Applied Categorical Structures **10:** 469–480, 2002.

On the Amnestic Modification of the Category of State Property Systems

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(Received: 6 July 1999; accepted: 20 November 1999)

Abstract. State property systems were created on the basis of physical intuition in order to describe a mathematical model for physical systems. A state property system consists of a triple: a set of states, a complete lattice of properties and a specified function linking the other two components. The definition of morphisms between such objects was inspired by the physical idea of a subsystem. In this paper we give an isomorphic description of the category of state property systems, thus introducing a category **SP**, which is concrete over **Set**. This isomorphic description enables us to investigate further categorical properties of **SP**. It turns out that the category **SP** is not amnestic. In our main theorem we prove that the amnestic modification of **SP** is the construct **Cls** of closure spaces and continuous maps. Moreover we observe that the categorical product and coproduct in **Cls** find, through **SP**, application in physics.

Mathematics Subject Classifications (2000): 54A05, 54B30, 81P05.

Key words: closure space, state property system, amnestic modification.

1. Introduction

The construct **Cls** of closure spaces and continuous maps is getting more and more attention in recent years. An object (X, cl) of the construct **Cls** is a set X structured by a closure operator $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following conditions. For $A, B \subseteq X$:

$$A \subseteq cl\,A, \tag{1}$$
$$A \subset B \Rightarrow cl\,A \subseteq cl\,B, \tag{2}$$
$$cl\,A = cl(cl\,A), \tag{3}$$
$$cl\,\emptyset = \emptyset. \tag{4}$$

A couple (X, cl) is called a closure space. A morphism $f : (X, cl_X) \to (Y, cl_Y)$ is a function that preserves the closure in the sense that $f(cl_X A) \subseteq cl_Y f(A)$, for all $A \subseteq X$. The morphisms are called continuous maps.

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An isomorphic description of the construct of closures via collections of closed sets, so called Moore families, has been known for quite some time and is for instance contained (in a slightly different form) in Birkhoff's *Lattice Theory* [9]. This description is presented in Section 3 below.

In [10] it was proved that the construct of closure spaces and continuous maps is topological. In the same paper the construction of initial and final structures in the setting of closures is explicitely described.

In various fields of mathematics, closures arise very naturally. In the paper just mentioned, closures play an essential role in the description of epimorphisms of certain categories.

Another example is the link between closures and projective and affine geometry which is explained in a recent book by M. K. Bennett [8]. The explicit formulation of the equivalence between categories consisting of geometric objects and certain subconstructs of the category of closures is based on the investigation of suitable geometric morphisms as studied by C. A. Faure and A. Frölicher [12, 13].

For quite some time closures have been considered in relation with lattice theory. The link between certain topological closures and their corresponding lattices of closed sets has been investigated by several authors. A long list of references to this field can be found in [11]. A treatment of this correspondence in the setting of closures and a description of a general representation theory was worked out by M. Erné [11]. His results, when applied to T_D (respectively T_1) closures, describe an equivalence between the categories of T_D (respectively T_1) closures and complete molecular (respectively complete atomistic) lattices. These equivalences were also established in [14, 17].

Not only within mathematics, but also in related fields, such as the foundations of physics, the relevance of closures has recently been stressed by several authors [4, 6, 16, 17, 19, 21, 23]. In particular, certain models of physical systems are built on a well defined lattice of properties of the system. Moreover there is a natural closure corresponding to the lattice, and this closure plays a key role. The physical interpretation of the correspondence is that a closed set corresponding to a certain property represents the set of states that make the property actual.

Recently, the first author took up the idea of combining two basic concepts of a physical entity in the formalisation of a mathematical model. The basic concepts are the states and the properties of the physical system. In the formalism, as proposed by D. Aerts in [5], the mathematical structure modelling a physical entity S consists of the following. First with an entity S there corresponds a well defined set Σ of states of the entity. These are the modes of being of the entity in the sense that at any moment the entity is *in* one and only one state. The second component is a well defined set $\mathcal{L}$ of properties. This set is endowed with the structure of a complete lattice. The components defined so far are linked by means of a function. This function $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ maps a state p to the set $\xi(p)$ of all properties that are actual in state p. The axioms for the triple $(\Sigma, \mathcal{L}, \xi)$ are given in the following definition. The objects thus defined are called 'State Property Systems'.

DEFINITION 1. A triple $(\Sigma, \mathcal{L}, \xi)$ is called a state property system if Σ is a set, $\mathcal{L}$ is a complete lattice and $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ is a function such that for $p \in \Sigma$, 0 the minimum of $\mathcal{L}$ and $(a_i)_i \in \mathcal{L}$, we have:

$$0 \notin \xi(p), \tag{5}$$

$$a_i \in \xi(p)\ \forall i \Rightarrow \bigwedge_i a_i \in \xi(p) \tag{6}$$

and for $a, b \in \mathcal{L}$ we have:

$$a \leq b \Leftrightarrow \forall r \in \Sigma : a \in \xi(r) \text{ then } b \in \xi(r). \tag{7}$$

Considering the interpretation of $\xi(p)$, it follows that $\leq$ is 'implication' and $\wedge$ is 'conjunction' with respect to actuality.

There is a dual way of describing the connection between states and properties. If $(\Sigma, \mathcal{L}, \xi)$ is a state property system, we can associate to each property a the set of states that make a actual. This transition is formally described by means of the so called Cartan map κ [2]:

$$\kappa : \mathcal{L} \to \mathcal{P}(\Sigma) : a \mapsto \kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\}. \tag{8}$$

In fact, ξ and κ express the *physical duality* between states and properties [18]. It is easily seen that if we order $\mathcal{P}(\Sigma)$ by inclusion and use the induced order on the image that $\kappa : \mathcal{L} \to \kappa(\mathcal{L})$ is an isomorphism of complete lattices.

The motivation for the definition of morphisms between state property systems comes from the investigation of subsystems. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$, respectively describing entities S and S' and suppose S is a subentity of S'. In that case, the following three natural requirements seem plausible:

(i) If the entity S' is in a state p' then the state $m(p')$ of S is determined. This defines a function m from the set of states of S' to the set of states of S;

(ii) if we consider a property a of the entity S, then to a corresponds a property $n(a)$ of the 'bigger' entity S'. This defines a function n from the set of properties of S to the set of properties of S';

(iii) we want a and $n(a)$ to be two descriptions of the 'same' property of S, once considered as an entity on itself, once as a subentity of S'. In other words we want a and $n(a)$ to be actual at once. This means that for a state p' of S' (and a corresponding state $m(p')$ of S) we want the following 'covariance principle' to hold:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p'). \tag{9}$$

This intuitive idea was formalised in [5] in the following way.

DEFINITION 2. If $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$ are state property systems then $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ is called an **SP**-morphism if $m : \Sigma' \to \Sigma$ and $n : \mathcal{L} \to \mathcal{L}'$ are functions such that for $a \in \mathcal{L}$ and $p' \in \Sigma'$:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p'). \tag{10}$$

The category of state property systems and **SP**-morphisms is denoted by **SP**.

The physical intuition about state property systems also led to certain natural constructions. In [6, 21] the product of state property systems was introduced. In Section 6 we recall its interpretation as a coarsest description of a compound physical system. The coproduct of closure spaces also finds application in the foundations of physics, and especially in a category of 'Orthogonality Spaces', which is related to **Cls** [16].

In this paper we are concerned with the investigation of **SP** in a categorical way. We prove that the category **SP** is isomorphic to a concrete category **C** over **Set**. Our main result states that the construct **C** is not amnestic and that its amnestic modification is (isomorphic to) the construct of closure spaces. It will follow that in **C** constructions of initial and final structures are not unique, but are essentially performed as in **Cls**. In some sense the main result of this paper improves the categorical equivalence between **SP** and **Cls** formulated by the same authors in [6].

For all categorical terminology we refer the reader to [1, 20].

2. The Construct C

In this section we introduce an isomorphic description of the category of state property systems.

DEFINITION 3. **C** is the category whose objects are given by pairs $(X, (\alpha, s))$ where X is a set, $\alpha = (\alpha(x))_{x \in X}$ is a family of sets indexed by X and s is a 'point', such that the following conditions are satisfied:

$$\forall x \in X : s \notin \alpha(x) \tag{11}$$

$$\forall a, b \in \bigcup_{x \in X} \alpha(x) : \text{If } (\forall x \in X : a \in \alpha(x) \Leftrightarrow b \in \alpha(x)) \text{ then } a = b \tag{12}$$

$$\forall A \subseteq Y\ \exists c \in Y \text{ such that } \forall x \in X : A \subseteq \alpha(x) \Leftrightarrow c \in \alpha(x). \tag{13}$$

We will call $Y = \bigcup_{x \in X} \alpha(x) \cup \{s\}$ the associated set of $(X, (\alpha, s))$ and we define the Cartan map of $(X, (\alpha, s))$ as:

$$\kappa : Y \to \mathcal{P}(X) : a \mapsto \kappa(a) = \{x \in X \mid a \in \alpha(x)\}. \tag{14}$$

If $(X, (\alpha, s))$ and $(X', (\alpha', s'))$ are objects of **C**, then a function $f : (X', (\alpha', s')) \to (X, (\alpha, s))$ is a **C**-morphism provided that $\forall a \in Y\ \exists a' \in Y'$ such that $f^{-1}(\kappa(a)) = \kappa'(a')$ where Y and Y' are the associated sets.

We note that since, by (12), κ is injective, this a' is unique.

THEOREM 1. **C** *is a concrete category over* **Set**.

Proof. The correspondence $U : \mathbf{C} \to \mathbf{Set}$ consisting of

(1) the mapping

$$|\mathbf{C}| \to |\mathbf{Set}| \tag{15}$$
$$(X, (\alpha, s)) \mapsto U(X, (\alpha, s)) = X, \tag{16}$$

(2) for every pair of objects $(X, (\alpha, s)), (X', (\alpha', s'))$ of **C** the mapping

$$\mathbf{C}((X', (\alpha', s')), (X, (\alpha, s))) \to \mathbf{Set}(U(X', (\alpha', s')), U(X, (\alpha, s))) \tag{17}$$
$$f : (X', (\alpha', s')) \to (X, (\alpha, s)) \mapsto f : X' \to X \tag{18}$$

is a faithful functor. □

In the following propositions we show how we can associate with each **C**-object a state property system and with each **C**-morphism an **SP**-morphism.

PROPOSITION 1. *Let* $(X, (\alpha, s))$ *be an object of* **C**. *Then* $(X, \bigcup_{x\in X} \alpha(x) \cup \{s\}, \varphi_\alpha) = (X, Y, \varphi_\alpha)$ *where* $\varphi_\alpha : X \to \mathcal{P}(Y) : x \mapsto \alpha(x)$, *is a state property system.*

Proof. Define the following relation on Y : for $a, b \in Y$

$$a \le b \Leftrightarrow \forall x \in X : a \in \varphi_\alpha(x) \Rightarrow b \in \varphi_\alpha(x). \tag{19}$$

Condition (12) of a **C**-object implies this preorder is a partial order. Next we show $(Y, \le)$ is a complete lattice. Let $A \subseteq Y$. Since condition (13) is satisfied, there exists a $c \in Y$ such that $\forall x \in X : c \in \alpha(x) \Leftrightarrow A \subseteq \alpha(x)$. We have $\bigwedge A = c$. Indeed, for $d \in Y$, $d \le c \Leftrightarrow (d \le a, \forall a \in A)$. Following Birkhoff [9] we have that $\forall A \subseteq Y : \bigvee A = \bigwedge\{d \in Y \mid d \text{ is an upper bound of } A\}$. Hence $(Y, \le, \wedge, \vee)$ is a complete lattice with minimum s. Condition (6) follows immediately from the definition of $\wedge$. □

PROPOSITION 2. *Let* $(X, (\alpha, s))$ *and* $(X', (\alpha', s'))$ *be objects of* **C** *and* $f : (X', (\alpha', s')) \to (X, (\alpha, s))$ *a morphism of* **C**. *Consider the state property systems* (X, Y, φ_α) *and* $(X', Y', \varphi_{\alpha'})$ *corresponding to these two objects, as proposed in Proposition* 1. *Then* $(f, (\kappa')^{-1} \circ f^{-1} \circ \kappa) : (X', Y', \varphi_{\alpha'}) \to (X, Y, \varphi_\alpha)$ *is an* **SP**-*morphism.*

Proof. It is clear that $f : X' \to X$ and $(\kappa')^{-1} \circ f^{-1} \circ \kappa : Y \to Y'$ are functions. $(\kappa')^{-1}$ is the inverse of $Y \to \kappa'(Y) : y \mapsto \kappa'(y)$. Further, for all $a \in Y$ and for all $x' \in X'$ we have: $((\kappa')^{-1} \circ f^{-1} \circ \kappa)(a) \in \varphi_{\alpha'}(x') \Leftrightarrow ((\kappa')^{-1} \circ f^{-1} \circ \kappa)(a) \in \alpha'(x') \Leftrightarrow x' \in \kappa'((\kappa')^{-1} \circ f^{-1} \circ \kappa)(a)) \Leftrightarrow x' \in f^{-1}(\kappa(a)) \Leftrightarrow f(x') \in \kappa(a) \Leftrightarrow a \in \alpha(f(x')) \Leftrightarrow a \in \varphi_\alpha(f(x'))$. □

Conversely we can associate a **C**-object to each state property system.

PROPOSITION 3. *Let* $(\Sigma, \mathcal{L}, \xi)$ *be a state property system. If* $\alpha_\xi = (\xi(p))_{p\in\Sigma}$, *then* $(\Sigma, (\alpha_\xi, 0))$ *is an object of* **C**.

Proof. Condition (11) follows immediately from the definition of a state property system. Since $(\mathcal{L}, \leq, \wedge, \vee)$ is a complete lattice with $\forall a, b \in \mathcal{L} : a \leq b \Leftrightarrow \forall p \in \Sigma : a \in \xi(p) \Rightarrow b \in \xi(p)$ conditions (12) and (13) are also satisfied. □

PROPOSITION 4. *Let* $(\Sigma, \mathcal{L}, \xi)$ *and* $(\Sigma', \mathcal{L}', \xi')$ *be state property systems and* $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ *an* **SP**-*morphism. Consider the* **C**-*objects* $(\Sigma, (\alpha_\xi, 0))$ *and* $(\Sigma', (\alpha_{\xi'}, 0'))$ *corresponding to these two objects, as proposed in Proposition* 3. *Then* $m : (\Sigma', (\alpha_{\xi'}, 0')) \to (\Sigma, (\alpha_\xi, 0))$ *is a* **C**-*morphism.*

Proof. Let $a \in \mathcal{L}$. Then $n(a) \in \mathcal{L}'$ and $m^{-1}(\kappa(a)) = \kappa'(n(a))$. □

Now we will formalize this connection between the category **SP** of state property systems and the category **C** into an isomorphism of categories.

THEOREM 2. *The category* **SP** *of state property systems is isomorphic to the concrete category* **C**.

Proof. From the previous propositions it follows that the correspondence $K : \mathbf{C} \to \mathbf{SP}$ consisting of

(1) the mapping

$$|\mathbf{C}| \to |\mathbf{SP}| \tag{20}$$

$$(X, (\alpha, s)) \mapsto K(X, (\alpha, s)) = \left(X, \bigcup_{x \in X} \alpha(x) \cup \{s\}, \varphi_\alpha\right), \tag{21}$$

(2) for every pair of objects $(X, (\alpha, s)), (X', (\alpha', s'))$ of **C** the mapping

$$\mathbf{C}((X', (\alpha', s)), (X, (\alpha, s))) \to \mathbf{SP}(K(X', (\alpha', s')), K(X, (\alpha, s))) \tag{22}$$

$$f \mapsto (f, (\kappa')^{-1} \circ f^{-1} \circ \kappa) \tag{23}$$

and the correspondence $L : \mathbf{SP} \to \mathbf{C}$ consisting of

(1) the mapping

$$|\mathbf{SP}| \to |\mathbf{C}| \tag{24}$$

$$(\Sigma, \mathcal{L}, \xi) \mapsto L(\Sigma, \mathcal{L}, \xi) = (\Sigma, (\alpha_\xi, 0)), \tag{25}$$

where 0 is the minimum of the complete lattice $\mathcal{L}$,

(2) for every pair of objects $(\Sigma, \mathcal{L}, \xi), (\Sigma', \mathcal{L}', \xi')$ of **SP** the mapping

$$\mathbf{SP}((\Sigma', \mathcal{L}', \xi'), (\Sigma, \mathcal{L}, \xi)) \to \mathbf{C}(L(\Sigma', \mathcal{L}', \xi'), L(\Sigma, \mathcal{L}, \xi)) \tag{26}$$

$$(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi) \mapsto m : (\Sigma', (\alpha_{\xi'}, 0')) \to (\Sigma, (\alpha_\xi, 0)) \tag{27}$$

are functors. It is easy to verify that $K \circ L = Id_{\mathbf{SP}}$ and $L \circ K = Id_{\mathbf{C}}$. Hence $\mathbf{C} \cong \mathbf{SP}$. □

As a consequence of Theorem 2 we can consider the **C**-objects as state property systems and the state property systems as **C**-objects. The same holds for the morphisms. In the present article it is our main goal to investigate the category **SP** categorically. That is why from now on we will use the notation $(X, (\alpha, s))$ for a state property system as defined in Definition 3. We call (α, s) an **SP**-structure on X.

3. Closure Spaces

In [6] we proved that the category **SP** of state property systems is equivalent to the category **Cls** of closure spaces. After recalling the definition of a closure space, we will recall the equivalence functors, but in the current notation.

DEFINITION 4. A closure space $(X, \mathcal{F})$ consists of a set X and a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following conditions:

$$\emptyset \in \mathcal{F}, \tag{28}$$

$$(F_i)_i \in \mathcal{F} \Rightarrow \bigcap_i F_i \in \mathcal{F}. \tag{29}$$

The closure operator corresponding to the closure space $(X, \mathcal{F})$ is defined as

$$cl : \mathcal{P}(X) \to \mathcal{P}(X) : A \mapsto \bigcap\{F \in \mathcal{F} \mid A \subseteq F\}. \tag{30}$$

If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are closure spaces then a function $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is called continuous if $\forall B \in \mathcal{G} : f^{-1}(B) \in \mathcal{F}$. The category of closure spaces and continuous maps is denoted **Cls**.

This description of **Cls** is isomorphic to the one in the introduction. In the next theorem we describe two natural functors, F from state property systems to closures and G from closures to state property systems. The functor F is based on Birkhoff's polarities [9, Ch. 5, §7]. This idea was also used by Aumann [7] to investigate an example from sociology and is the cornerstone of Formal Concept Analysis [15]. Moreover G corresponds to Aumann's transition from closures to relations. A more detailed description of the link with Aumann's work will be worked out in [22].

THEOREM 3. *The functor* $F : \mathbf{SP} \to \mathbf{Cls}$ *consisting of*
(1) *the mapping*

$$|\mathbf{SP}| \to |\mathbf{Cls}| \tag{31}$$

$$(X, (\alpha, s)) \mapsto F(X, (\alpha, s)) = (X, \mathcal{F}_\alpha), \tag{32}$$

where $\mathcal{F}_\alpha = \kappa(Y)$ *with* Y *the associated set of* $(X, (\alpha, s))$,
(2) *for every pair of objects* $(X, (\alpha, s))$, $(X', (\alpha', s'))$ *of* **SP** *the mapping*

$$\mathbf{SP}((X', (\alpha', s')), (X, (\alpha, s))) \to \mathbf{Cls}(F(X', (\alpha', s')), F(X, (\alpha, s))) \tag{33}$$

$$f : (X', (\alpha', s')) \to (X, (\alpha, s)) \mapsto f : (X', \mathcal{F}_{\alpha'}) \to (X, \mathcal{F}_\alpha) \tag{34}$$

and the functor $G : \mathbf{Cls} \to \mathbf{SP}$ *consisting of*

(1) *the mapping*

$$|\mathbf{Cls}| \to |\mathbf{SP}| \tag{35}$$

$$(X, \mathcal{F}) \mapsto G(X, \mathcal{F}) = (X, (\alpha_{\mathcal{F}}, \emptyset)), \tag{36}$$

where $\alpha_{\mathcal{F}}(x) = \{F \in \mathcal{F} \mid x \in F\}, \forall x \in X,$

(2) *for every pair of objects* $(X, \mathcal{F}), (X', \mathcal{F}')$ *of* **Cls** *the mapping*

$$\mathbf{Cls}((X', \mathcal{F}'), (X, \mathcal{F})) \to \mathbf{SP}(G(X', \mathcal{F}'), G(X, \mathcal{F})) \tag{37}$$

$$f : (X', \mathcal{F}') \to (X, \mathcal{F}) \mapsto f : (X', (\alpha_{\mathcal{F}'}, \emptyset)) \to (X, (\alpha_{\mathcal{F}}, \emptyset)) \tag{38}$$

establish an equivalence of categories.

The functor G clearly is injective on objects and so G is a full embedding and thus **Cls** can be considered as a full subconstruct of **SP**.

4. The Preorder Relation on the Fibers

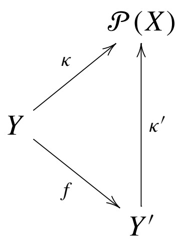
We consider the usual preorder relation on the fiber of an **SP**-object and we study the associated equivalence relation.

DEFINITION 5. Let X be a set and let (α, s) and (α', s') be **SP**-structures on X. The **SP**-structure (α', s') is called finer than (α, s) (and (α, s) coarser than (α', s')) if $Id_X : (X, (\alpha', s')) \to (X, (\alpha, s))$ is an **SP**-morphism. We then write $(\alpha, s) \leq (\alpha', s')$. The **SP**-structures (α, s) and (α', s') are said to be equivalent provided $(\alpha, s) \leq (\alpha', s')$ and $(\alpha', s') \leq (\alpha, s)$, noted $(\alpha, s) \sim (\alpha', s')$.

PROPOSITION 5. *Let X be a set. If (α, s) and (α', s') are* **SP**-*structures on X and Y, Y' are the associated sets, then the following are equivalent:*

(1) $(\alpha, s) \leq (\alpha', s')$;

(2) $\exists f : Y \to Y'$ *such that the following diagram commutes.*



Proof. (1) ⇒ (2) Assume that $(\alpha, s) \leq (\alpha', s')$. Then for all $a \in Y$ there exists $a' \in Y'$ such that $\kappa(a) = \kappa'(a')$. Since κ' is injective this a' is unique. We define the function $f : Y \to Y' : a \mapsto f(a) = a'$.

(2) ⇒ (1) Trivial. □

EXAMPLE 1. Let $(X, (\alpha, s))$ be a state property system with Y the associated set. Then, with the notations of Theorem 3, $F(X, (\alpha, s)) = (X, \mathcal{F}_\alpha)$ is a closure space and $GF(X, (\alpha, s)) = (X, (\alpha_{\mathcal{F}_\alpha}, \emptyset))$ is a state property system for which the associated set is $\mathcal{F}_\alpha$. Then the **SP**-structures (α, s) and $(\alpha_{\mathcal{F}_\alpha}, \emptyset)$ are equivalent.

The Cartan map of $(X, (\alpha_{\mathcal{F}_\alpha}, \emptyset))$ is the injection $i : \mathcal{F}_\alpha \hookrightarrow \mathcal{P}(X)$. Since $i \circ \kappa = \kappa$ it follows from Proposition 5 that $(\alpha, s) \leq (\alpha_{\mathcal{F}_\alpha}, \emptyset)$. Since $\kappa \circ (\kappa|^{\mathcal{F}_\alpha})^{-1} = i$ where $(\kappa|^{\mathcal{F}_\alpha})^{-1} : \mathcal{F}_\alpha \to Y$, we find that $(\alpha_{\mathcal{F}_\alpha}, \emptyset) \leq (\alpha, s)$.

This example shows that the category **SP** is not amnestic, i.e. the fibres are not partially ordered. Indeed, we have $(\alpha, s) \leq (\alpha_{\mathcal{F}_\alpha}, \emptyset)$ and $(\alpha_{\mathcal{F}_\alpha}, \emptyset) \leq (\alpha, s)$, but $(\alpha, s) \neq (\alpha_{\mathcal{F}_\alpha}, \emptyset)$ if $(X, (\alpha, s))$ is not $G(X, \mathcal{F})$ for some closure structure $\mathcal{F}$ on X.

Every concrete category which is not amnestic, has an amnestic modification [1]. In the next section we investigate the amnestic modification of **SP**.

5. The Amnestic Modification of SP

In this section we state our main result, describing the link between the construct **Cls** and the physically inspired construct **SP**.

THEOREM 4. **Cls** *is the amnestic modification of* **SP**.

Proof. The relation for **SP**-structures on a set X: '(α, s) is equivalent to (α', s')' as defined in Definition 5 is an equivalence relation on the fiber of X. Since we know that **Cls** is a full subcategory of **SP** we only have to show that **Cls** contains precisely one member from each equivalence class. Consider a state property system $(X, (\alpha, s))$ and the equivalence class $[(\alpha, s)] = \{(\alpha', s') \mid (\alpha', s') \sim (\alpha, s)\}$. In Example 1 we showed that $(\alpha_{\mathcal{F}_\alpha}, \emptyset) \in [(\alpha, s)]$. Since **Cls** is topological, $(\alpha_{\mathcal{F}_\alpha}, \emptyset)$ is the only closure structure in the equivalence class $[(\alpha, s)]$. □

We can interpret the result of Theorem 4 as follows: In the non-amnestic category **SP** there are, quoting from [1], 'too many **SP**-structures floating around in the fibers'. For each closure structure on a given set there is a proper class of equivalent state property structures on this set. So **SP** is neither fiber small, nor does it satisfy the terminal separator property. In Propositions 6 and 7 we shall show how to construct this class of state property systems, but first we give a physical illustration of this mathematical situation.

EXAMPLE 2. We consider a physical system consisting of a vessel filled with water in a laboratory at 277 K. The system has two possible states: p is 'the vessel is filled with 0.5 liter of water', q is 'the vessel contains 1.5 liter of water'. We consider two nontrivial properties of the system. The first one, a, is actual iff the vessel contains less than 1 liter of liquid. The other, a', is actual iff the content of the vessel weighs less than 1 kg. We also introduce 1, a property which is actual in every state, and 0 which is never actual. Motivated by highschool physics, we define the following two families indexed by Σ (recall the interpretation: $\alpha(p)$ is the set of properties actual in state p).

$$\alpha(p) = \{a, 1\}, \qquad \alpha(q) = \{1\}, \tag{39}$$

$$\alpha'(p) = \{a', 1\}, \qquad \alpha(q) = \{1\}. \tag{40}$$

Obviously $(\alpha, 0)$ and $(\alpha', 0)$ are equivalent **SP**-structures on $\Sigma = \{p, q\}$. In the corresponding closure space, a and a' are identified: both are represented by $\{p\}$. Physically this means that these two properties are always actual together for the considered system.

If, on the other hand, we reinterpret p as 'the vessel contains 0.5 liter of mercury' and q as 'the vessel contains 1.5 liter of mercury' in the same laboratory, it is physically obvious that a is actual in p but not in q, and a' is actual in neither of the two states. If we redefined α and α' accordingly, we would of course obtain non-equivalent structures on Σ.

PROPOSITION 6. *Let X be a set and let $\mathcal{F}$ be a closure system on X. Let $(X, (\alpha_{\mathcal{F}}, \emptyset))$ the associated state property system with associated set $Y = \bigcup_{x\in X} \alpha_{\mathcal{F}}(x) \cup \emptyset = \mathcal{F}$. Choose a set Y' which has the same cardinality as Y. Consider a bijective function $h : Y \to Y'$ and put $h(\emptyset) = s' \in Y'$. We can define the following function:*

$$\tilde{h} : \mathcal{P}(Y) \to \mathcal{P}(Y') : A \mapsto \tilde{h}(A) = \{h(a) \mid a \in A\}. \tag{41}$$

Define α' as follows:

$$\forall x \in X : \alpha'(x) = \tilde{h}(\alpha_{\mathcal{F}}(x)). \tag{42}$$

*Then $(X, (\alpha', s'))$ is a state property system and the **SP**-structures $(\alpha_{\mathcal{F}}, \emptyset)$ and (α', s') are equivalent.*

PROPOSITION 7. *Let X be a set. Every state property structure on X can be constructed in the way described in Proposition 6.*

Proof. Let $(X, (\beta, t))$ be a state property sytem with associated set Z. Then $(X, \mathcal{F}_\beta)$ is a closure space. $\kappa_\beta|^{\mathcal{F}_\beta} : Z \to \mathcal{F}_\beta$ is a bijection. Let $h = (\kappa_\beta|^{\mathcal{F}_\beta})^{-1}$. Then we have $\beta(x) = \tilde{h}(\alpha_{\mathcal{F}_\beta}(x)), \forall x \in X$. □

6. Limits in SP

As we mentioned in the introduction, physical intuition about states and properties has led to certain natural constructions of products and coproducts [6, 16, 21]. One of the motivations for our categorical study of **SP** was the question of whether physically inspired constructions correspond to categorical limits and colimits. As an application of the results in the previous sections we easily obtain a description of limits and colimits in **SP**. In fact, in the next proposition we give a description of initial structures for arbitrary sources with codomains in **SP**. The construction first associates a corresponding source with codomains in **Cls**. The fact that **Cls** is a topological construct then guarantees the existence of an initial closure structure which gives rise to an initial structure for the given source with codomains in **SP**. The same procedure is applicable for constructing final structures in **SP**.

PROPOSITION 8. *Let X be a set, $(X_i, (\alpha_i, s_i))_{i \in I}$ be a family of state property systems and $(f_i : X \to X_i)_{i \in I}$ a family of maps. To find an initial* **SP***-structure on X with respect to $(X, f_i, (X_i, (\alpha_i, s_i)))$ we consider the equivalence class $[(\alpha_i, s_i)]$, $\forall i \in I$. For each $i \in I$, we take the unique closure $\mathcal{F}_i$ such that $(\alpha_{\mathcal{F}_i}, \emptyset) \in [(\alpha_i, s_i)]$. We consider the initial closure structure $\mathcal{F}$ with respect to $(X, f_i, (X_i, \mathcal{F}_i))$. Then $(\alpha_{\mathcal{F}}, \emptyset)$ is an initial* **SP***-structure with respect to $(X, f_i, (X_i, (\alpha_i, s_i)))$. Moreover every* **SP***-structure which is equivalent to $(\alpha_{\mathcal{F}}, \emptyset)$ is also an initial structure for the given source.*

The physical significance of the product of closure spaces (state property systems) will next be explained. For more information we refer to [3]. Consider two physical entities S_1 and S_2 described by two closure spaces $(\Sigma_1, \mathcal{F}_1)$ and $(\Sigma_2, \mathcal{F}_2)$. Recall that the product of these closure spaces is given by $(\Sigma_1 \times \Sigma_2, \mathcal{F})$, where

$$\mathcal{F} = \{(F_1 \times \Sigma_2) \cap (\Sigma_1 \times F_2) \mid F_i \in \mathcal{F}_i\} = \{F_1 \times F_2 \mid F_i \in \mathcal{F}_i\}. \tag{43}$$

This is a minimal description of a compound system containing these two entities, in the following sense. Suppose $(X, \mathcal{G})$ is a closure space associated to an entity S 'containing' S_1 and S_2 as subsystems. Then (see the physical motivation of morphisms), there are continuous maps $f_1 : (X, \mathcal{G}) \to (\Sigma_1, \mathcal{F}_1)$ and $f_2 : (X, \mathcal{G}) \to (\Sigma_2, \mathcal{F}_2)$. Hence, there is a unique map $f : (X, \mathcal{G}) \to (\Sigma_1 \times \Sigma_2, \mathcal{F})$ such that $\pi_i \circ f = f_i$, where $\pi_i : \Sigma_1 \times \Sigma_2 \to \Sigma_i$ is the projection. Moreover, $\mathcal{F}$ contains the properties (closed sets) of S_1, those of S_2, their conjunction (intersection) and *nothing more*. Therefore, the product can be interpreted as a 'coarsest composition' of S_1 and S_2.

The coproduct's physical significance can be found in [16]. It requires some more axioms on the state property systems.

Acknowledgments

The authors thank Horst Herrlich for pointing out the reference to the Aumann paper and a referee for drawing our attention to FCA.

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