventional non-contextual state hidden variable theories.

7 References

[1] J. Von Neumann, *Grudlehren*, Math. Wiss. XXXVIII, 1932. [2] J.S. Bell, Rev. Mod. Phys. **38**, 447, 1966.

[3] A.M. Gleason, J. Math. Mech. **6**, 885, 1957. [4] S. Kochen and E.P. Specker, J. Math. Mech. **17**, 59, 1967. [5] S.P. Gudder, J. Math. Phys **11**, 431 (1970).

[6] D. Aerts, *A possible explanation for the probabilities of quantum mechanics and a macroscopic situation that violates Bell inequalities*, in *Recent Developments in Quantum Logic*, eds. P. Mittelstaedt et al., in Grundlagen der Exacten Naturwissenschaften, vol. **6**, Wissenschaftverlag, Bibliographischen Institut, Mannheim, 235, 1984. [7] D. Aerts, *A possible explanation for the probabilities of quantum mechanics*, J. Math. Phys. **27**, 202, 1986. [8] D. Aerts, *The origin of the non-classical character of the quantum probability model*, in Information, Complexity and Control in Quantum Physics, A. Blanquiere, et al., eds., Springer-Verlag, 1987. [9] B. Coecke, Found. Phys. Lett. **8**, 437 (1995). [10] B. Coecke, Helv. Phys. Acta. **68**, 396 (1995).

[11] D. Aerts, Found. Phys. **24**, 1227 (1994). [12] D. Aerts, Int. J. Theor. Phys. **34**, 1165 (1995). [13] D. Aerts, *The Entity and Modern Physics* in *Identity and Individuality of Physical Objects*, ed. T. Peruzzi, Princeton University Press, Princeton, (1995). [14] J.M. Jauch and C. Piron, Helv. Phys. Acta. **36**, 827 (1963). [15] J.S. Bell, Rev. Mod. Phys. **38**, 447 (1966).