A Quantum Structure Description of the Liar Paradox[*]

Diederik Aerts, Jan Broekaert and Sonja Smets

Center Leo Apostel (CLEA),
Brussels Free University,
Krijgskundestraat 33, 1160 Brussels
diraerts@vub.ac.be, jbroekae@vub.ac.be
sonsmets@vub.ac.be

Abstract

In this article we propose an approach that models the truth behavior of cognitive entities (i.e. sets of connected propositions) by taking into account in a very explicit way the possible influence of the cognitive person (the one that interacts with the considered cognitive entity). Hereby we specifically apply the mathematical formalism of quantum mechanics because of the fact that this formalism allows the description of real contextual influences, i.e. the influence of the measuring apparatus on the physical entity. We concentrated on the typical situation of the liar paradox and have shown that (1) the truth-false state of this liar paradox can be represented by a quantum vector of the non-product type in a finite dimensional complex Hilbert space and the different cognitive interactions by the actions of the corresponding quantum projections, (2) the typical oscillations between false and truth - the paradox -is now quantum dynamically described by a Schrödinger equation. We analyse possible philosophical implications of this result.

1 Introduction.

The liar paradox is the oldest semantical paradox we find in literature. In its simplest forms we trace the paradox back to Eubulides - a pupil of Euclid - and to the Cretan Epimenides. From the Greeks on till today, different

---

alternative forms of the liar emerged. We now encounter variations of the one sentence paradox (the simplest form of the liar) but also of the two or more sentence paradox. The two sentence paradox is known as the postcard paradox of Jourdain, which goes back to Buridan in 1300. On one side of a postcard we read 'the sentence on the other side of this card is true' and on the other side of it we read 'the sentence on the other side of this card is false'.

In this paper we will not work with the original forms of the paradox, but in the version in which we use an index or sentence pointer followed by the sentence this index points at :

<u>Single Liar :</u>

(1)   sentence (1) is false

<u>Double Liar :</u>

(1)   sentence (2) is false

(2)   sentence (1) is true

2   Applying the Quantum Mechanical Formalism.

The theories of chaos and complexity have shown that similar patterns of behaviour can be found in very different layers of reality. The success of these theories demonstrates that interesting conclusions about the nature of reality can be inferred from the encountered structural similarities of dynamical behaviour in different regions of reality. Chaos and complexity theories are however deterministic theories that do not take into account the fundamental contextuality that is introduced by the influence of the act of observation on the observed. Most of the regions of reality are highly contextual (e.g. the social layer, the cognitive layer, the pre- material quantum layer), rather with exception of the material layer of reality were contextuality is minimal. In this sense it is strange that no attempts have been undertaken to find similarities using contextual theories, such as quantum mechanics, in the different regions of reality. The study that we present in this paper should be classified as such an attempt, and is part of one of the projects in our center focusing on the layered structure of reality (Clea Research Project,1997-; Aerts, 1994; Aerts, 1999)

We justify the use of the mathematical formalism of quantum mechanics to model context dependent entities, because a similar approach has already

been developed by some of us for the situation of an opinion pole within the social layer of reality (Aerts, 1998; Aerts and Aerts, 1994, 1997; Aerts, Broekaert and Smets, 1999; Aerts, Coecke and Smets, 1999). In such an opinion pole specific questions are put forward that introduce a real influence of the interviewer on the interviewee, such that the situation is contextual. It is shown explicitly in (Aerts and Aerts, 1995, 1997) that the probability model that results is this situation is of a quantum mechanical nature.

By means of a model we will present the liar - one sentence - or the double liar - a group of sentences - as one entity that we consider to 'exist' within the cognitive layer of reality. The existence is being expressed by the possibility of influencing other cognitive entities, and by the different states that it can be in. Indeed it has been shown that the concept of entity can be introduced rigorously and founded on the previously mentioned properties. In this way we justify the present use (Aerts, 1992).

3   Measuring Coginitive Entities : Modeling Truth Behavior.

In this paragraph we will explore the context dependence of cognitive entities like the liar paradox. We introduce the explicit dependence of the truth and falsehood of a sentence on the cognitive interaction with the cognitive person. Reading a sence, or with other words 'making a sentence true or false' will be modeled as 'performing a measurement' on the sentence within the cognitive layer of reality. This means that in our description a sentence within the cognitive layer of reality is 'in general' neither true, nor false. The 'state true' and the 'state false' of the sentence are 'eigenstates' of the measurement. During the act of measurement the state of the sentence changes in such a way that it is true or that it is false. This general 'neither true nor false state' will be called a superposition state in analogy with the quantum mechanical concept. We shall see that it is effectively a superposition state in the mathematical sense after we have introduced the complex Hilbert space description.

We proceed operationally as follows. Before the cognitive measurement (this means before we start to interact with the sentence, read it and make a hypothesis about its truth or falsehood) the sentence is considered to be neither true nor false and hence in a superposition state. If we want to start to analyse the cognitive inferences entailed, we make one of the two possible hypothesis, that it is true or that it is false. The making of one of these two

3

hypothesis - this is part of the act of measurement - changes the state of the sentence to one of the two eigenstates - true or false. As a consequence of the act of measurement the sentence becomes true or false (is in the state true or false) within the cognitive entity were the sentence is part of. This change influences the state of this complete cognitive entity. We will see that if we apply this approach to the double liar, that the change of state puts into work a dynamic process that we can describe by a Schrödinger equation. We have to consider three situations:

$$
A \quad \begin{cases} (1) & \text{sentence (2) is false} \\ (2) & \text{sentence (1) is true} \end{cases}
$$

$$
B \quad \begin{cases} (1) & \text{sentence (2) is true} \\ (2) & \text{sentence (1) is true} \end{cases}
$$

$$
C \quad \begin{cases} (1) & \text{sentence (2) is false} \\ (2) & \text{sentence (1) is false} \end{cases}
$$

4 The Double Liar: A Full Quantum Description.

The resemblance of the truth values of single sentences and the two-fold eigenvalues of a spin-1/2 state is used to construct a dynamical representation; the measurement evolution as well as a continuous time evolution are included.

We recall some elementary properties of a spin state. Elementary particles - like the electron - are bestowed with a property referred to as an intrinsic angular momentum or spin. The spin of a particle is quantised: upon measurement the particle only exposes a finite number of distinct spin values. For the spin-1/2 particle, the number of spin states is two, they are commonly referred to as the 'up' and 'down' state. This two-valuedness can adequately describe the truth function of a liar type cognitive entity. Such a sentence supposedly is either true or false. The quantum mechanical description on the other hand allows a superposition of the 'true' and 'false' state. This corresponds to our view of allowing cognitive entities before measurement - i.e. reading and hypothetising - to reside in a non-determinate state of truth or falsehood. In quantum mechanics such a state Ψ is described by

a poundered superposition of the two states:

$$\Psi = c_{true} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_{false} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The operation of finding whether such a cognitive entity is true or false, is done by applying respectively the true-projector P_{true} or false-projector P_{false}.

$$P_{true} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{false} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In practice in the context of the cognitive entity, this corresponds to the assignment of either truth or falsehood to a sentence after its reading. In quantum mechanics, the true-measurement on the superposed state Ψ results in the true state ;

$$P_{true}\Psi = c_{true} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

while the square modulus of the corresponding pounderation factor c_{true} gives the statistical probability of finding the entity in the true-state. An unequivocal result is therefore not obtained when the superposition does not leave out one of the states completely, i.e. either c_{true} or c_{false} is zero. Only in those instances do we attribute to a sentence its truth or falsehood.

The coupled sentences of the two-sentence liar paradox (C) for instance are precisely described by the so called 'singlet state'. This global state combines, using the tensor product $\otimes$, states of sentence one with states of sentence two:

$$\frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

The appropriate true-projectors for sentence one and two are now:

$$P_{1,true} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1_2 \qquad P_{2,true} = 1_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The false-projectors are obtained by switching the diagonal elements 1 and 0 on the diagonal of the matrix.

In the same manner the coupled sentences of the liar paradox (B) can be constructed:

$$\frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

5

Our final aim is to describe the real double liar paradox (A) quantum mechanically and even more to show how the true-false cycle originates from the Schrödinger time-evolution of the appropriate initial state. The description of this system necessitates the coupled Hilbert space $C^4 \otimes C^4$, a larger space than for the previous systems. In this case the truth and falsehood values from measurement and semantical origin must be discerned, the dimension for each sentence therefore must be 4.

The initial unmeasured state - i.e. Ψ_0 - of the real double liar paradox is:

$$\frac{1}{2}\left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\otimes\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$$

Each next term in this sum is actually the consecutive state which is reached in the course of time, when the paradox is read through. This can be easily verified by applying the appropriate truth-operators:

$$P_{1,true}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\otimes 1_2 \qquad P_{2,true}=1_1\otimes\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

The projectors for the false-states are constructed by placing the 1 on the final diagonal place.

The explicit construction of the unitary evolution operator is accomplished through an intermediary equivalent representation in C^{16}. The complex space $C^4 \otimes C^4$ is isomorphic to C^{16}. In this aim the basis of the C^{16} is constructed as (i and j from 1 to 4) :

$$e_i \otimes e_j = e_{\kappa(i,j)} \quad \text{and} \quad \kappa(i,j) = 4(i-1)+j$$

In C^{16} the unmeasured state Ψ_0 is then given by:

$$\Psi_0 = \frac{1}{2}\{e_{10}+e_8+e_{13}+e_3\}$$

The 4 by 4 submatrix - U_D - of the discrete unitary evolution operator, which describes the time-evolution at instants of time when a sentence has changed truth value, is:

$$U_D=\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

In order to obtain a description at every instance of time, a procedure of diagonalisation on the submatrix U_D was performed, i.e. $U_D|_{\mathrm{diag}}$. From the Schrödinger evolution and Stone's Theorem we obtain:

$$H_{sub}|_{\mathrm{diag}} = i \ln U_D|_{\mathrm{diag}}$$

Now inverting the procedure of diagonalisation, the infinitesimal generator of the time-evolution - the submatrix hamiltonian - is obtained :

$$H_{sub} = \begin{pmatrix} -1/2 & -1/2 & (1-i)/2 & (1+i)/2 \\ -1/2 & -1/2 & (1+i)/2 & (1-i)/2 \\ (1+i)/2 & (1-i)/2 & 1/2 & 1/2 \\ (1-i)/2 & (1+i)/2 & 1/2 & 1/2 \end{pmatrix}$$

The submatrix of the evolution operator $U(t)$, valid at all times is then given by the expression:

$$U_{sub}(t) = e^{-iH_{sub}t}$$

The time evolution operator $U_{sub}(t)$ in the 4 by 4 subspace of C^{16} becomes (modulo a numerical factor $\frac{1}{4}$ for all elements):

$$\begin{pmatrix} 1+e^{-it}+e^{it}+e^{2it} & 1-e^{-it}-e^{it}+e^{2it} & 1-ie^{-it}+ie^{it}-e^{2it} & 1+ie^{-it}-ie^{it}-e^{2it} \\ 1-e^{-it}-e^{it}+e^{2it} & 1+e^{-it}+e^{it}+e^{2it} & 1+ie^{-it}-ie^{it}-e^{2it} & 1-ie^{-it}+ie^{it}-e^{2it} \\ 1+ie^{-it}-ie^{it}-e^{2it} & 1-ie^{-it}+ie^{it}-e^{2it} & 1+e^{-it}+e^{it}+e^{2it} & 1-e^{-it}-e^{it}+e^{2it} \\ 1-ie^{-it}+ie^{it}-e^{2it} & 1+ie^{-it}-ie^{it}-e^{2it} & 1-e^{-it}-e^{it}+e^{2it} & 1+e^{-it}+e^{it}+e^{2it} \end{pmatrix}$$

The hamiltonian H as well as the time-evolution operator $U(t)$ in $C^4 \otimes C^4$ is immediately obtained by inverting the basis transformation function κ:

$$H = \sum_{\kappa,\lambda=1}^{16} H_{sub\kappa(i,j)\lambda(u,v)} O_{iu} \otimes O_{jv}$$

and

$$U(t) = \sum_{\kappa,\lambda=1}^{16} U_{sub\kappa(i,j)\lambda(u,v)}(t) O_{iu} \otimes O_{jv}$$

with;

$$O_{iu} \otimes O_{jv} = \{e_i.e_u^t\} \otimes \{e_j.e_v^t\}$$

For example, term $\kappa = 3$, $\lambda = 10$ of the time evolution operator $U(t)$ is;

$$\frac{1}{4}(1 - ie^{-it} + ie^{-it} - ie^{2it}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Starting from the initial state Ψ_0 the constructed dynamical evolution leaves the system unchanged; Ψ_0 is a time invariant state:

$$\Psi_0(t) = \Psi_0$$

As soon as a measurement for truth or falsehood on either of the sentences is made, the dynamical evolution sets of in a cyclical mode, attributing alternatively thruth and falsehood to the consecutively read sentences.

The quantum formalism therefore seems an appropriate tool to describe the liar paradox. Could the formalism be applied to more intricate cognitive entities? Given the procedure we applied - an adaptation of the formalism of two interacting spin-3/2 particles - it is possible to extend the liar paradox to more complex variants of multiple sentences refering to one another in a truth confirming or denying manner. The minimal dimension to represent quantummechanically such a paradoxical set of n sentences will not be less than 2^n. The exact dimension of the appropriate Hilbert space depends on the specific n-sentence liar paradox described.

5 Conclusion.

We analysed how cognitive entities behave by using the formalism of quantum mechanics where the influence of the cognitive observer on the cognitive entity can be taken into account. In the same way as we described the double liar we can also represent the n-dimensional liar. The vector in the Hilbert space that we used to represent the state of the double liar is an eigenvector of the Hamiltonian of the system. This shows that we can consider the double liar as a cognitive entity without being measured on as an invariant of the time evolution. Once a measurement - a cognitive act - on one of the sub-elements is performed, the whole cognitive entity changes into a state that is no eigenstate anymore of the Hamiltonian. And after this measurement this state will start to change dynamically in the typical way of the liar paradox, sentences becoming true and false, and staying constantly coupled. This behaviour is exactly described by the Schrödinger equation that we have derived. In this way we have given a description of the internal dynamics within self-referring cognitive entities as the liar paradox. Our aim is to develop this approach further and to analyse in which way we can describe other examples of cognitive entities. We also want to analyse in further research in which way this result can be interpreted within a general scheme that connects different layers of reality structurally. Some profound

philosophical questions, still very speculative at this stage of our research, but certainly stimulating, can be put forward: e.g. Can we learn something about the nature and origin of dynamical change by considering this example of the liar paradox? Could the cognitive layer be considered being in a very early structuring stage, such that we trace down very primitive dynamical and contextual processes that could throw some light on primitive dynamical and contextual processes encountered in the pre-material layer (e.g. spin processes)? Apart from these speculative but stimulating philosophical questions, we also would like to investigate further in which way our quantum mechanical model for the cognitive layer of reality could be an inspiration for the development of a general interactive logic that can take into acount more subtle dynamical and contextual influences than just those of the cognitive person on the truth behavior of the cognitive entities.

6 Acknowledgements

7 References.

Aerts, D., 1992, "Construction of reality and its influence on the understanding of quantum structures", *Int. J. Theor. Phys.*, **31**, 1813.

Aerts, D., 1994, "The Biomousa: a new view of discovery and creation", in *Perspectives on the World, an interdisciplinary reflection*, eds. Aerts, D., et al., VUBPress.

Aerts, D., 1998, "The entity and modern physics: the creation-discovery view of reality", in *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*, ed. Castellani, E., Princeton University Press, Princeton.

Aerts, D., 1999, "The game of the biomousa: a view of discovery and creation, in *Worldviews and the problem of synthesis*, eds. Aerts, D.,

Van Belle, H. and Van der Veken, J., KLuwer Academic, Dordecht.

Aerts, D. and Aerts, S., 1994, "Applications of quantum statistics in psychological studies of decision processes", *Foundations of Science* **1**, 85.

Aerts, D. and Aerts, S., 1997, "Applications of quantum statistics in psychological studies of decision processes", in *Foundations of Statistics*, eds. Van Fraassen B., Kluwer Academic, Dordrecht.

Aerts, D. Broekaert, J. and Smets, S., 1999, "The liar paradox in a quantum mechanical perspective", *Foundations of Science*, **4**, 115.

Aerts, D. Coecke B. and Smets S., 1999, "On the origin of probabilities in quantum mechanics: creative and contextual aspects", in *Metadebates On Science*, eds. Cornelis, G., Smets, S. and Van Bendegem, J.P., Kluwer Academic, Dordrecht.

Clea Research Project (1997-2000), "Integrating Worldviews: Research on the Interdisciplinary Construction of a Model of Reality with Ethical and Practial Relevance" Ministry of the Flemish Community, dept. Science, Innovation and Media.