State Property Systems and Closure Spaces: Extracting the Classical en Nonclassical Parts*

Diederik Aerts
Center Leo Apostel for Interdisciplinary Studies (CLEA)
Foundations of the Exact Sciences (FUND)
Department of Mathematics
Vrije Universiteit Brussel, 1160 Brussels, Belgium
E-Mail: diraerts@vub.ac.be

Didier Deses
Topology (TOPO)
Foundations of the Exact Sciences (FUND)
Department of Mathematics
Vrije Universiteit Brussel, 1160 Brussels, Belgium
E-Mail: diddesen@vub.ac.be

Abstract

In [1] an equivalence of the categories **SP** and **Cls** was proven. The category **SP** consists of the state property systems [2] and their morphisms, which are the mathematical structures that describe a physical entity by means of its states and properties [3, 4, 5, 6, 7, 8]. The category **Cls** consists of the closure spaces and the continuous maps. In earlier work it has been shown, using the equivalence between **Cls** and **SP**, that some of the axioms of quantum axiomatics are equivalent with separation axioms on the corresponding closure space. More particularly it was proven that the axiom of atomicity is equivalent to the T_1 separation axiom [9]. In the present article we analyze the intimate relation that exists between classical and nonclassical in the state property systems and disconnected and connected in the corresponding closure space, elaborating results that appeared in [10, 11]. We introduce classical properties using the concept of super selection rule, i.e. two properties are separated by a superselection rule iff there do not exist 'superposition states' related to these two properties. Then we show that the classical properties of a state property system correspond exactly to the clopen subsets of the corresponding closure space. Thus connected closure spaces correspond precisely to state property systems for which the elements 0 and I are the only classical properties, the so called pure nonclassical state property systems. The main result is a decomposition theorem, which allows us to split a state property system into a number of 'pure nonclassical state property systems' and a 'totally classical state property system'. This decomposition theorem for a state property system is the translation of a decomposition theorem for the corresponding closure space into its connected components.

*Published as: Aerts, D. and Deses, D., 2002, State property systems and closure spaces: extracting the classical and nonclassical parts, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Probability and Axiomatics*, eds. Aerts, D, Czachor, M. and Durt, T, World Scientific, Singapore.

1 State Property Systems and Closure Spaces

The general approaches to quantum mechanics make use of mathematical structures that allow the description of pure quantum entities and pure classical entities, as well as mixtures of both. In this article we study the Geneva-Brussels approach, where the basic physical concepts are the one of state and property of a physical entity [3, 4, 5, 6, 7, 8]. Traditionally the collection of properties is considered to be a complete lattice, partially ordered by the implication of properties, with an orthocomplementation, representing the quantum generalization of the 'negation' of a property. A state is represented by the collections of properties that are actual whenever the entity is in this state. We mention however that in these earlier approaches [3, 4, 5, 6, 7, 8] the mathematical structure that underlies the physical theory had not completely been identified. To identify the mathematical structure in a complete way, the structure of a state property system was introduced in [2].

Suppose that we consider a physical entity S, and we denote its set of states by Σ and its set of properties by $\mathcal{L}$. The state property system corresponding to this physical entity S is a triple $(\Sigma, \mathcal{L}, \xi)$, where Σ is the set of states of S, $\mathcal{L}$ the set of properties of S, and ξ a map from Σ to $\mathcal{P}(\mathcal{L})$, that makes correspond to each state $p \in \Sigma$ the set of properties $\xi(p) \in \mathcal{P}(\mathcal{L})$ that are actual if the entity S is in state p. Some additional requirements, that express exactly how the physicists perceives a physical entity in relation with its states and properties, are satisfied in a state property system. Let us introduce the formal definition of a state property system and then explain what these additional requirements mean.

Definition 1 (State Property System) *A triple $(\Sigma, \mathcal{L}, \xi)$ is called a state property system if Σ is a set, $\mathcal{L}$ is a complete lattice and $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$ is a function such that for $p \in \Sigma$, 0 the minimal element of $\mathcal{L}$ and $(a_i)_i \in \mathcal{L}$, we have:*

$$0 \quad \notin \quad \xi(p) \tag{1}$$

$$a_i \in \xi(p)\ \forall i \quad \Rightarrow \quad \wedge_i a_i \in \xi(p) \tag{2}$$

and for $a, b \in \mathcal{L}$ we have:

$$a < b \quad \Leftrightarrow \quad \forall r \in \Sigma : a \in \xi(r) \text{ then } b \in \xi(r) \tag{3}$$

We demand that $\mathcal{L}$, the set of properties, is a complete lattice. This means that the set of properties is partially ordered, with the physical meaning of the partial order relation $<$ being the following: $a, b \in \mathcal{L}$, such that $a < b$ means that whenever property a is actual for the entity S, also property b is actual for the entity S. If $\mathcal{L}$ is a complete lattice, it means that for an arbitrary family of properties $(a_i)_i \in \mathcal{L}$ also the infimum $\wedge_i a_i$ of this family is a property. The property $\wedge_i a_i$ is the property that is actual iff all of the properties a_i are actual. Hence the infimum represents the logical 'and'. The minimal element 0 of the lattice of properties is the property that is never actual (*e.g.* the physical entity does not exist). Requirement (1) expresses that a property that is in the image by ξ of an arbitrary state $p \in \Sigma$ can never be the 0 property. Requirement (2) expresses that if for a state $p \in \Sigma$ all the properties a_i are actual, this implies that for this state p also the 'and' property $\wedge_i a_i$ is actual. Requirement (3) expresses the meaning of the partial order relation that we gave already: $a < b$ iff whenever p is a state of S such that a is actual if S is in this state, then also b is actual if S is in this state.

Along the same lines, just traducing what the physicist means when he imagines the situation of two physical entities, of which one is a sub entity of the other, the morphisms of state property systems can be deduced. More concretely, suppose that S is a sub entity of S'. Then each state p' of S' determines a state p of S, namely the state p where the sub entity S is in when S' is in state p'.

This defines a map $m : \Sigma' \rightarrow \Sigma$. On the other hand, each property a of S determines a property a' of S, namely the property of the sub entity, but now conceived as a property of the big entity. This defines a map $n : \Sigma \rightarrow \Sigma'$. Suppose that we consider now a state p' of S', and a property a of S, such that $a \in \xi(m(p'))$. This means that the property a is actual if the sub entity S is in state $m(p')$. This state of affairs can be expressed equally by stating that the property $n(a)$ is actual when the big entity is in state p'. Hence, as a basic physical requirement of merological covariance we should have:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{4}$$

This all gives rise to the following definition of morphism for state property systems.

Definition 2 (Morphisms of State Property Systems) *Suppose that* $(\Sigma, \mathcal{L}, \xi)$ *and* $(\Sigma', \mathcal{L}', \xi')$ *are state property systems then*

$$(m, n) : (\Sigma', \mathcal{L}', \xi') \rightarrow (\Sigma, \mathcal{L}, \xi)$$

is called an **SP***-morphism if* $m : \Sigma' \rightarrow \Sigma$ *and* $n : \mathcal{L} \rightarrow \mathcal{L}'$ *are functions such that for* $a \in \mathcal{L}$ *and* $p' \in \Sigma'$ *:*

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{5}$$

Using the previous definitions we can use these concept to generate a category of state property systems, in the mathematical sense.

Definition 3 (The Category SP) *The category of state property systems and their morphisms is denoted by* **SP***.*

Definition 4 (The Cartan Map) *If* $(\Sigma, \mathcal{L}, \xi)$ *is a state property system then its Cartan map is the mapping* $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$ *defined by :*

$$\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma) : a \mapsto \kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\} \tag{6}$$

It was amazing to be able to prove (see [1]) that this category of states property systems and its morphisms is equivalent to a category which arises as a generalization of the category of topological spaces and continuous maps, namely to the category of closure spaces and the continuous maps. We will now introduce this category of closure spaces.

A topological space consists of a set X, and a collection of 'open' subsets, such that X is open, any union of open subsets is again open and any finite intersection of open subsets is again open. A subset of X is called closed if it's complement is open. Therefore we have that in a topological space the empty set is closed, any intersection of closed sets is closed and any finite union of closed sets is again closed. Hence a topological space is also defined by it's closed sets.

In mathematics the concept topological space is very useful and arises in many different areas. However there are occasions when we 'almost' have a topological space. Let's take the following example. Consider the plane $\mathbb{R}^2$ and the collection of all convex subsets of $\mathbb{R}^2$ (A is convex if the segment between any two points of A lies completely within A). Clearly $\emptyset$ is convex and every intersection of convex sets is again convex. However a finite union of convex sets does not need to be convex. Hence the convex subsets of the plane can 'almost' be considered as closed sets, but they do not form a topological space. To be able to consider such structures one has introduced the notion of closure spaces.

Definition 5 (Closure Space) *A closure space $(X, \mathcal{F})$ consists of a set X and a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following conditions:*

$$\begin{array}{rcl} \emptyset & \in & \mathcal{F} \\ (F_i)_i \in \mathcal{F} & \Rightarrow & \cap_i F_i \in \mathcal{F} \end{array}$$

The closure operator corresponding to the closure space $(X, \mathcal{F})$ is defined as

$$cl : \mathcal{P}(X) \to \mathcal{P}(X) : A \mapsto \bigcap\{F \in \mathcal{F} \mid A \subseteq F\} \tag{7}$$

If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are closure spaces then a function $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is called a continuous map if $\forall B \in \mathcal{G} : f^{-1}(B) \in \mathcal{F}$. The category of closure spaces and continuous maps is denoted by **Cls**.

The following theorem shows how we can associate with each state property system a closure space and with each morphism a continuous map, hence we get the categorical equivalence described in [1].

Theorem 1 *The correspondence $F : \mathbf{SP} \longrightarrow \mathbf{Cls}$ consisting of*
(1) the mapping

$$\begin{array}{rcl} |\mathbf{SP}| & \to & |\mathbf{Cls}| \\ (\Sigma, \mathcal{L}, \xi) & \mapsto & F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L})) \end{array}$$

(2) for every pair of objects $(\Sigma, \mathcal{L}, \xi), (\Sigma', \mathcal{L}', \xi')$ of **SP** *the mapping*

$$\begin{array}{rcl} \mathbf{SP}((\Sigma', \mathcal{L}', \xi'), (\Sigma, \mathcal{L}, \xi)) & \to & \mathbf{Cls}(F(\Sigma', \mathcal{L}', \xi'), F(\Sigma, \mathcal{L}, \xi)) \\ (m, n) & \mapsto & m \end{array}$$

is a covariant functor.

We can also connect a state property system to a closure space and a morphism to a continuous map.

Theorem 2 *The correspondence $G : \mathbf{Cls} \longrightarrow \mathbf{SP}$ consisting of*
(1) the mapping

$$\begin{array}{rcl} |\mathbf{Cls}| & \to & |\mathbf{SP}| \\ (\Sigma, \mathcal{F}) & \mapsto & G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \bar{\xi}) \end{array}$$

where $\bar{\xi} : \Sigma \to \mathcal{P}(\mathcal{F}) : p \mapsto \{F \in \mathcal{F} \mid p \in F\}$
(2) for every pair of objects $(\Sigma, \mathcal{F}), (\Sigma', \mathcal{F}')$ of **Cls** *the mapping*

$$\begin{array}{rcl} \mathbf{Cls}((\Sigma', \mathcal{F}'), (\Sigma, \mathcal{F})) & \to & \mathbf{SP}(G(\Sigma', \mathcal{F}'), G(\Sigma, \mathcal{F})) \\ m & \mapsto & (m, m^{-1}) \end{array}$$

is a covariant functor.

Theorem 3 (Equivalence of SP and Cls) *The functors*

$$F : \mathbf{SP} \to \mathbf{Cls}$$

$$G : \mathbf{Cls} \to \mathbf{SP}$$

establish an equivalence of categories.

The above equivalence is a very powerful tool for studying state property systems. It states that the lattice $\mathcal{L}$ of properties can be seen as the lattice of closed sets of a closure space on the states Σ, conversely every closure space on X can be considered as a set of states (X) and a lattice of properties (the lattice of closed sets).

Recall that closure spaces are in fact a generalization of topological spaces, hence a number of topological properties have been generalized to closure spaces. Moreover with the previous equivalence, a concept which can be defined using closed sets on a closure space can be translated in an equivalent concept for state property systems. At first sight this translation does not need to be meaningful in the context of physical systems. However it turned out that many such translations actually coincided with well known physical concepts.

We shall give one example which was studied in [9]. A topological space is called T_1 if the following separation axiom is satisfied. For every two points x, y there are open sets which contain x resp. y but do not contain y resp. x. This is equivalent to stating that all singletons are closed sets. Hence the following definition.

Definition 6 (T_1 Closure Space) *A closure space $(X, \mathcal{F})$ is a T_1 closure space iff $\forall x \in X : \{x\} \in \mathcal{F}$.*

In the theory of state property systems, or more general of property lattices the concept of atomistic lattice is quite fundamental. In [9] it was proven that using the equivalence between state property systems and closure spaces both concepts are in fact related.

Definition 7 (Atomistic State Property System) *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. Then the map s_ξ maps a state p to the strongest property it makes actual, i.e.*

$$s_\xi : \Sigma \to \mathcal{L} : p \mapsto \wedge \xi(p) \tag{8}$$

T.F.A.E.

(1) $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ is injective and $\forall p \in \Sigma : s_\xi(p)$ is an atom of $\mathcal{L}$.

(2) $\forall p, q \in \Sigma : \xi(p) \subset \xi(q) \Rightarrow p = q$

(3) $F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}))$ is a T_1 closure space.

If a state property system satisfies one, and hence all of the above conditions it is called an atomistic state property system, in this case $\mathcal{L}$ is a complete atomistic lattice.

If we write $\mathbf{Cls_1}$ for the full subcategory of $\mathbf{Cls}$ given by T_1 closure spaces, and $\mathbf{SP_a}$ for the full subcategory of $\mathbf{SP}$ given by the atomistic state property systems, then the general equivalence can be reduced.

Theorem 4 (Equivalence of $\mathbf{SP_a}$ and $\mathbf{Cls_1}$) *The functors*

$$F : \mathbf{SP_a} \to \mathbf{Cls_1}$$

$$G : \mathbf{Cls_1} \to \mathbf{SP_a}$$

establish an equivalence of categories.

For a more extensive study of separation axioms and their relation with state property systems we refer to [12]. In the present text our final aim is to use the described equivalence to translate the concept of connectedness in closure spaces into terms of state property systems. It will give us a means to distinguish ‘classical’ and ‘quantum mechanical’ properties of a physical entity. First we will need a more precise concept of classical property.

2 Super Selection Rules

In this section we start to distinguish the classical aspects of the structure from the quantum aspects. We all know that the concept of superposition state is very important in quantum mechanics. The superposition states are the states that do not exist in classical physics and hence their appearance is one of the important quantum aspects. To be able to define properly a superposition state we need the linearity of the set of states. On the level of generality that we work now, we do not necessarily have this linearity, which could indicate that the concept of superposition state cannot be given a meaning on this level of generality. This is however not really true: the concept can be traced back within this general setting, by introducing the idea of 'superselection rule'. Two properties are separated by a superselection rule iff there do not exist 'superposition states' related to these two properties. This concept will be the first step towards a characterization of classical properties of a physical system.

Definition 8 (Super Selection Rule) *Consider a state property system $(\Sigma, \mathcal{L}, \xi)$. For $a, b \in \mathcal{L}$ we say that a and b are separated by a super selection rule, and denote a ssr b, iff for $p \in \Sigma$ we have:*

$$a \vee b \in \xi(p) \Rightarrow a \in \xi(p) \text{ or } b \in \xi(p) \tag{9}$$

We again use the equivalence between state property systems and closure spaces to translate the concept of 'separation by a superposition rule' into a concept for the closed sets of a closure space. Amazingly we find that properties that are 'separated by a superselection rule' (i.e. they are 'classical' properties in a certain sense) correspond to closed sets that also behave in a classical way, where classical now refers to classical topology.

Theorem 5 *Consider a state property system $(\Sigma, \mathcal{L}, \xi)$ and its corresponding closure space $\mathcal{F} = \kappa(\mathcal{L})$. For $a, b \in \mathcal{L}$ we have:*

$$a \ ssr\ b \Leftrightarrow \kappa(a \vee b) = \kappa(a) \cup \kappa(b) \Leftrightarrow \kappa(a) \cup \kappa(b) \in \mathcal{F} \tag{10}$$

Proof: Suppose that $a, b \in \mathcal{L}$ such that a *ssr* b. If $p \in \kappa(a \vee b)$, then $a \vee b \in \xi(p)$. Then it follows that $a \in \xi(p)$ or $b \in \xi(p)$. So we have $p \in \kappa(a)$ or $p \in \kappa(b)$, which shows that $p \in \kappa(a) \cup \kappa(b)$. This proves that $\kappa(a \vee b) \subseteq \kappa(a) \cup \kappa(b)$. We obviously have the other inclusion and hence $\kappa(a \vee b) = \kappa(a) \cup \kappa(b)$. It follows immediately that $\kappa(a) \cup \kappa(b) \in \mathcal{F}$. Conversely, if $\kappa(a) \cup \kappa(b) \in \mathcal{F}$, then there exists a property $c \in \mathcal{L}$ such that $\kappa(c) = \kappa(a) \cup \kappa(b)$. From $\kappa(a) \subseteq \kappa(c)$ it follows that $a < c$, and in a similar way we have $b < c$. So it follows that $a \vee b < c$. As a consequence we have $\kappa(a \vee b) \subseteq \kappa(c) = \kappa(a) \cup \kappa(b)$. Since $\kappa(a) \cup \kappa(b) \subseteq \kappa(a \vee b)$, we have $\kappa(a \vee b) = \kappa(a) \cup \kappa(b)$. Consider now an arbitrary $p \in \Sigma$ such that $a \vee b \in \xi(p)$. Then $p \in \kappa(a \vee b) = \kappa(a) \cup \kappa(b)$. As a consequence $p \in \kappa(a)$ or $p \in \kappa(b)$. This proves that $a \in \xi(p)$ or $b \in \xi(p)$ which shows that a ssr b. □

This theorem shows that the properties that are separated by a super selection rule are exactly the ones that behave also classically within the closure system. In the sense that their set theoretical unions are closed. This also means that if our closure system reduces to a topology, and hence all finite unions of closed subsets are closed, all finite sets of properties are separated by super selection rules.

Corollary 1 *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. T.F.A.E.:*

(1) Every two properties of $\mathcal{L}$ are separated by a super selection rule.

(2) The corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$ is a topological space.

A state property satisfying one, and hence both of the above conditions will be called a 'super selection classical' state property system or 's-classical' state property system. The full subcategory of **SP** *given by the s-classical state property systems will be written as* $^{\mathbf{sc}}\mathbf{SP}$.

Hence the equivalence between state property systems and closure space can be reduced to an equivalence between s-classical state property systems, in which no two properties have 'superposition states' related to them, and topological spaces.

Theorem 6 (Equivalence of $^{\mathbf{sc}}\mathbf{SP}$ and Top) *The functors*

$$F : {}^{\mathbf{sc}}\mathbf{SP} \to \mathbf{Top}$$

$$G : \mathbf{Top} \to {}^{\mathbf{sc}}\mathbf{SP}$$

establish an equivalence of categories.

3 D-classical Properties

We are ready now to introduce the concept of a 'deterministic classical property' or 'd-classical property'. To make clear what we mean by this we have to explain shortly how properties are tested. For each property $a \in \mathcal{L}$ there exists a test α, which is an experiment that can be performed on the physical entity under study, and that can give two outcomes, 'yes' and 'no'. The property a tested by the experiment α is actual iff the state p of S is such that we can predict with certainty (probability equal to 1) that the outcome 'yes' will occur for the test α. If the state p of S is such that we can predict with certainty that the outcome 'no' will occur, we test in some way a complementary property of the property a, let us denote the complementary property by a^c. Now we have three possibilities: (1) the state of S is such that α gives 'yes' with certainty; (2) the state of S is such that α gives 'no' with certainty; and (3) the state of S is such that neither the outcome 'yes' nor the outcome 'no' is certain for the experiment α. The third case represents the situations of 'quantum indeterminism'. That is the reason that a property a tested by an experiment α where the third case is absent will be called a 'deterministic classical' property or 'd-classical' property.

Definition 9 (D-classical Property) *Consider a state property system $(\Sigma, \mathcal{L}, \xi)$. We say that a property $a \in \mathcal{L}$ is a 'deterministic classical property' or 'd-classical' property, if there exists a property $a^c \in \mathcal{L}$ such that $a \vee a^c = I$, $a \wedge a^c = 0$ and a ssr a^c.*

Remark that for every state property system $(\Sigma, \mathcal{L}, \xi)$ the properties 0 and I are d-classical properties. Note also that if $a \in \mathcal{L}$ is a d-classical property, we have for $p \in \Sigma$ that $a \in \xi(p) \Leftrightarrow a^c \notin \xi(p)$ and $a \notin \xi(p) \Leftrightarrow a^c \in \xi(p)$. This follows immediately from the definition of a d-classical property.

Theorem 7 *Consider a state property system $(\Sigma, \mathcal{L}, \xi)$. If $a \in \mathcal{L}$ is a d-classical property, then a^c is unique and is a d-classical property. We will call it the complement of a. Further we have:*

$$\begin{aligned} (a^c)^c &= a \\ a < b &\Rightarrow b^c < a^c \\ \kappa(a^c) &= \kappa(a)^C \end{aligned}$$

Proof: Suppose that we have another property $b \in \mathcal{L}$ such that $a \vee b = I$, $a \wedge b = 0$ and a ssr b. Consider an arbitrary state $p \in \Sigma$ such that $a^c \in \xi(p)$. This means that $a \notin \xi(p)$. We have however $a \vee b \in \xi(p)$, which implies, since a ssr b, that $a \in \xi(p)$ or $b \in \xi(p)$. As a consequence we have $b \in \xi(p)$. This means that we have proven that $a^c < b$. In a completely analogous way we can show that also $b < a^c$, which shows that a^c is unique. Obviously a^c is a d-classical property. Then the idempotency follows from the fact that a is the complement of a^c and from the uniqueness of the complement. Consider $a < b$ and an arbitrary state $p \in \Sigma$ such that $b^c \in \xi(p)$. This means that $b \notin \xi(p)$, which implies that $a \notin \xi(p)$. As a consequence we have $a^c \in \xi(p)$. So we have shown that $b^c < a^c$. Further we have $p \in \kappa(a^c)$ iff $a^c \in \xi(p)$. From the above mentioned remark this is equivalent with $a \notin \xi(p)$. and $p \notin \kappa(a)$ which is the same as saying that $p \in \kappa(a)^C$. So we have $\kappa(a^c) = \kappa(a)^C$. □

Definition 10 (Connected Closure Space) *A closure space $(X, \mathcal{F})$ is called connected if the only clopen (i.e. closed and open) sets are $\emptyset$ and X.*

We shall see now that these subsets that make closure systems disconnected are exactly the subsets corresponding to d-classical properties.

Theorem 8 *Consider a state property system $(\Sigma, \mathcal{L}, \xi)$ and its corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$. For $a \in \mathcal{L}$ we have:*

$$a \text{ is d} - \text{classical } \Leftrightarrow \kappa(a) \text{ is clopen} \quad (11)$$

Proof: ¿From the previous propositions it follows that if a is d-classical, then $\kappa(a)$ is clopen. So now consider a clopen subset $\kappa(a)$ of Σ. This means that $\kappa(a)^C$ is closed, and hence that there exists a property $b \in \mathcal{L}$ such that $\kappa(b) = \kappa(a)^C$. We clearly have $a \wedge b = 0$ since there exists no state $p \in \Sigma$ such that $p \in \kappa(a)$ and $p \in \kappa(b)$. Since $\Sigma = \kappa(a) \cup \kappa(b)$ we have $a \vee b = I$. Further we have that for an arbitrary state $p \in \Sigma$ we have $a \in \xi(p)$ or $b \in \xi(p)$ which shows that a ssr b. This proves that $b = a^c$ and that a is d-classical. □

This means that the d-classical properties correspond exactly to the clopen subsets of the closure system.

Corollary 2 *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. T.F.A.E.*
(1) The properties 0 and I are the only d-classical ones.
(2) $F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}))$ is a connected closure space.

We now introduce 'completely quantum mechanical' or pure nonclassical state property systems, in the sense that there are no (non-trivial) d-classical properties.

Definition 11 (Pure Nonclassical State Property System) *A state property system $(\Sigma, \mathcal{L}, \xi)$ is called a pure nonclassical state property system if the properties 0 and I are the only d-classical properties.*

Theorem 9 *Let $(\Sigma, \mathcal{F})$ be a closure space. T.F.A.E.*
(1) $(\Sigma, \mathcal{F})$ is a connected closure space.
(2) $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \bar{\xi})$ is a pure nonclassical state property system.

Proof: Let $(\Sigma, \mathcal{F})$ be a connected state property system. Then $\emptyset$ and Σ are the only clopen sets in $(\Sigma, \mathcal{F})$. Since the Cartan map associated to ξ is given by $\kappa : \mathcal{F} \to \mathcal{P}(\Sigma) : F \mapsto F$, we have $\kappa(\emptyset) = \emptyset$ and $\kappa(\Sigma) = \Sigma$. Applying proposition 8, we find that $\emptyset$ and Σ are the only d-classical properties of $\mathcal{F}$.

Conversely, let $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \bar{\xi})$ be a pure nonclassical state property system. Then by corollary 2, $(\Sigma, \mathcal{F}) = FG(\Sigma, \mathcal{F})$ is a connected closure space. $\square$

If we define $\mathbf{SP_Q}$ as the full subcategory of $\mathbf{SP}$ where the objects are the pure nonclassical state property systems and we define $\mathbf{Cls_C}$ as the full subcategory of $\mathbf{Cls}$ where the objects are the connected closure spaces, then the previous propositions and theorem 3 imply an equivalence of the categories $\mathbf{SP_Q}$ and $\mathbf{Cls_C}$.

Theorem 10 (Equivalence of $\mathbf{SP_Q}$ and $\mathbf{Cls_C}$) *The functors*

$$F : \mathbf{SP_Q} \to \mathbf{Cls_C}$$

$$G : \mathbf{Cls_C} \to \mathbf{SP_Q}$$

establish an equivalence of categories.

Again we have found using the equivalence 3 that a physical concept (i.e. nonclassicality) translates to a known topological property (i.e. connectedness). In the next section we will use topological methods to construct a decomposition of a state property system into pure nonclassical components.

4 Decomposition Theorem

As for topological spaces, every closure space can be decomposed uniquely into connected components. In the following we say that, for a closure space $(X, \mathcal{F})$, a subset $A \subseteq X$ is connected if the induced subspace is connected. It can be shown that the union of any family of connected subsets having at least one point in common is also connected. So the component of an element $x \in X$ defined by

$$K_{\mathbf{Cls}}(x) = \bigcup \{A \subseteq X \mid x \in A, A \text{ connected } \} \tag{12}$$

is connected and therefore called the connection component of x. Moreover, it is a maximal connected set in X in the sense that there is no connected subset of X which properly contains $K_{\mathbf{Cls}}(x)$. From this it follows that for closure spaces $(X, \mathcal{F})$ the set of all distinct connection components in X form a partition of X. So we can consider the following equivalence relation on X : for $x, y \in X$ we say that x is equivalent with y iff the connection components $K_{\mathbf{Cls}}(x)$ and $K_{\mathbf{Cls}}(y)$ are equal. Further we remark that the connection components are closed sets.

In the following we will try to decompose state property systems similarly into different components.

Theorem 11 *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system and let $(\Sigma, \kappa(\mathcal{L}))$ be the corresponding closure space. Consider the following equivalence relation on Σ :*

$$p \sim q \Leftrightarrow K_{\mathbf{Cls}}(p) = K_{\mathbf{Cls}}(q) \tag{13}$$

with equivalence classes $\Omega = \{\omega(p) | p \in \Sigma\}$. If $\omega \in \Omega$ we define the following :

$$\begin{aligned} \Sigma_\omega &= \omega = \{p \in \Sigma \mid \omega(p) = \omega\} \\ s(\omega) &= s(\omega(p)) = a, \text{ such that } \kappa(a) = \omega(p) \\ \mathcal{L}_\omega &= [0, s(\omega)] = \{a \in \mathcal{L} \mid 0 \leq a \leq s(\omega)\} \subset \mathcal{L} \\ \xi_\omega &: \Sigma_\omega \to \mathcal{P}(\mathcal{L}_\omega) : p \mapsto \xi(p) \cap \mathcal{L}_\omega \end{aligned}$$

then $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ is a state property system.

Proof: Since $\mathcal{L}_\omega$ is a sublattice (segment) of $\mathcal{L}$, it is a complete lattice with maximal element $I_\omega = s(\omega)$ and minimal element $0_\omega = 0$. Let $p \in \Sigma_\omega$. Then $0 \notin \xi(p)$. So $0 \notin \xi(p) \cap \mathcal{L}_\omega = \xi_\omega(p)$. If $a_i \in \xi_\omega(p)$, $\forall i$, then $a_i \in \mathcal{L}_\omega$ and $a_i \in \xi(p)$, $\forall i$. Hence $\wedge a_i \in \mathcal{L}_\omega \cap \xi(p) = \xi_\omega(p)$. Finally, let $a, b \in \mathcal{L}_\omega$ with $a <_\omega b$ and let $r \in \Sigma_\omega$. If $a \in \xi_\omega(r)$, then $a \in \mathcal{L}_\omega$ and $a \in \xi(r)$, thus $b \in \mathcal{L}_\omega$ and $b \in \xi(r)$. So $b \in \xi_\omega(r)$. Conversely, if $a, b \in \mathcal{L}_\omega$ and $\forall r \in \Sigma_\omega : a \in \xi_\omega(r) \Rightarrow b \in \xi_\omega(r)$ then we consider a q such that $a \in \xi(q)$ (q must be in Σ_ω by definition of $\mathcal{L}_\omega$). Then $a \in \xi_\omega(q)$ implies that $b \in \xi_\omega(q)$. So $b \in \xi(q)$ and $a < b$. Thus $a <_\omega b$. □

Moreover we can show that the above introduced state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ have no proper d-classical properties, and hence are pure nonclassical state property systems.

Theorem 12 *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. If $\omega \in \Omega$, then $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ is a pure nonclassical state property system.*

Proof: If a is classical element of $\mathcal{L}_\omega$, then $\kappa(a)$ must be a clopen set of the associated closure space $(\Sigma_\omega, \kappa(\mathcal{L}_\omega))$ which is a connected subspace of $(\Sigma, \kappa(\mathcal{L}))$. Hence there are no proper classical elements of $\mathcal{L}_\omega$. □

Theorem 13 *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. If we introduce the following :*

$$\begin{aligned} \Omega &= \{\omega(p) \mid p \in \Sigma\} \\ \mathcal{C} &= \{\vee s(\omega_i) \mid \omega_i \in \Omega\} \\ \eta &: \Omega \to \mathcal{P}(\mathcal{C}) : \omega = \omega(p) \mapsto \xi(p) \cap \mathcal{C} \end{aligned}$$

then $(\Omega, \mathcal{C}, \xi)$ is an atomistic state property system.

Proof: First we remark that η is well defined because if $\omega(p) = \omega(q)$, then $\xi(p) \cap \mathcal{C} = \xi(q) \cap \mathcal{C}$. Indeed, if $\vee s(\omega_i) \in \xi(p)$ then $p \in \kappa(\vee s(\omega_i)) = cl(\cup \omega_i)$ in the corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$. Since $cl(\cup \omega_i)$ is not connected we have that $K_{\mathbf{Cls}}(p) = \omega(p) = \omega(q) \subset cl(\cup \omega_i)$ so $q \in cl(\cup \omega_i) = \kappa(\vee s(\omega_i))$ and $\vee s(\omega_i) \in \xi(q)$. Now, since $\mathcal{C}$ is a sublattice of $\mathcal{L}$ it is a complete lattice with $1_\mathcal{C} = 1$ and $0_\mathcal{C} = 0$. By definition $\mathcal{C}$ is generated by its atoms $\{s(\omega) \mid \omega \in \Omega\}$. Clearly $0 \notin \eta(\omega(p))$ because $0 \notin \xi(p)$. If $a_i \in \eta(\omega(p)) = \xi(p) \cap \mathcal{C}$, $\forall i$, then $\wedge a_i \in \xi(p) \cap \mathcal{C} = \eta(\omega(p))$. Finally, let $a, b \in \mathcal{C}$ with $a <_\mathcal{C} b$. Let $\omega(p) \in \Omega$ with $a \in \eta(\omega(p))$. Thus $a \in \xi(p)$. $a <_\mathcal{C} b$ implies $a < b$. So we have $b \in \xi(p) \cap \mathcal{C} = \eta(\omega(p))$. Conversely, let $a, b \in \mathcal{C}$ and assume that $\forall p \in \Sigma : a \in \eta(\omega(p)) \Rightarrow b \in \eta(\omega(p))$. Then we have $\forall p \in \Sigma : a \in \xi(p) \Rightarrow b \in \xi(p)$. Thus $a < b$ and $a <_\mathcal{C} b$. In order to prove that $(\Omega, \mathcal{C}, \eta)$ is an atomistic state property, we show that η is injective. So consider $p, q \in \Sigma$ such that $\omega(p) \neq \omega(q)$. Since $p \in \omega(p) = \kappa(s(\omega(p)))$, we have $s(\omega(p)) \in \xi(p) \cap \mathcal{C}$ and since $q \notin \omega(p) = \kappa(s(\omega(p)))$ we have $s(\omega(p)) \notin \xi(q) \cap \mathcal{C}$. This implies that $\xi(p) \cap \mathcal{C} \neq \xi(q) \cap \mathcal{C}$, i.e. $\eta(\omega(p)) \neq \eta(\omega(q))$. Thus η is injective and $(\Omega, \mathcal{C}, \eta)$ is an atomistic state property system. □

Theorem 14 *$(\Omega, \mathcal{C}, \eta)$ is a totally classical state property system, in the sense that the only pure nonclassical segments (i.e. segments with no proper classical elements) are trivial, i.e. $\{0, s(\omega)\}$.*

Proof: Suppose $[0, a]$ is a pure nonclassical segment of $\mathcal{C}$, then in the corresponding closure space $(\Sigma, \kappa(\mathcal{L}))$ the subset $\kappa(a)$ is connected hence $\kappa(a) \subset \omega$ for some $\omega \in \Omega$, hence $a < s(\omega)$. Since $s(\omega)$ is an atom, $a = s(\omega)$. Thus $[0, a] = \{0, s(\omega)\}$. □

Corollary 3 *The closure space associated with $(\Omega, \mathcal{C}, \eta)$ is a totally disconnected closure space.*

Summarizing the previous results we get:

Theorem 15 (decomposition theorem) *Any state property system* $(\Sigma, \mathcal{L}, \xi)$ *can be decomposed into:*

- *a number of pure nonclassical state property systems* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega), \omega \in \Omega$
- *and a totally classical state property system* $(\Omega, \mathcal{C}, \eta)$

Thus the decomposition of a closure space into its maximal connected components yields a way to decompose a state property system $(\Sigma, \mathcal{L}, \xi)$ into pure nonclassical state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega), \omega \in \Omega$. In the context of closure spaces the maximal connected components are subspaces of the given space. However we do not yet have that the pure nonclassical state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ are subsystems of $(\Sigma, \mathcal{L}, \xi)$. To show this we introduce a new concept of subsystem.

5 Closed Subspaces and *ap*-Subsystems

Definition 12 (AP-subsystem) *Let* $(\Sigma, \mathcal{L}, \xi)$ *be a state property system and let* $a \in \mathcal{L}$. *Consider the following:*

- $\Sigma' = \kappa(a)$
- $\mathcal{L}' = [0, a]$
- $\xi' = \xi_{|\Sigma'}$

We now have a new state property system $(\Sigma', \mathcal{L}', \xi')$ *which we shall call an 'actual property' (ap-) subsystem of* $(\Sigma, \mathcal{L}, \xi)$ *generated by a .*

The name 'actual property' subsystem comes from the physical interpretation of this construction: give a property a of the physical system, we consider only those states Σ' for which a is always actual.

Theorem 16 *Let* $(\Sigma', \mathcal{L}', \xi')$ *be an ap-subsystem of* $(\Sigma, \mathcal{L}, \xi)$*, generated by a. Consider the corresponding closure spaces* $(\Sigma', \kappa(\mathcal{L}'))$ *and* $(\Sigma, \kappa(\mathcal{L}))$*, we have that* $(\Sigma', \kappa(\mathcal{L}'))$ *is a closed subspace of* $(\Sigma, \kappa(\mathcal{L}))$.

Proof: Follows immediately from the definition. □

Theorem 17 *Consider a closed subspace* $(\Sigma', \mathcal{F}')$ *of the closure space* $(\Sigma, \mathcal{F})$*, we have that* $(\Sigma', \mathcal{F}', \bar{\xi}')$ *is an ap-subsystem of* $(\Sigma, \mathcal{F}, \bar{\xi})$ *generated by* Σ'.

Proof: Follows immediately from the definition. □

¿From the above two theorems we see that *ap*-subsystems correspond exactly to closed subspaces of the associated closure space.

Any closed subspace Σ' of a closure space $(\Sigma, \mathcal{F})$ induces in a natural way a canonical inclusion map:

$$i : (\Sigma', \mathcal{F}') \to (\Sigma, \mathcal{F})$$

which in turn, by the functional equivalence between the category of closure spaces and state property systems gives a morphism:

$$(i, i^{-1}) : (\Sigma', \mathcal{F}', \bar{\xi}') \to (\Sigma, \mathcal{F}, \xi)$$

Theorem 18 *Let* $(\Sigma', \mathcal{L}', \xi')$ *be an ap-subsystem of* $(\Sigma, \mathcal{L}, \xi)$*, generated by* a*. We now define the following maps:*

$$m : \Sigma' \to \Sigma : p \mapsto p$$

$$n : \mathcal{L} \to \mathcal{L}' : c \mapsto a \wedge c$$

then $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ *is a morphism in the category of state property systems which reduces to the canonical inclusion between the underlying closure spaces.*

Proof: We have to show that for $c \in \mathcal{L}$ and $p' \in \Sigma' : c \in \xi(m(p')) \Leftrightarrow n(c) \in \xi'(p')$. Let's start with $c \in \xi(m(p')) \Leftrightarrow c \in \xi(p') \Leftrightarrow c \in \xi'(p')$. Because $\kappa(a) = \Sigma'$ we know that $a \in \xi'(p') = \xi(p')$, therefore $n(c) = c \wedge a \in \xi'(p')$. Conversely, if $n(c) = c \wedge a \in \xi'(p')$ then $p' \in \kappa'(c \wedge a) = \kappa'(c) \cap \kappa'(a) = \kappa'(c) \cap \Sigma' = \kappa'(c)$ therefore $c \in \xi'(p)$. □

We shall apply these results to the pure nonclassical state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega), \omega \in \Omega$ that we have introduced in the previous section. Recall that we started with a state property system $(\Sigma, \mathcal{L}, \xi)$ with associated closure space $(\Sigma, \kappa(\mathcal{L}))$. By means of the connection relation on $(\Sigma, \kappa(\mathcal{L}))$ we obtained a partition $\Omega = \{\omega(p) = K_{\mathbf{Cls}}(p) | p \in \Sigma\}$ of Σ. Moreover each $w \in \Omega$ with $\omega = \omega(p) = K_{\mathbf{Cls}}(p)$ was a closed subset of $(\Sigma, \kappa(\mathcal{L}))$. Hence there was an $a = s(\omega)$ such that $\kappa(a) = \omega$. We will now use this property $a = s(\omega)$ to create an *ap*-subsystem.

$$\Sigma' = \kappa(a) = \omega$$

$$\mathcal{L}' = [0, a] = [0, s(\omega)]$$

$$\xi' = \xi^{|\mathcal{L}'}_{|\Sigma'} : p' \mapsto \xi(p) \cap \mathcal{L}'$$

We easily see that for an $\omega \in \Omega$ this *ap*-subsystem is in fact $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. Let

$$m : \Sigma_\omega \to \Sigma : p \mapsto p$$

$$n : \mathcal{L} \to \mathcal{L}_\omega : c \mapsto s(\omega) \wedge c$$

then $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ is a morphism in the category of state property systems which reduces to the canonical inclusion between the underlying closure spaces. In this way $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega), \omega \in \Omega$ is always an *ap*-subsystem of $(\Sigma, \mathcal{L}, \xi)$.

6 The D-classical Part of a State Property System

In this section we want to show how it is possible to extract the d-classical part of a state property system. First of all we have to define the d-classical property lattice related to the entity S that is described by the state property system $(\Sigma, \mathcal{L}, \xi)$.

Definition 13 (D-classical Property Lattice) *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$*. We call* $\mathcal{C}' = \{\wedge_i a_i | a_i \text{ is a d} - \text{classical property}\}$ *the d-classical property lattice corresponding to the state property system* $(\Sigma, \mathcal{L}, \xi)$*.*

Theorem 19 $\mathcal{C}'$ *is a complete lattice with the partial order relation and infimum inherited from* $\mathcal{L}$ *and the supremum defined as follows: for* $a_i \in \mathcal{C}'$*,* $\vee_i a_i = \wedge_{b \in \mathcal{C}', a_i \leq b \ \forall i}\ b$*.*

Remark that the supremum in the lattice $\mathcal{C}'$ is not the one inherited from $\mathcal{L}$.

Theorem 20 *Consider a state property system* $(\Sigma, \mathcal{L}, \xi)$. *Let* $\xi'(q) = \xi(q) \cap \mathcal{C}'$ *for* $q \in \Sigma$, *then* $(\Sigma, \mathcal{C}', \xi')$ *is a state property system which we shall refer to as the d-classical part of* $(\Sigma, \mathcal{L}, \xi)$.

Proof: Clearly $0 \notin \xi'(p)$ for $p \in \Sigma$. Consider $a_i \in \xi'(p)\ \forall i$. Then $a_i \in \xi(p) \cap \mathcal{C}'\ \forall i$, from which follows that $\wedge_i a_i \in \xi(p) \cap \mathcal{C}'$ and hence $\wedge_i a_i \in \xi'(p)$. Consider $a, b \in \mathcal{C}'$. Let us suppose that $a \leq b$ and consider $r \in \Sigma$ such that $a \in \xi'(r)$. This means that $a \in \xi(r) \cap \mathcal{C}'$. From this follows that $b \in \xi(r) \cap \mathcal{C}'$ and hence $b \in \xi'(r)$. On the other hand let us suppose that $\forall r \in \Sigma : a \in \xi'(r)$ then $b \in \xi'(r)$. Since $a, b \in \mathcal{C}'$, this also means that $\forall r \in \Sigma : a \in \xi(r)$ then $b \in \xi(r)$. From this follows that $a \leq b$. $\square$

Since $(\Sigma, \mathcal{C}', \xi')$ is a state property system, it has a corresponding closure space $(\Sigma, \kappa(\mathcal{C}'))$. In order to check some property of this space we introduce the following concepts.

Definition 14 (Weakly Zero-dimensional Closure Space) *Let* $(X, \mathcal{F})$ *be a closure space and* $\mathcal{B} \subset \mathcal{F}$. $\mathcal{B}$ *is called a base of* $(X, \mathcal{F})$ *iff* $\forall F \in \mathcal{F} : \exists B_i \in \mathcal{B} : F = \cap B_i$. $(X, \mathcal{F})$ *is called weakly zero-dimensional iff there is a base consisting of clopen sets.*

Theorem 21 *The closure space* $(\Sigma, \kappa(\mathcal{C}'))$ *corresponding to the state property system* $(\Sigma, \mathcal{C}', \xi')$ *is weakly zero-dimensional.*

Proof: To see this recall that a is classical iff $\kappa(a)$ is clopen in $(\Sigma, \kappa(\mathcal{L}))$, hence $\kappa(\mathcal{C}')$ is a family of closed sets on Σ which consists of all intersections of the clopen sets of $(\Sigma, \kappa(\mathcal{L}))$. $\square$

In general $(\Sigma, \mathcal{C}', \xi')$ does not need to be atomistic, hence it is different from the totally classical state property system $(\Omega, \mathcal{C}, \eta)$ associated with $(\Sigma, \mathcal{L}, \xi)$. To illustrate this we give an example.
Let's consider the following state property system.

$$\begin{aligned} \Sigma &= \{p, q, r, s, t\} \\ \mathcal{L} &= \{0, a, b, c, d, I\} \\ \xi &: \Sigma \to \mathcal{P}(\mathcal{L}) \end{aligned}$$

with $\xi(p) = \xi(q) = \{b, d, I\}, \xi(r) = \{a, d, I\}$ and $\xi(s) = \xi(t) = \{c, I\}$. The structure for the lattice $\mathcal{L}$ is given by figure 1.

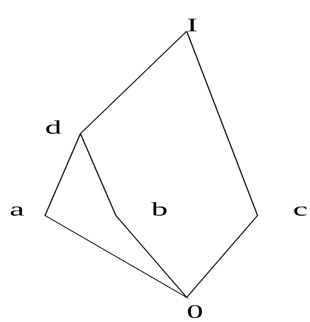

Figure 1: The lattice $\mathcal{L}$

The corresponding closure space (see figure 2) is

$$\begin{aligned} \Sigma &= \{p, q, r, s, t\} \\ \kappa(\mathcal{L}) &= \{\emptyset, \{r\}, \{p, q\}, \{s, t\}, \{p, q, r\}, \Sigma\} \end{aligned}$$

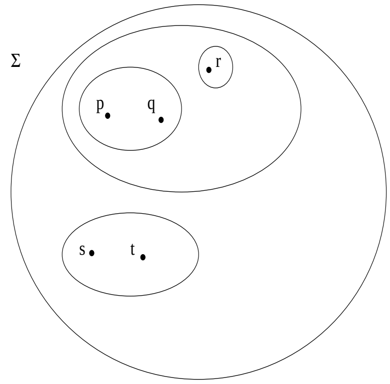

Figure 2: The closure space $\Sigma, \kappa(\mathcal{L})$

Determining the connectedness components in this closure space, we find the following:

$$K_{\mathbf{Cls}}(p) = K_{\mathbf{Cls}}(q) = \{p, q\}$$

$$K_{\mathbf{Cls}}(r) = \{r\}$$

$$K_{\mathbf{Cls}}(s) = K_{\mathbf{Cls}}(t) = \{s, t\}$$

We have three pure nonclassical state property systems: $(\Sigma_{\omega_1}, \mathcal{L}_{\omega_1}, \xi_{\omega_1})$, $(\Sigma_{\omega_2}, \mathcal{L}_{\omega_2}, \xi_{\omega_2})$ and $(\Sigma_{\omega_3}, \mathcal{L}_{\omega_3}, \xi_{\omega_3})$.

$$\begin{array}{ll} \Sigma_{\omega_1} = \{p, q\}, & \mathcal{L}_{\omega_1} = [0, b] \\ \Sigma_{\omega_2} = \{r\}, & \mathcal{L}_{\omega_2} = [0, a] \\ \Sigma_{\omega_3} = \{s, t\}, & \mathcal{L}_{\omega_3} = [0, c] \end{array}$$

$$\begin{array}{l} \xi_{\omega_1}(p) = \xi_{\omega_1}(q) = \{b\} \\ \xi_{\omega_2}(r) = \{a\} \\ \xi_{\omega_3}(s) = \xi_{\omega_3}(t) = \{c\} \end{array}$$

The atomistic totally classical state property system $(\Omega, \mathcal{C}, \eta)$ is given by:

$$\begin{array}{rcl} \Omega & = & \{\{p, q\}, \{r\}, \{s, t\}\} \\ \mathcal{C} & = & \mathcal{L} \\ \eta & : & \Omega \to \mathcal{P}(\mathcal{C}) \end{array}$$

where $\eta(\{p, q\}) = \{b, d, 1\}$, $\eta(\{r\}) = \{a, d, 1\}$ and $\eta(\{s, t\}) = \{c, 1\}$. The classical part is given by $(\Sigma, \mathcal{C}', \xi')$ where

$$\xi'(p) = \xi(p) \cap \mathcal{C}' \text{ for } p \in \Sigma$$

$$\mathcal{C}' = \{0, c, d, I\}$$

Acknowledgments

Part of the research for this article took place in the framework of the bilateral Flemish-Polish project 127/E-335/S/2000. D. Deses is Research Assistent at the FWO Belgium.

References

[1] D. Aerts, E. Colebunders, A. Van der Voorde and B. Van Steirteghem, State property systems and closure spaces: a study of categorical equivalence, *International Journal of Theoretical Physics*, **38**, 359–385, 1999. Preprint at `http://arXiv.org/abs/quant-ph/0105108`

[2] D. Aerts, Foundations of quantum physics: a general realistic and operational approach, *International Journal of Theoretical Physics*, **38**, 289–358, 1999. Preprint at `http://arXiv.org/abs/quant-ph/0105109`

[3] D. Aerts, Description of many physical entities without the paradoxes encountered in quantum mechanics, *Foundations of Physics*, **12**, 1131–1170, 1982.

[4] D. Aerts, Classical theories and non classical theories as special case of a more general theory, *Journal of Mathematical Physics*, **24**, 2441–2453, 1983.

[5] D. Aerts, Quantum structures, separated physical entities and probability, *Foundations of Physics*, **24** 1227–1259, 1994.

[6] C. Piron, *Foundations of Quantum Physics*, W. A. Benjamin, Reading, Mass., 1976.

[7] C. Piron, Recent developments in quantum mechanics, *Helvetica Physica Acta*, **62**, 82, 1989.

[8] C. Piron, *Mécanique Quantique: Bases et Applications*, Presses Polytechniques et Universitaires Romandes, Lausanne, 1990.

[9] A. Van der Voorde, A categorical approach to T_1 separation and the product of state property systems, *International Journal of Theoretical Physics*, **39**, 943, 2000.

[10] D. Aerts, D. Deses and A. Van der Voorde, Connectedness applied to closure spaces and state property systems, *Journal of Electrical Engineering*, **52**, 18–21, 2001. Preprint at `http://arXiv.org/abs/quant-ph/0205163`

[11] D. Aerts, D. Deses and A. Van der Voorde, Classicality and connectedness for state property systems and closure spaces, *International Journal of Theoretical Physics*, in press.

[12] A. Van der Voorde, *Separation axioms in extension theory for closure spaces and their relevance to state property systems*, PhD. Thesis, Vrije Universiteit Brussel, 2001.