Quantum axiomatics and a theorem of M.P. Solèr*

Diederik Aerts, Bart Van Steirteghem

Foundations of the Exact Sciences (FUND),
Department of Mathematics, Brussels Free University,
Pleinlaan 2, B–1050 Brussels, Belgium;
diraerts@vub.ac.be, bvsteirt@vub.ac.be

Abstract

Three of the traditional quantum axioms (orthocomplementation, orthomodularity and the covering law) show incompatibilities with two products introduced by Aerts for the description of joint entities. Inspired by Solèr's theorem and Holland's AUG axiom, we propose a property of 'plane transitivity', which also characterizes classical Hilbert spaces among infinite–dimensional orthomodular spaces, as a possible partial substitute for the 'defective' axioms.

1 Introduction

In his axiomatization of standard quantum mechanics Holland (1995) introduces the Ample Unitary Group axiom (cf. (2) in Proposition 1 of this paper). It hints at an evolution axiom but has the shortcoming that it is not lattice theoretical. In particular, it cannot be formulated for *property lattices* —complete, atomistic and orthocomplemented lattices— which play a central role in the Geneva–Brussels approach to the foundations of physics (Piron, 1976, 1989, 1990; Aerts, 1982, 1983, 1984; Moore, 1995, 1999). Inspired by this axiom, we propose a property, called 'plane transitivity' (section 4), which does not have *this* imperfection. Like the AUG axiom it characterizes classical Hilbert spaces among infinite–dimensional orthomodular spaces and it still looks like 'demanding enough symmetries or evolutions'.

The traditional quantum axiomatics show some shortcomings in the description of compound systems (Aerts, 1982, 1984; Pulmannovà, 1983, 1985). In particular, orthocomplementation, orthomodularity and the covering law are not compatible with two products —'separated' and 'minimal'— introduced by Aerts (section 3). Plane transitivity, on the other hand, 'survives' these two products (section 4). That way it is a candidate to help fill the gap left by the failing axioms.

*Published as: Aerts, D. and Van Steirteghem, B., 2000, "Quantum axiomatics and a theorem of M.P. Solèr", *Int. J. Theor. Phys.*, **39**, 497-502.

2 Alternatives to Solèr's theorem

Consider a complete, atomistic, orthocomplemented and irreducible lattice $\mathcal{L}$ satisfying the covering law. Suppose moreover that its length is at least 4. Then there exist a division ring K with an involutorial anti–automorphism $\lambda \mapsto \lambda^*$ and a vector space E over K with a Hermitian form $< \cdot, \cdot >$ such that $\mathcal{L}$ is ortho–isomorphic to the lattice $\mathcal{L}(E)$ of closed (biorthogonal) subspaces of E. Moreover, $\mathcal{L}$ is orthomodular if and only if $(E, K, < \cdot, \cdot >)$ is orthomodular: $M + M^\perp = E$ for every $\emptyset \neq M \subset E$ with $M = M^{\perp\perp}$ (Maeda and Maeda, 1970; Piron, 1976; Faure and Frölicher, 1995). Solèr (1995) has proven the following characterization of classical Hilbert spaces: if E contains an infinite orthonormal sequence, then $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $(E, K, < \cdot, \cdot >)$ is the corresponding Hilbert space. Holland (1995) has shown that it is enough to demand the existence of a nonzero $\lambda \in K$ and an infinite orthogonal sequence $(e_n)_n \in E$ such that $< e_n, e_n >= \lambda$ for every n. To be precise, either $(E, K, < \cdot, \cdot >)$ or $(E, K, - < \cdot, \cdot >)$ is then a classical Hilbert space. We shall not make this precision explicitly in what follows.

In the first proposition, we summarize some alternatives to Solèr's result, by means of automorphisms of $\mathcal{L}(E)$.

Proposition 1 *Let $(E, K, < \cdot, \cdot >)$ be an orthomodular space and let $\mathcal{L}(E)$ be the lattice of its closed subspaces. The following are equivalent:*

(1) $(E, K, < \cdot, \cdot >)$ is an infinite–dimensional Hilbert space over $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

(2) E is infinite–dimensional and given two orthogonal atoms p, q in $\mathcal{L}(E)$, there is a unitary operator U such that $U(p) = q$.

(3) There exist $a, b \in \mathcal{L}(E)$, where b is of dimension at least 2, and an ortholattice automorphism f of $\mathcal{L}(E)$ such that $f(a) \lneq a$ and $f|_{[0,b]}$ is the identical map.

(4) E is infinite–dimensional and given two orthogonal atoms p, q in $\mathcal{L}(E)$ there exist distinct atoms p_1, p_2 and an ortholattice automorphism f of $\mathcal{L}(E)$ such that $f|_{[0,p_1 \vee p_2]}$ is the identity and $f(p) = q$.

Condition (2) is Holland's Ample Unitary Group axiom (1995) and (3) is due to Mayet (1998). Using the properties listed in section 2 of (Mayet, 1998), one can easily prove that (4) implies (2). We will use (4) to formulate a lattice theoretical alternative to the AUG axiom (section 4).

3 Compound entities and the axioms

Aerts has introduced two 'products' for the description of compound entities. We shall present them 'mathematically' and recall their 'interaction' with the axioms of quantum mechanics proposed by Piron (1976). For an operational justification of these products we refer to (Aerts, 1982, 1984).

First we recall some notions and results due to Moore (1995). A *state space* is a pair $(\Sigma, \perp)$, where Σ is a set (of states) and $\perp$ (orthogonality) is a symmetric antireflexive binary relation which separates the points of Σ (if $p \neq q$ then $\exists r$ such that $p \perp r$ and $q \not\perp r$). For $A \subset \Sigma$, put $A^\perp = \{q \in \Sigma \,|\, q \perp p \ \forall p \in A\}$. Then $(\mathcal{L}_\Sigma, \subset, ^\perp)$ is a property lattice with $\bigwedge\{A_r\} = \bigcap\{A_r\}$, where $\mathcal{L}_\Sigma = \{A \subset$

$\Sigma \mid A = A^{\perp\perp}\}$. In particular, $(\Sigma, \mathcal{L}_\Sigma)$ is a T_1–*closure space*: $\mathcal{L}_\Sigma \ni \emptyset$ is a family of subsets of Σ closed under arbitrary intersections and $\{p\} \in \mathcal{L}_\Sigma, \forall p \in \Sigma$.

Next, consider two entities S_1, S_2 described by their state spaces $(\Sigma_1, \perp_1)$ and $(\Sigma_2, \perp_2)$. Denote the corresponding property lattices by $\mathcal{L}_1$ and $\mathcal{L}_2$. Suppose S_1 and S_2 are 'separated'. Aerts (1982) suggests the *separated product* $\mathcal{L}_1 \owedge \mathcal{L}_2$ for the description of S_1 and S_2 taken together. Its state space is $(\Sigma_1 \times \Sigma_2, \perp_{\owedge})$ where

$$(p_1, p_2) \perp_{\owedge} (q_1, q_2) \Leftrightarrow p_1 \perp q_1 \text{ or } p_2 \perp q_2$$

$\mathcal{L}_1 \owedge \mathcal{L}_2$ is then the corresponding property lattice (Piron, 1989). This product is not 'compatible' with orthomodularity and the covering law in the following sense: if $\mathcal{L}_1 \owedge \mathcal{L}_2$ satisfies one of these properties, then $\mathcal{L}_1$ or $\mathcal{L}_2$ is Boolean (Aerts, 1982).

Aerts (1984) proposes another lattice as the 'coarsest' description of a compound entity containing the two (not necessarily separated) entities S_1, S_2. We give a slightly different, but equivalent construction of this *minimal product* $\mathcal{L}_1 \coprod \mathcal{L}_2$. Consider the closure spaces $(\Sigma_i, \mathcal{L}_i)$. Since $\mathbf{Cls}_1$, the category of T_1–closure spaces and continuous maps, is closed under products (cf. Dikranjan et al., 1988), $(\Sigma_1, \mathcal{L}_1)$ and $(\Sigma_2, \mathcal{L}_2)$ have a $\mathbf{Cls}_1$–product, which we denote $(\Sigma_1 \times \Sigma_2, \mathcal{L}_1 \coprod \mathcal{L}_2)$. This notation is for consistency with (Aerts et al., 1999). Of course, $\mathcal{L}_1 \coprod \mathcal{L}_2$ is a complete atomistic lattice, but the orthocomplementation is problematic. Indeed, if we define the following —operationally justified by Aerts (1984)— orthogonality on $\Sigma_1 \times \Sigma_2$: $(p_1, p_2) \perp (q_1, q_2) \Leftrightarrow p_1 \perp q_1$ or $p_2 \perp q_2$, then $\mathcal{L}_1 \coprod \mathcal{L}_2$ cannot have an orthocomplementation compatible with $\perp$ unless $\mathcal{L}_1 \subset \{0, 1\}$ or $\mathcal{L}_2 \subset \{0, 1\}$. Moreover, the same is true for the covering law: if $\mathcal{L}_1 \coprod \mathcal{L}_2$ satisfies the covering law, then $\mathcal{L}_1$ or $\mathcal{L}_2$ is trivial. For completeness, we mention that this product is compatible with a suitable form of orthomodularity.

These problems with the traditional axioms in the description of joint entities have made it desirable to find (nice) properties compatible with the separated and minimal product. If we slightly generalize condition (4) of Proposition 1, we obtain a property which survives both products.

4 Plane transitivity

To seize both products with the same terminology, we introduce *pseudo property lattices*. $(\mathcal{L}, \Sigma, \perp)$ is a *p.p.l.* if $\mathcal{L}$ is a complete atomistic lattice and $\perp$ is an orthogonality on its set of atoms Σ. Using the well–known correspondence between atomistic lattices and T_1–closure spaces (Faure, 1994), every ppl has an associated closure space $(\Sigma, \mathcal{F}_\mathcal{L})$ where

$$\mathcal{F}_\mathcal{L} = \{F \subset \Sigma \mid p \in \Sigma, p < \vee F \Rightarrow p \in F\}$$

It easily follows that the above construction of the minimal product generalizes to a minimal product of ppl's. To be precise, the minimal product of $(\mathcal{L}_1, \Sigma_1, \perp)$ and $(\mathcal{L}_2, \Sigma_2, \perp)$ is $(\mathcal{L}_1 \coprod \mathcal{L}_2, \Sigma_1 \times \Sigma_2, \perp)$, where $(\Sigma_1 \times \Sigma_2, \mathcal{L}_1 \coprod \mathcal{L}_2)$ is the $\mathbf{Cls}_1$–product of $(\Sigma_1, \mathcal{F}_{\mathcal{L}_1})$ and $(\Sigma_2, \mathcal{F}_{\mathcal{L}_2})$ and the orthogonality is defined as above.

We call $f : \mathcal{L} \to \mathcal{L}$ a *symmetry* (of ppl's) if it is an order–automorphism, such that $\forall p, q \in \Sigma$ we have $p \perp q \Leftrightarrow f(p) \perp f(q)$. We remark that for state spaces, symmetries are nothing else than permutations conserving the orthogonality in both directions (Piron 1989). Indeed, if α is such a permutation of $(\Sigma, \perp)$, then

$$f : \mathcal{L}_\Sigma \to \mathcal{L}_\Sigma : A \mapsto \alpha(A)$$

is the unique ortho–automorphism of $\mathcal{L}_\Sigma$ such that $f\{p\} = \alpha(p)$ for every p in Σ. In particular, f is a symmetry of the ppl $(\mathcal{L}_\Sigma, \Sigma, \perp)$ associated to $(\Sigma, \perp)$.

We call a ppl $(\mathcal{L}, \Sigma, \perp)$ *plane transitive* if for all atoms $p, q \in \Sigma$ there exist two distinct atoms p_1, p_2 and a symmetry f such that $f|_{[0, p_1 \vee p_2]}$ is the identity and $f(p) = q$. Looking at Proposition 1, it is obvious that if $\mathcal{L}$ is the lattice of biorthogonal subspaces of an infinite–dimensional orthomodular space E, E is a classical Hilbert space iff (with a slight abuse of language) $\mathcal{L}$ is plane transitive.

Proposition 2 *Let $(\mathcal{L}_1, \Sigma_1, \perp)$ and $(\mathcal{L}_2, \Sigma_2, \perp)$ be ppl's. If both are plane transitive, then so is their minimal product $(\mathcal{L}_1 \coprod \mathcal{L}_2, \Sigma_1 \times \Sigma_2, \perp)$.*

Indeed, consider (r_1, r_2) and (s_1, s_2) in $\Sigma_1 \times \Sigma_2$. Choose a symmetry f_1 and an atom $p_1 \in \Sigma_1$ such that $f_1(r_1) = s_1$ and $f_1(p_1) = p_1$. Next, choose $p_2 \neq q_2$ in Σ_2 and a symmetry f_2 of $(\mathcal{L}_2, \Sigma_2, \perp)$ such that $f_2(r_2) = s_2$ and $f_2|_{[0, p_2 \vee q_2]}$ is the identical map. Then $f_i|_{\Sigma_i}$ is a $\mathbf{Cls}_1$–automorphism of $(\Sigma_i, \mathcal{F}_{\mathcal{L}_i})$. It follows that $(t_1, t_2) \mapsto (f_1(t_1), f_2(t_2))$ is a $\mathbf{Cls}_1$–automorphism of $(\Sigma_1 \times \Sigma_2, \mathcal{L}_1 \coprod \mathcal{L}_2)$ and hence generates an order–automorphism $f_1 \times f_2$ of $\mathcal{L}_1 \coprod \mathcal{L}_2$. Trivially, $f_1 \times f_2(r_1, r_2) = (s_1, s_2)$. Also, $f_1 \times f_2|_{[0, (p_1, p_2) \vee (p_1, q_2)]}$ is the identity. Finally, it is straightforward to verify that $f_1 \times f_2$ conserves the orthogonality on $\Sigma_1 \times \Sigma_2$ in both directions.

Using a similar argument, one easily shows the same holds for the separated product. Note that a state space $(\Sigma, \perp)$ is called plane transitive if its associated ppl $(\mathcal{L}_\Sigma, \Sigma, \perp)$ is plane transitive.

Proposition 3 *If two state spaces $(\Sigma_1, \perp)$ and $(\Sigma_2, \perp)$ are plane transitive, then so is their separated product $(\Sigma_1 \times \Sigma_2, \perp_{\textcircled{\wedge}})$.*

5 Questions

Several questions remain. Plane transitivity does not have the necessary elegance to be a fundamental axiom: what is the physical significance of this invariant plane? Another question is: can the unitary operators of an orthomodular space be characterized at the lattice level? In other words, can Holland's AUG axiom be formulated lattice theoretically? Maybe, it can be generalized to the transitivity of the whole group of ortholattice automorphisms and still characterize classical Hilbert spaces among infinite–dimensional orthomodular spaces. This would be an elegant symmetry (or evolution) axiom.

Acknowledgements

D.A. is a senior research associate and B.V.S. is a research assistant of the Fund for Scientific Research—Flanders.

References

Aerts, D. (1982). Description of many physical entities without the paradoxes encountered in quantum mechanics, *Found. Phys.*, **12**, 1131-1170.

Aerts, D. (1983). Classical theories and non classical theories as a special case of a more general theory, *J. Math. Phys.*, **24**, 2441-2454.

Aerts, D. (1984). Construction of the tensor product for the lattices of properties of physical entities, *J. Math. Phys.*, **25**, 1434-1441.

Aerts, D., Colebunders, E., Van der Voorde, A., and Van Steirteghem, B. (1999). State property systems and closure spaces: a study of categorical equivalence, *Int. J. Theor. Phys.*, **38**, 359-385.

Dikranjan, D., Giuli, E., and Tozzi, A. (1988). Topological categories and closure operators, *Quaestiones Mathematicae*, **11**, 323-337.

Faure, Cl.-A. (1994). Categories of closure spaces and corresponding lattices, *Cahier de top. et géom. diff. catég.*, **35**, 309-319.

Faure, Cl.-A., and Frölicher, A. (1995). Dualities for infinite–dimensional projective geometries, *Geom. Ded.*, **56**, 225-236.

Holland, S.S. Jr. (1995). Orthomodularity in infinite dimensions; a theorem of M. Solèr, *Bull. Amer. Math. Soc.*, **32**, 205-234.

Maeda, F., and Maeda, S. (1970). *Theory of symmetric lattices*, Springer–Verlag, Berlin.

Mayet, R. (1998). Some characterizations of the underlying division ring of a Hilbert lattice by automorphisms, *Int. J. Theor. Phys.*, **37**, 109-114.

Moore, D.J. (1995). Categories of representations of physical systems, *Helv. Phys. Acta*, **68**, 658-678.

Moore, D.J. (1999). On state spaces and property lattices, *Stud. Hist. Phil. Mod. Phys.*, to appear.

Piron, C. (1976). *Foundations of quantum physics*, Benjamin, New York.

Piron, C. (1989). Recent developments in quantum mechanics, *Helv. Phys. Acta*, **62**, 82-90.

Piron, C. (1990). *Mécanique quantique bases et applications*, Presses polytechniques et universitaires romandes, Lausanne.

Pulmannovà, S. (1983). Coupling of quantum logics, *Int. J. Theor. Phys.*, **22**, 837.

Pulmannovà, S. (1985). Tensor product of quantum logics, *J. Math. Phys.*, **26**, 1.

Solèr, M.P. (1995). Characterization of Hilbert spaces by orthomodular spaces, *Comm. Algebra*, **23**, 219-243.