

The Liar-paradox in a Quantum Mechanical Perspective*

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Abstract

We propose a method to model the truth behaviour of cognitive entities taking into account the possible influence of the cognitive person on the truth behaviour of the entity. Hereby we specifically apply the mathematical formalism of quantum mechanics because of the fact that this formalism allows the description of real contextual influences, i.e. the influence of the measuring apparatus on the physical entity. We concentrated on the typical situation of the liar paradox and have shown that (1) the truth-false state of this liar paradox can be represented by a quantum vector in a finite dimensional complex Hilbert space and the different interpretative interactions by the actions of the corresponding quantum projections, (2) the typical oscillations between false and truth - the paradox - is now quantum dynamically described by a Schrödinger equation. We analyse possible philosophical implications of this result.

1 Introduction.

The Liar paradox is the oldest semantical paradox we find in literature. In its simplest forms we trace the paradox back to Eubulides - a pupil of Euclid - and to the Cretan Epimenides. From the Greeks on till today, different alternative forms of the liar emerged. We now encounter variations of the one sentence paradox (the simplest form of the liar) but also of the two or more sentence paradox. The two sentence paradox is known as the postcard paradox of Jourdain, which goes back to Buridan in 1300. On one side of a postcard we read ‘the sentence on the other side of this card is true’ and

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on the other side of it we read ‘the sentence on the other side of this card is false’.

In this paper we will not work with the original forms of the paradox, but in the version in which we use an index or sentence pointer followed by the sentence this index points at :

Single Liar :

(1) sentence (1) is false

Double Liar :

(1) sentence (2) is false

(2) sentence (1) is true

2 The Origin of our Approach.

Our aim is to show that the liar paradox can be described in a meaningful way by the quantum mechanical formalism. The theories of chaos and complexity have shown that similar patterns of behaviour can be found in very different layers of reality. The success of these theories demonstrates that interesting conclusions about the nature of reality can be inferred from the encountered structural similarities of dynamical behaviour in these different regions of reality. Chaos and complexity theories are however deterministic theories that do not take into account the fundamental contextuality that is introduced by the influence of the act of observation on the observed. Most of the regions of reality are highly contextual (e.g. the social layer, the cognitive layer, the pre- material quantum layer), rather with exception of the material layer of reality where contextuality is minimal. In this sense it is strange that no attempts have been undertaken to find similarities using contextual theories, such as quantum mechanics, in the different regions of reality. The study that we present in this paper should be classified as such an attempt, and is part of one of the projects in our center focusing on the layered structure of reality [1].

We will model one sentence or a group of sentences - e.g. the Double Liar - as one entity that we consider to ‘exist’ within the cognitive layer of reality. The existence is being expressed by the possibility of influencing other cognitive entities, and by the different states that it can be in. Indeed it has been shown that the concept of entity can be introduced rigorously and founded on the previously mentioned properties. In this way we justify the present use [2].

In this paper we will treat only the special case of the liar paradox situation, but we refer to [3,4] where a similar approach has been developed for the situation of an opinion pole within the cognitive layer of reality. In such an opinion pole specific questions are put forward that introduce a real influence of the interviewer on the interviewee, such that the situation is contextual. It is shown explicitly in [3] that the probability model that results is of a quantum mechanical nature.

3 Modelisation of Truth Behaviour of Cognitive Entities.

Referring to the previous paragraph, we introduce in our approach the explicit dependence of the truth and falsehood of a sentence, on the cognitive interaction with the cognitive person. 'Making a sentence true or false' will be modeled as 'performing a measurement' on the sentence within the cognitive layer of reality. This means that in our description a sentence within the cognitive layer of reality is 'in general' neither true, nor false. The 'state true' and the 'state false' of the sentence are 'eigenstates' of the measurement. During the act of measurement the state of the sentence changes in such a way that it is true or that it is false. This general 'neither true nor false state' will be called a superposition state in analogy with the quantum mechanical concept. We shall see that it is effectively a superposition state in the mathematical sense after we have introduced the complex Hilbert space description.

We proceed operationally as follows. Before the cognitive measurement (this means before we start to interact with the sentence, read it and make a hypothesis about its truth or falsehood) the sentence is considered to be neither true nor false and hence in a superposition state. If we want to start to analyse the cognitive inferences entailed, we make one of the two possible hypothesis, that it is true or that it is false. The making of one of these two hypothesis - this is part of the act of measurement - changes the state of the sentence to one of the two eigenstates - true or false. As a consequence of the act of measurement the sentence becomes true or false (is in the state true or false) within the cognitive entity where the sentence is part of. This change influences the state of this complete cognitive entity. We will see that if we apply this approach to the Double Liar, that the change of state puts into work a dynamic process that we can describe by a Schrödinger

equation. We have to consider three situations:

$$A \quad \begin{cases} (1) & \text{sentence (2) is false} \\ (2) & \text{sentence (1) is true} \end{cases}$$

$$B \quad \begin{cases} (1) & \text{sentence (2) is true} \\ (2) & \text{sentence (1) is true} \end{cases}$$

$$C \quad \begin{cases} (1) & \text{sentence (2) is false} \\ (2) & \text{sentence (1) is false} \end{cases}$$

4 A Quantum Description of the Truth Behaviour of the Double Liar Paradox.

The resemblance of the truth values of single sentences and the two-fold eigenvalues of a spin-1/2 state is used to construct a dynamical representation; the measurement evolution as well as a continuous time evolution are included.

Let us recall some elementary properties of a spin-1/2 state. Elementary particles - like the electron - are bestowed with a property referred to as an intrinsic angular momentum or spin. It has been shown that the spin of a particle is quantised: upon measurement the particle only exposes a finite number of distinct spin values. For the spin-1/2 particle, the number of spin states is two, they are commonly referred to as the ‘up’ and ‘down’ state. This two-valuedness can adequately describe the truth function of a liar type cognitive entity. Such a sentence supposedly is either true or false. The quantum mechanical description on the other hand allows a superposition of the ‘true’ and ‘false’ state. This corresponds to our view of allowing cognitive entities before measurement - i.e. reading and hypothetising - to reside in a non-determinate state of truth or falsehood. In quantum mechanics such a state Ψ is described by a pondered superposition of the two states:

$$\Psi = c_{true} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_{false} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The operation of finding whether such a cognitive entity is true or false, is done by applying respectively the true-projector P_{true} or false-projector P_{false} .

$$P_{true} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_{false} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In practice in the context of the cognitive entity, this corresponds to the assignment of either truth or falsehood to a sentence after its reading. In quantum mechanics, the true-measurement on the superposed state Ψ results in the true state pondered by the factor c_{true} ;

$$P_{true}\Psi = c_{true} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

An unequivocal result is therefore not obtained when the superposition does not leave out one of the states completely, i.e. either c_{true} or c_{false} is zero. Only in those instances do we attribute to a sentence its truth or falsehood. The coupled sentences of the two-sentence Liar paradox (C) for instance are precisely described by the so called ‘singlet state’. This global state combines, using the tensor product \otimes , states of sentence one with states of sentence two:

$$\frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

The appropriate true-projectors for sentence one and two are now:

$$P_{1,true} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1_2 \quad P_{2,true} = 1_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The false-projectors are obtained by switching the diagonal elements 1 and 0 on the diagonal of the matrix.

In the same manner the coupled sentences of the Liar paradox (B) can be constructed:

$$\frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Our final aim is to describe the real Double Liar paradox (A) quantum mechanically and even more to show how the true-false cycle originates from the Schrödinger time- evolution of the appropriate initial state. We found it necessary to describe this system in the coupled Hilbert space $C^4 \otimes C^4$ [5].

$$\frac{1}{2} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Each next term in this sum is actually the consecutive state which is reached in the course of time, when the paradox is read through. This can be easily verified by applying the appropriate truth-operators:

$$P_{1,true} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1_2 \quad P_{2,true} = 1_1 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The projectors for the false-states are constructed by placing the 1 on the final diagonal place.

The discrete unitary evolution operator U_D , which describes the time-evolution at instants of time when a sentence has changed truth value, is given by:

$$U_D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In order to obtain a description at every instance of time, a procedure of diagonalisation on U_D was performed, i.e. $U_D|_{\text{diag}}$. From the Schrödinger evolution and Stone's Theorem we obtain:

$$H|_{\text{diag}} = i \ln U_D|_{\text{diag}}$$

Now inverting the procedure of diagonalisation, the infinitesimal generator of the time evolution - the hamiltonian - is obtained. The evolution operator $U(t)$, valid at all times is then given by:

$$U(t) = e^{-iHt}$$

In this manner simple realisations of the time evolution operator $U(t)$ can be obtained [5].

5 Conclusion.

We analysed how cognitive entities behave by using the formalism of quantum mechanics where the contextuality (the influence of the cognitive observer on the cognitive entity) can be taken into account. In the same way as we represented the Double Liar we can represent also the n-dimensional Liar [5]. The vector in the Hilbert space that we used to represent the Double Liar is an eigenvector of the Hamiltonian of the system. This shows that we

can consider the Double Liar as a cognitive entity without being measured on as an invariant of the time evolution. Once a measurement - a cognitive act - on one of the sub-elements is performed, the whole cognitive entity changes into a state that is no eigenstate anymore of the Hamiltonian. And after this measurement this state will start to change in the typical way of the Liar paradox, sentences becoming true and false, and staying constantly coupled. This behaviour is exactly described by the Schrödinger equation that we have derived. In this way we have given a description of the internal dynamics within self-referring cognitive entities as the Liar paradox. Our aim is to develop this approach further and to analyse in which way we can describe other examples of cognitive entities.

6 References.

- [1] **Clea Research Project**, “Integrating Worldviews: Research on the Interdisciplinary Construction of a Model of Reality with Ethical and Practical Relevance” Ministry of the Flemish Community, dept. Science, Innovation and Media.
- [2] **Aerts, D.**, “ Construction of Reality and its Influence on the Understanding of Quantum Structures”, *Int. J. Theor. Phys.*, Vol 31, 10, p. 1813, 1992
- [3] **Aerts, D. and Aerts, S.**: “Applications of Quantum Statistics in Psychological Studies of Decision Processes”, *Foundations of Science* **1**, p. 85, 1995-1996.
- [4] **Aerts, D. Coecke B. and Smets S.**, “On the Origin of Probabilities in Quantum Mechanics: Creative and Contextual Aspects”, *Einstein-meets-Magritte proceedings*, Dordrecht-Kluwer, 1997, accepted for publication.
- [5] **Aerts, D. Broekaert, J. and Smets, S.** : “A Quantum Structure Description of the Liar-paradox” , to be published.