# Connectedness applied to closure spaces and state property systems*

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**Abstract**

In [1] a description of a physical entity is given by means of a state property system and in [2] it is proven that any state property system is equivalent to a closure space. In the present paper we investigate the relations between classical properties and connectedness for closure spaces. The main result is a decomposition theorem, which allows us to split a state property system into a number of 'pure nonclassical state property systems' and a 'totally classical state property system'.

## 1 Introduction

In [1] a physical entity is represented by a mathematical model called a state property system. This model contains a complete lattice of properties of the physical entity. In [2] it is shown that the lattice can viewed as the lattice of closed sets of a closure space. We introduce the concept of classical property of the entity, and show that these correspond exactly to the clopen (open and closed) subsets of the associated closure space. Using the concept of connectedness for closure space we decompose the state property system into smaller ones which will be 'completely quantum mechanical' (no classical properties)

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and another one which will be 'totally classical'. Finally we introduce a way to extract the classical properties of the entity, and interpret this in term of the closure space. Let us first introduce the basic definitions and concepts.

**Definition 1.** *A triple $(\Sigma, \mathcal{L}, \xi)$ is called a state property system if $\Sigma$ is a set, $\mathcal{L}$ is a complete lattice and $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ is a function such that for $p \in \Sigma$, 0 the minimal element of $\mathcal{L}$ and $(a_i)_i \in \mathcal{L}$, we have:*

$$0 \notin \xi(p) \text{ and } \forall i \ a_i \in \xi(p) \ \Rightarrow \wedge_i a_i \in \xi(p)$$

*and for $a, b \in \mathcal{L}$ we have: $a < b \Leftrightarrow \forall r \in \Sigma : a \in \xi(r)$ then $b \in \xi(r)$.*
*If $(\Sigma, \mathcal{L}, \xi)$ is a state property system then its Cartan map is the mapping $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ defined by :*

$$\kappa : \mathcal{L} \to \mathcal{P}(\Sigma) : a \mapsto \kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\}$$

The physical interpretation of this mathematical structure (introduced in [1]) is the following. Considering an entity $S$, the set $\Sigma$ consists of states of $S$ while the set $\mathcal{L}$ consists of properties of $S$. These two sets are linked by means of a function $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ which maps a state $p$ to the set $\xi(p)$ of all properties that are actual in state $p$. This means that the statement 'a state $p$ makes the property $a$ actual' is mathematically expressed by the formula: $a \in \xi(p)$.

**Definition 2.** *A closure space $(X, \mathcal{F})$ consists of a set $X$ and a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following two conditions:*

$$\emptyset \in \mathcal{F} \text{ and } (F_i)_i \in \mathcal{F} \Rightarrow \cap_i F_i \in \mathcal{F}$$

*The closure operator corresponding to the closure space $(X, \mathcal{F})$ is defined as*

$$cl : \mathcal{P}(X) \to \mathcal{P}(X) : A \mapsto \bigcap \{F \in \mathcal{F} \mid A \subseteq F\}$$

The following theorem shows how we can associate with each state property system a closure space and vice versa.

**Proposition 1.** *If $(\Sigma, \mathcal{L}, \xi)$ is a state property system then $F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}))$ is a closure space. Conversely, if $(\Sigma, \mathcal{F})$ is a closure space then $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi')$, where $\xi'(x) = \{F \in \mathcal{F} | x \in F\}$, is a state property system.*

This is a consequence of the fact that there is an categorical equivalence between state property systems and closure spaces, as proven in [2] .

## 2 Super selection rules and classical properties

Superposition states in quantum mechanics are those states that do not exist in classical physics and hence their appearance is one of the important quantum characteristics. This concept can be traced back within this general setting, by introducing the idea of 'super selection rule'. Two properties are separated by a super selection rule iff there do not exist 'superposition states' related to these two properties.

**Definition 3.** *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$. *For* $a, b \in \mathcal{L}$ *we say that* $a$ *and* $b$ *are separated by a super selection rule, and denote* $a$ *ssr* $b$, *iff for* $p \in \Sigma$ *we have:* $a \vee b \in \xi(p) \Rightarrow a \in \xi(p)$ or $b \in \xi(p)$

**Lemma 1.** *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$ *and its corresponding closure system* $\mathcal{F} = \kappa(\mathcal{L})$. *For* $a, b \in \mathcal{L}$ *we have:*

$$a \ ssr \ b \Leftrightarrow \kappa(a \vee b) = \kappa(a) \cup \kappa(b) \Leftrightarrow \kappa(a) \cup \kappa(b) \in \mathcal{F}$$

*Proof.* This is an easy verification. □

We are ready now to introduce the concept of a 'classical property'.

**Definition 4.** *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$. *We say that a property* $a \in \mathcal{L}$ *is a 'classical property', if there exists a property* $a^c \in \mathcal{L}$ *such that* $a \vee a^c = I$, $a \wedge a^c = 0$ *and* $a$ *ssr* $a^c$.

Remark that for every state property system $(\Sigma, \mathcal{L}, \xi)$ the properties $0$ and $I$ are classical properties. Note also that if $a \in \mathcal{L}$ is a classical property, we have for $p \in \Sigma$ that $a \in \xi(p) \Leftrightarrow a^c \notin \xi(p)$ and $a \notin \xi(p) \Leftrightarrow a^c \in \xi(p)$. This follows immediately from the definition of a classical property. From the previous definition and lemma 1 one can prove the following lemma.

**Lemma 2.** *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$. *If* $a \in \mathcal{L}$ *is a classical property, then* $a^c$ *is unique and is a classical property. We call it the complement of* $a$. *Further the following is satisfied:*

$$(a^c)^c = a, a < b \Rightarrow b^c < a^c, \kappa(a^c) = \kappa(a)^C$$

**Definition 5.** *A closure space* $(X, \mathcal{F})$ *is called connected if the only clopen (i.e. closed and open) sets are* $\emptyset$ *and* $X$.

We can show now that the subsets that make closure systems disconnected are exactly the subsets corresponding to classical properties.

**Proposition 2.** *Consider a state property system $(\Sigma, \mathcal{L}, \xi)$ and its corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$. For $a \in \mathcal{L}$ we have: $a$ is classical $\Leftrightarrow \kappa(a)$ is clopen.*

*Proof.* From the previous lemmas it follows that if $a$ is classical, then $\kappa(a)$ is clopen. So now consider a clopen subset $\kappa(a)$ of $\Sigma$. This means that $\kappa(a)^C$ is closed, and hence that there exists a property $b \in \mathcal{L}$ such that $\kappa(b) = \kappa(a)^C$. We clearly have $a \wedge b = 0$ since there exists no state $p \in \Sigma$ such that $p \in \kappa(a)$ and $p \in \kappa(b)$. Since $\Sigma = \kappa(a) \cup \kappa(b)$ we have $a \vee b = I$. Further we have that for an arbitrary state $p \in \Sigma$ we have $a \in \xi(p)$ or $b \in \xi(p)$ which shows that $a$ ssr $b$. This proves that $b = a^c$ and that $a$ is classical. □

This means that the classical properties correspond exactly to the clopen subsets of the closure system.

**Corollary 1.** *Let $(\Sigma, \mathcal{L}, \xi)$ be a state propery system. T.F.A.E.*
*(1) The properties $0$ and $I$ are the only classical ones.*
*(2) $F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}))$ is a connected closure space.*

**Definition 6.** *A state property system $(\Sigma, \mathcal{L}, \xi)$ is called a 'pure nonclassical state property system' if the properties $0$ and $I$ are the only classical properties.*

**Proposition 3.** *Let $(\Sigma, \mathcal{F})$ be a closure space. T.F.A.E.*
*(1) $(\Sigma, \mathcal{F})$ is a connected closure space.*
*(2) $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi)$ is a pure nonclassical state property system.*

*Proof.* Let $(\Sigma, \mathcal{F})$ be a connected state property system. Then $\emptyset$ and $\Sigma$ are the only clopen sets in $(\Sigma, \mathcal{F})$. Since the Cartan map associated to $\xi$ is given by $\kappa : \mathcal{F} \to \mathcal{P}(\Sigma) : F \mapsto F$, we have $\kappa(\emptyset) = \emptyset$ and $\kappa(\Sigma) = \Sigma$. Applying proposition 2, we find that $\emptyset$ and $\Sigma$ are the only classical properties of $\mathcal{F}$. Conversely, let $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi)$ be a pure nonclassical state property system. Then by corollary 1, $(\Sigma, \mathcal{F}) = FG(\Sigma, \mathcal{F})$ is a connected closure space. □

## 3 Decomposition theorem

As for topological spaces, every closure space can be decomposed uniquely into connected components. In the following we say that, for a closure space $(X, \mathcal{F})$, a subset $A \subseteq X$ is connected if the induced subspace is connected. It can be shown that the union of any family of connected subsets having at least one point in common is also connected. So the component of an element $x \in X$ defined by $K_{\mathbf{Cls}}(x) = \bigcup\{A \subseteq X \mid x \in A, A \text{ connected }\}$ is connected and therefore called the connection component of $x$. Moreover, it is a maximal connected set in $X$ in the sense that there is no connected subset of $X$ which properly contains $K_{\mathbf{Cls}}(x)$. From this it follows that for closure spaces $(X, \mathcal{F})$ the set of all distinct connection components in $X$ form a partition of $X$. In the following we will decompose state property systems similarly into different components.

**Proposition 4.** *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system and let $(\Sigma, \kappa(\mathcal{L}))$ be the corresponding closure space. Consider the equivalence relation on $\Sigma$ given by: $p \sim q \Leftrightarrow K_{\mathbf{Cls}}(p) = K_{\mathbf{Cls}}(q)$ with equivalence classes $\Omega = \{\omega(p) | p \in \Sigma\}$. If $\omega \in \Omega$ we define:*

$$
\begin{aligned}
\Sigma_\omega &= \omega = \{p \in \Sigma \mid \omega(p) = \omega\} \\
s(\omega) &= s(\omega(p)) = a, \text{ such that } \kappa(a) = \omega(p) \\
\mathcal{L}_\omega &= [0, s(\omega)] = \{a \in \mathcal{L} \mid 0 \leq a \leq s(\omega)\} \subset \mathcal{L} \\
\xi_\omega &: \Sigma_\omega \to \mathcal{P}(\mathcal{L}_\omega) : p \mapsto \xi(p) \cap \mathcal{L}_\omega
\end{aligned}
$$

*then $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ is a state property system.*

*Proof.* Since $\mathcal{L}_\omega$ is a sublattice (segment) of $\mathcal{L}$, it is a complete lattice with maximal element $I_\omega = s(\omega)$ and minimal element $0_\omega = 0$. Let $p \in \Sigma_\omega$. Then $0 \notin \xi(p)$. So $0 \notin \xi(p) \cap \mathcal{L}_\omega = \xi_\omega(p)$. If $a_i \in \xi_\omega(p), \ \forall i$, then $a_i \in \mathcal{L}_\omega$ and $a_i \in \xi(p), \ \forall i$. Hence $\wedge a_i \in \mathcal{L}_\omega \cap \xi(p) = \xi_\omega(p)$. Finally, let $a, b \in \mathcal{L}_\omega$ with $a <_\omega b$ and let $r \in \Sigma_\omega$. If $a \in \xi_\omega(r)$, then $a \in \mathcal{L}_\omega$ and $a \in \xi(r)$, thus $b \in \mathcal{L}_\omega$ and $b \in \xi(r)$. So $b \in \xi_\omega(r)$. Conversely, if $a, b \in \mathcal{L}_\omega$ and $\forall r \in \Sigma_\omega : a \in \xi_\omega(r) \Rightarrow b \in \xi_\omega(r)$ then we consider a $q$ such that $a \in \xi(q)$ ($q$ must be in $\Sigma_\omega$ by definition of $\mathcal{L}_\omega$). Then $a \in \xi_\omega(q)$ implies that $b \in \xi_\omega(q)$. So $b \in \xi(q)$ and $a < b$. Thus $a <_\omega b$. □

Moreover we can show that the above introduced state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ have no proper classical properties, and hence are pure nonclassical state property systems.

**Proposition 5.** *Let* $(\Sigma, \mathcal{L}, \xi)$ *be a state property system. If* $\omega \in \Omega$*, then* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *is a pure nonclassical state property system.*

*Proof.* If $a$ is classical element of $\mathcal{L}_\omega$, then $\kappa(a)$ must be a clopen set of the associated closure space $(\Sigma_\omega, \kappa(\mathcal{L}_\omega))$ which is a connected subspace of $(\Sigma, \kappa(\mathcal{L}))$. Hence there are no proper classical elements of $\mathcal{L}_\omega$. □

**Proposition 6.** *Let* $(\Sigma, \mathcal{L}, \xi)$ *be a state property system. If we introduce the following :*

$$\begin{aligned} \Omega &= \{\omega(p) \mid p \in \Sigma\} \\ \mathcal{C} &= \{\vee s(\omega_i) \mid \omega_i \in \Omega\} \\ \eta &: \Omega \to \mathcal{P}(\mathcal{C}) : \omega = \omega(p) \mapsto \xi(p) \cap \mathcal{C} \end{aligned}$$

*then* $(\Omega, \mathcal{C}, \xi)$ *is a state property system.*

*Proof.* First we remark that $\eta$ is well defined because if $\omega(p) = \omega(q)$, then $\xi(p) \cap \mathcal{C} = \xi(q) \cap \mathcal{C}$. Indeed, if $\vee s(\omega_i) \in \xi(p)$ then $p \in \kappa(\vee s(\omega_i)) = cl(\cup \omega_i)$ in the corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$. Since $cl(\cup\omega_i)$ is not connected we have that $K_{\mathbf{Cls}}(p) = \omega(p) = \omega(q) \subset cl(\cup\omega_i)$ so $q \in cl(\cup\omega_i) = \kappa(\vee s(\omega_i))$ and $\vee s(\omega_i) \in \xi(q)$. Now, since $\mathcal{C}$ is a sublattice of $\mathcal{L}$ it is a complete lattice with $1_\mathcal{C} = 1$ and $0_\mathcal{C} = 0$. By definition $\mathcal{C}$ is generated by its atoms $\{s(\omega) \mid \omega \in \Omega\}$. Clearly $0 \notin \eta(\omega(p))$ because $0 \notin \xi(p)$. If $a_i \in \eta(\omega(p)) = \xi(p) \cap \mathcal{C}$, $\forall i$, then $\wedge a_i \in \xi(p) \cap \mathcal{C} = \eta(\omega(p))$. Finally, let $a, b \in \mathcal{C}$ with $a <_\mathcal{C} b$. Let $\omega(p) \in \Omega$ with $a \in \eta(\omega(p))$. Thus $a \in \xi(p)$. $a <_\mathcal{C} b$ implies $a < b$. So we have $b \in \xi(p) \cap \mathcal{C} = \eta(\omega(p))$. Conversely, let $a, b \in \mathcal{C}$ and assume that $\forall p \in \Sigma : a \in \eta(\omega(p)) \Rightarrow b \in \eta(\omega(p))$. Then we have $\forall p \in \Sigma : a \in \xi(p) \Rightarrow b \in \xi(p)$. Thus $a < b$ and $a <_\mathcal{C} b$. □

**Proposition 7.** $(\Omega, \mathcal{C}, \eta)$ *is a totally classical state property system, in the sense that the only quantum segments (i.e. segments with no proper classical elements) are trivial, i.e.* $\{0, s(\omega)\}$*.*

*Proof.* Suppose $[0, a]$ is a quantumsegment of $\mathcal{C}$, then in the corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$ the subset $\kappa(a)$ is connected hence $\kappa(a) \subset \omega$ for some $\omega \in \Omega$, hence $a < s(\omega)$. Since $s(\omega)$ is an atom, $a = s(\omega)$. Thus $[0, a] = \{0, s(\omega)\}$. □

**Corollary 2.** *The closure space associated with* $(\Omega, \mathcal{C}, \eta)$ *is a totally disconnected closure space.*

Summarizing the previous results we get:

**Proposition 8.** *Any state property system* $(\Sigma, \mathcal{L}, \xi)$ *can be decomposed into: a number of pure nonclassical state property systems* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega), \omega \in \Omega$ *and a totally classical state property system* $(\Omega, \mathcal{C}, \eta)$

# 4 The classical part of a state property system

In this section we want to show how it is possible to extract the classical part of a state property system. First of all we have to define the classical property lattice related to the entity $S$ that is described by the state property system $(\Sigma, \mathcal{L}, \xi)$.

**Definition 7 (Classical property lattice).** *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$. *We call* $\mathcal{C}' = \{\wedge_i a_i | a_i \text{ is a classical property}\}$ *the classical property lattice corresponding to the state property system* $(\Sigma, \mathcal{L}, \xi)$.

**Proposition 9.** $\mathcal{C}'$ *is a complete lattice with the partial order relation and infimum inherited from* $\mathcal{L}$ *and the supremum defined as follows: for* $a_i \in \mathcal{C}'$, $\vee_i a_i = \wedge_{b \in \mathcal{C}', a_i \leq b\ \forall i}\ b$.

Remark that the supremum in the lattice $\mathcal{C}'$ is not the one inherited from $\mathcal{L}$.

**Proposition 10.** *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$. *Let* $\xi'(q) = \xi(q) \cap \mathcal{C}'$ *for* $q \in \Sigma$, *then* $(\Sigma, \mathcal{C}', \xi')$ *is a state property system which we shall refer to as the classical part of* $(\Sigma, \mathcal{L}, \xi)$.

*Proof.* Clearly $0 \notin \xi'(p)$ for $p \in \Sigma$. Consider $a_i \in \xi'(p)\ \forall i$. Then $a_i \in \xi(p) \cap \mathcal{C}'\ \forall i$, from which follows that $\wedge_i a_i \in \xi(p) \cap \mathcal{C}'$ and hence $\wedge_i a_i \in \xi'(p)$. Consider $a, b \in \mathcal{C}'$. Let us suppose that $a \leq b$ and consider $r \in \Sigma$ such that $a \in \xi'(r)$. This means that $a \in \xi(r) \cap \mathcal{C}'$. From this follows that $b \in \xi(r) \cap \mathcal{C}'$ and hence $b \in \xi'(r)$. On the other hand let us suppose that $\forall r \in \Sigma : a \in \xi'(r)$ then $b \in \xi'(r)$. Since $a, b \in \mathcal{C}'$, this also means that $\forall r \in \Sigma : a \in \xi(r)$ then $b \in \xi(r)$. From this follows that $a \leq b$. □

Since $(\Sigma, \mathcal{C}', \xi')$ is a state property system, it has a corresponding closure space $(\Sigma, \kappa(\mathcal{C}'))$. In order to check some property of this space we introduce the following concepts.

**Definition 8.** *Let $(X, \mathcal{F})$ be a closure space and $\mathcal{B} \subset \mathcal{F}$. $\mathcal{B}$ is called a base of $(X, \mathcal{F})$ iff $\forall F \in \mathcal{F} : \exists B_i \in \mathcal{B} : F = \cap B_i$. $(X, \mathcal{F})$ is called weakly zero-dimensional iff there is a base consisting of clopen sets.*

**Proposition 11.** *The closure space $(\Sigma, \kappa(\mathcal{C}'))$ corresponding to the state property system $(\Sigma, \mathcal{C}', \xi')$ is weakly zero-dimensional.*

*Proof.* To see this recall that $a$ is classical iff $\kappa(a)$ is clopen in $(\Sigma, \kappa(\mathcal{L}))$, hence $\kappa(\mathcal{C}')$ is a family of closed sets on $\Sigma$ which consists of all intersections of the clopen sets of $(\Sigma, \kappa(\mathcal{L}))$. □

In general the lattice of $(\Sigma, \mathcal{C}', \xi')$ does not need to be atomistic, hence it is different from the totally classical state property system $(\Omega, \mathcal{C}, \eta)$ associated with $(\Sigma, \mathcal{L}, \xi)$.

# References


[1] Aerts, D. (1999). Foundations of quantum physics: a general realistic and operational approach, *Int. J. Theor. Phys.*, **38**, 289-358.

[2] Aerts, D., Colebunders, E., Van der Voorde, A. and Van Steirteghem, B. (1999). State property systems and closure spaces: a study of categorical equivalence, *Int. J. Theor. Phys.*, **38**, 359.



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