

The Linearity of Quantum Mechanics at Stake: The Description of Separated Quantum Entities*

Diederik Aerts and Frank Valckenborgh

Center Leo Apostel (CLEA) and
Foundations of the Exact Sciences (FUND),
Brussels Free University, Krijgskundestraat 33,
1160 Brussels, Belgium.
diraerts@vub.ac.be, fvalcken@vub.ac.be

Abstract

We consider the situation of a physical entity that is the compound entity consisting of two ‘separated’ quantum entities. In earlier work it has been proved by one of the authors that such a physical entity cannot be described by standard quantum mechanics. More precisely, it was shown that two of the axioms of traditional quantum axiomatics are at the origin of the impossibility for standard quantum mechanics to describe this type of compound entity. One of these axioms is equivalent with the superposition principle, which means that separated quantum entities put the linearity of quantum mechanics at stake. We analyze the conceptual steps that are involved in this proof, and expose the necessary material of quantum axiomatics to be able to understand the argument.

1 Introduction

It is often stated that quantum mechanics is basically a linear theory. Let us reflect somewhat about what one usually means when expressing this statement.

The Schrödinger equation that describes the change of the state of a quantum entity under the influence of the external world is a linear equation.

*Published as: D. Aerts and F. Valckenborgh, “The linearity of quantum mechanics at stake: the description of separated quantum entities”, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore (2002).

This means that if the wave function $\psi_1(x)$ and the wave function $\psi_2(x)$ are both solutions of the Schrödinger equation, then, for $\lambda_1, \lambda_2 \in \mathbb{C}$, the wave function $\lambda_1\psi_1(x) + \lambda_2\psi_2(x)$ is also a solution. Hence the set of solutions of the Schrödinger equation forms a vector space over the field of complex numbers: solutions can be added and multiplied by a complex number and the results remain solutions. This is the way how the linearity of the evolution equation is linked to the linearity, or vector space structure, of the set of states.

There is another type of ‘change of state’ in quantum mechanics, namely the one provoked by a measurement or an experiment. This type of change, often called the ‘collapse of the wave function’, is nonlinear. It is described by the action of a projection operator associated with the self-adjoint operator that represents the considered measurement – and hence for this part it is linear, because a projection operator is a linear function – followed by a renormalization of the state. The two effects, projection and renormalization, one after the other, give rise to a nonlinear transformation.

The fundamental nature of the linearity of the vector space used to represent the states of a quantum mechanical entity is expressed by adopting the ‘superposition principle’ as one of the basic principles of quantum mechanics. Because linearity in general appears very often as an idealized case of the real situation, some suspicion towards the fundamental linear nature of quantum mechanics is at its place. Would there not be a more general theory that as a first order linear approximation gives rise to quantum mechanics?

In relation with this question it is good to know that the situation in quantum mechanics is very different from the situation in classical physics. Nonlinear situations in classical mechanics exist at all places, and it can easily be understood how the linearized version of the theory is an idealization of the general situations (e.g. the linearization used to study small movements around an equilibrium position). Quantum mechanics on the contrary was immediately formulated as a linear theory, and no nonlinear version of quantum mechanics has ever been proposed in a general way.

The fundamentally different way in which linearity presents itself in quantum mechanics as compared to classical mechanics makes that quite a few physicists believe that quantum linearity is a profound property of the world. The way in which classical mechanics works as a theory for the macroworld with at a ‘more basic level’ quantum mechanics as a description for the microworld, and additionally the hypothesis that this macroworld is built from building blocks that are quantum, makes that some physicists propose that the nonlinearity of macrophenomena should emerge from an underlying linearity of the microworld. This type of reflections is however

very speculative. Mostly because nobody has been able to solve in a satisfactory way the problem of the classical limit, and explain how the microworld described by quantum mechanics gives rise to a macroworld described by classical mechanics.

Because of the profound and unsolved nature of this problem it is worth to analyze a result that has been obtained by one of the authors in the eighties. The result is the following:

If we consider the physical entity that consists of two ‘separated’ quantum entities, then this physical entity cannot be described by standard quantum mechanics [1, 2].

The aspect of this result that we want to focus on in this article, is that the origin of the impossibility for standard quantum mechanics to describe the entity consisting of two separated quantum entities is the linearity of the vector space representing the states of a quantum entity.

We analyze the conceptual steps to arrive at this result in the present paper without giving proofs. For the proofs we refer to [1, 2].

2 Quantum Axiomatics

As we mentioned in the introduction, there is no straightforward way to conceive of a more general, possibly nonlinear, quantum mechanics if one starts conceptually from the standard quantum mechanical formalism. The reason is that standard quantum mechanics is elaborated completely around the vector space structure of the set of states of a quantum entity and the linear operator algebra on this vector space. If one tries to drop linearity starting from this structure one is left with nothing that remains mathematically relevant to work with.

We also mentioned that there is one transformation in standard quantum mechanics that is nonlinear, namely the transformation of a state under influence of a measurement. The nonlinearity here comes from the fact that also a renormalization procedure is involved, because states of a quantum entity are not represented by vectors, but by normalized vectors of the vector space. This fact gives us a first hint of where to look for possible ways to generalize quantum mechanics and free it from its very strict vector space strait jacket. This is also the way things have happened historically. Physicists and mathematicians noticed that the requirement of normalization and renormalization after projection means that quantum states ‘live’

in the projective geometry corresponding with the vector space. The standard quantum mechanical representation theory of groups makes full use of this insight: group representations are projective representations and not vector space representations, and experimental results confirm completely that it is the projective representations that are at work in the reality of the microworld and not the vector space representations.

Of course, there is a deep mathematical connection between a projective geometry and a vector space, through what is called the ‘fundamental representation theorem of projective geometry’ [3]. This theorem states that every projective geometry of dimension greater than two can be represented in a vector space over a division ring, where a ray of the vector space corresponds to a point of the projective geometry, and a plane through two different rays corresponds to a projective line. This means that a projective geometry entails the type of linearity that is encountered in quantum mechanics. Conceptually however a projective geometric structure is quite different from a vector space structure. The aspects of a projective geometry that give rise to linearity can perhaps more easily be generalized than this is the case for the aspects of a vector space related to linearity.

John von Neumann gave the first abstract mathematical formulation of quantum mechanics [4], and proposed an abstract complex Hilbert space as the basic mathematical structure to work with for quantum mechanics. If we refer to standard quantum mechanics we mean quantum mechanics as formulated in the seminal book of von Neumann. Some years later he wrote an important paper, together with Garrett Birkhoff, that initiated the research on quantum axiomatics [5]. In this paper Birkhoff and von Neumann propose to concentrate on the set of closed subspaces of the complex Hilbert space as the basic mathematical structure for the development of a quantum axiomatics. In later years George Mackey wrote an influential book on the mathematical foundations of quantum mechanics where he states explicitly that a physical foundation for the complex Hilbert structure should be looked for [6]. A breakthrough came with the work of Constantin Piron when he proved a fundamental representation theorem [7]. It had been noticed meanwhile that the set of closed subspaces of a complex Hilbert space forms a complete, atomistic, orthocomplemented lattice and Piron proved the converse, namely that a complete, atomistic orthocomplemented lattice, satisfying some extra conditions, could always be represented as the set of closed subspaces of a generalized Hilbert space [7, 8]. In his proof Piron first derives a projective geometry and then makes the step to the vector space. Piron’s representation theorem is exposed in detail in theorem 2 of the present article.

As we will see, it is exactly the extra conditions, needed to represent the lattice as the lattice of closed subspaces of a generalized Hilbert space, that are not satisfied for the description of the compound entity that consists of two separated quantum entities. Since the aim of this article is to put forward the conceptual steps that are involved in the failure of standard quantum mechanics to describe such an entity, we will start by explaining the general aspects of quantum axiomatics in some detail, omitting all proofs, for the sake of readability. For the reader who is interested in a more detailed exposition, references to the literature are given.

2.1 What Is a Complete Lattice?

A lattice \mathcal{L} is a set that is equipped with a partial order relation $<$. This means a relation such that for $a, b, c \in \mathcal{L}$ we have:

$$a < a \tag{1}$$

$$a < b \quad \text{and} \quad b < a \Rightarrow a = b \tag{2}$$

$$a < b \quad \text{and} \quad b < c \Rightarrow a < c \tag{3}$$

(1) is called reflexivity, (2) is called antisymmetry and (3) is called transitivity of the relation $<$. Hence a partial order relation is a reflexive, anti-symmetric and transitive relation.

Let us give two examples of partial order relations. First, consider a set Ω and let $\mathcal{P}(\Omega)$ be the set of all subsets of Ω . The inclusion operation \subset on $\mathcal{P}(\Omega)$ is a partial order on $\mathcal{P}(\Omega)$. Second, consider a complex Hilbert space \mathcal{H} , and let $\mathcal{P}(\mathcal{H})$ be the set of all closed subspaces of \mathcal{H} . The inclusion operation \subset on $\mathcal{P}(\mathcal{H})$ is a partial order on $\mathcal{P}(\mathcal{H})$. The two examples that we propose here are the archetypical examples for quantum axiomatics. The first example represents (part of) the mathematical structure related to a classical physical entity, where Ω corresponds with its state space, and the second example the one of a quantum physical entity, where \mathcal{H} is the complex Hilbert space representing the states of the quantum entity.

A lattice is a mathematical object that has some more structure than just this partial order relation. Suppose we consider two elements $a, b \in \mathcal{L}$ of the lattice, then we demand that there exists an infimum and a supremum for a and b for the partial order relation $<$ in \mathcal{L} . The infimum and supremum are denoted respectively $a \wedge b$ and $a \vee b$. An infimum of a and b is a greatest lower bound, this means a maximum of all the elements of \mathcal{L} that are smaller than a and smaller than b . A supremum is a least upper bound, this means a minimum of all the elements of \mathcal{L} that are greater than a and greater than

b. Let us repeat this in a formal way. For $a, b \in \mathcal{L}$ there exist $a \wedge b, a \vee b \in \mathcal{L}$ such that for $x, y \in \mathcal{L}$ we have:

$$x < a \text{ and } x < b \Leftrightarrow x < a \wedge b \quad (4)$$

$$a < y \text{ and } b < y \Leftrightarrow a \vee b < y \quad (5)$$

A structure $(\mathcal{L}, <, \wedge, \vee)$, where \mathcal{L} is a set, $<$ is a partial order relation satisfying (1), (2) and (3), and \wedge and \vee are an infimum and a supremum, satisfying (4) and (5) is a **lattice**.

Our two previous examples are both examples of lattices. For the case of $\mathcal{P}(\Omega)$ the infimum and supremum of two subsets $A, B \in \mathcal{P}(\Omega)$ are given respectively by the intersection $A \cap B \in \mathcal{P}(\Omega)$ and the union $A \cup B \in \mathcal{P}(\Omega)$ of subsets. For the case of $\mathcal{P}(\mathcal{H})$ the infimum of two closed subspaces $A, B \in \mathcal{P}(\mathcal{H})$ is given by the intersection $A \cap B \in \mathcal{P}(\mathcal{H})$. On the other hand, the union of two closed subspaces is in general not a closed subspace. This means that the union does not give us the supremum in this case. For two closed subspaces $A, B \in \mathcal{P}(\mathcal{H})$, the smallest closed subspace that contains both, is $\overline{A + B}$, the topological closure of the sum of the two subspaces. Hence this is the supremum of A and B in $\mathcal{P}(\mathcal{H})$.

It is worth remarking that we can see already at this level of the discussion one of the fundamental differences between a classical entity and a quantum entity. The vectors that are contained in the topological closure $\overline{A + B}$ of the sum of A and B are exactly the vectors that are superpositions of vectors in A and vectors in B . Hence for a quantum entity, described by $\mathcal{P}(\mathcal{H})$, there are additional vectors in the supremum of A and B , contained neither in A nor in B , while for the classical entity, described by $\mathcal{P}(\Omega)$, there are no such additional elements, because the supremum of A and B is the union $A \cup B$. This means that by changing our description to the level of the lattice, we can express both cases, the classical case and the quantum case, in the same mathematical language, which is not true if we describe the quantum entity in a Hilbert space and the classical entity in a state (or phase) space.

A **complete lattice** is a lattice that contains an infimum and a supremum for every subset of its elements, hence not only for each pair of elements as it is the case for a lattice. Let us express this requirement in a formal way. For $A \subset \mathcal{L}$ there exists $\wedge A$ and $\vee A$ such that:

$$x < a \forall a \in A \Leftrightarrow x < \wedge A \quad (6)$$

$$a < y \forall a \in A \Leftrightarrow \vee A < y \quad (7)$$

Note that a complete lattice \mathcal{L} contains always a minimal element, namely

the element $\wedge \mathcal{L}$, that we denote by 0 , and a maximal element, namely $\vee \mathcal{L}$, that we denote by I .

Our two examples, $(\mathcal{P}(\Omega), \subset, \cap, \cup)$ and $(\mathcal{P}(\mathcal{H}), \subset, \cap, \overline{+})$ are both complete lattices. The minimal elements of $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ are respectively \emptyset and $\{0\}$, and the maximal elements are Ω and \mathcal{H} .

In the axiomatic approach the elements of the lattice represent the **properties** of the physical entity under study. Let us explain how the states of a physical entity are represented in the axiomatic approach.

2.2 Atoms of a Lattice

An element of a (complete) lattice is called an atom, if it is a smallest element different from 0 . Let us define precisely what we mean. We say that $p \in \mathcal{L}$ is an atom of \mathcal{L} if for $x \in \mathcal{L}$ we have:

$$0 < x < p \Rightarrow x = 0 \text{ or } x = p \quad (8)$$

The atoms of $(\mathcal{P}(\Omega), \subset, \cap, \cup)$ are the singletons of the phase space Ω , and the atoms of $(\mathcal{P}(\mathcal{H}), \subset, \cap, \overline{+})$ are the one dimensional subspaces (rays) of the Hilbert space \mathcal{H} .

In the traditional versions of the axiomatic approach the states of the physical entity under study were represented by the atoms of the lattice \mathcal{L} . Of course, this is largely due to the fact that for the case of standard quantum mechanics, the rays can indeed be identified as atoms of the lattice of closed subspaces of the Hilbert space. On the other hand, from a physical point of view, it is obvious that a state is a completely different concept than a property, and hence for an operational approach states should not be properties. Already in [1, 2] it can be seen that when it comes to calculating and proving theorems, the states are treated differently from the properties, although the old tradition of representing both within the same mathematical structure, reminiscent of the ‘states are represented by atoms of the lattice’ idea, is maintained in [1, 2]. Only slowly the insight grew about how to handle this problem in a more profound way. The new way, that is fully operationally founded, is to introduce two sets from the start, the set of **states** of the physical entity, denoted by Σ , and the set of properties, denoted by \mathcal{L} . It follows from the operational part of the construction that additionally to these two sets one needs to consider a one-to-one function $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$, called the Cartan map, such that $p \in \kappa(a)$ expresses the following fundamental physical situation: “The property $a \in \mathcal{L}$ is ‘actual’ if the physical entity is in state $p \in \Sigma$ ”. Moreover it follows from the operational aspects of the axiomatic approach that κ satisfies the three following

additional requirements:

$$\kappa(\bigwedge_i a_i) = \bigcap_i \kappa(a_i) \quad (9)$$

$$\kappa(0) = \emptyset \quad (10)$$

$$\kappa(I) = \Sigma \quad (11)$$

for $(a_i)_i \in \mathcal{L}$ and 0 and I being respectively the minimal and maximal element of \mathcal{L} . The triple $(\Sigma, \mathcal{L}, \kappa)$, where Σ is a set, \mathcal{L} a complete lattice, and κ a map satisfying (9), (10), and (11) has been called a state-property system [9, 10], and it is this mathematical structure that can be derived from the operational aspects of the axiomatic approach.

Let us consider our two examples and see how this structure appears there. For the quantum case, Σ is the set of rays of the Hilbert space and \mathcal{L} the set of closed subspaces. The Cartan map maps each closed subspace on the set of rays that are contained in this closed subspace, and indeed (9), (10), and (11) are satisfied. For the classical case, Σ is the phase space and \mathcal{L} is the set of subsets of the phase space. The Cartan map is the identity. Of course, in the two examples, quantum and classical, the states correspond with atoms of the lattice of properties. By means of two axioms we regain this property for the general axiomatic situation. Since, as we mentioned already, the structure of a state-property system is derived from the operational aspects of the axiomatic approach, these two axioms should be considered as the first two axioms of the new axiomatic approach where states are not identified *a priori* with the atoms of the property lattice. Let us give these two axioms.

2.3 The First Two Axioms: State Determination and Atomisticity

A physical entity S is described by its state-property system $(\Sigma, \mathcal{L}, \kappa)$, where Σ is a set, its elements representing the states of S , \mathcal{L} is a complete lattice, its elements representing the properties of S , and κ is a map from \mathcal{L} to $\mathcal{P}(\Sigma)$, satisfying (9), (10), and (11), and expressing the physical situation: “The property $a \in \mathcal{L}$ is actual if the entity S is in state $p \in \Sigma$ ” by $p \in \kappa(a)$. This is the structure that we derive from only operational aspects of the axiomatic approach. The first axiom that we introduce consists in demanding that a state is determined by the set of properties that are actual in this state.

Axiom 1 (State Determination) For $p, q \in \Sigma$ such that

$$\bigwedge_{p \in \kappa(a)} a = \bigwedge_{q \in \kappa(b)} b \quad (12)$$

we have $p = q$.

We remark that in [1, 2, 8, 12] this axiom is considered to be satisfied *a priori*. The second axiom consists in demanding that the states can be considered as atoms of the property lattice.

Axiom 2 (Atomisticity) For $p \in \Sigma$ we have that

$$\bigwedge_{p \in \kappa(a)} a \tag{13}$$

is an atom of \mathcal{L} .

Obviously these two axioms are satisfied for the two examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ that we considered.

2.4 The Third Axiom: Orthocomplementation

For the third axiom it is already very difficult to give a complete physical interpretation. This third axiom introduces the structure of an orthocomplementation for the lattice of properties. At first sight the orthocomplementation could be seen as a structure that plays a similar role for properties as the negation in logic plays for propositions. But that is not a very careful way of looking at things. We cannot go into the details of the attempts that have been made to interpret the orthocomplementation in a physical way, and refer to [8, 12, 1, 2, 11] for those that are interested in this problem. Also in [13, 14, 15, 16] the problem is considered in depth.

Axiom 3 (Orthocomplementation) The lattice \mathcal{L} of properties of the physical entity under study is orthocomplemented. This means that there exists a function $' : \mathcal{L} \rightarrow \mathcal{L}$ such that for $a, b \in \mathcal{L}$ we have:

$$(a')' = a \tag{14}$$

$$a < b \Rightarrow b' < a' \tag{15}$$

$$a \wedge a' = 0 \quad \text{and} \quad a \vee a' = I \tag{16}$$

For $\mathcal{P}(\Omega)$ the orthocomplement of a subset is given by the complement of this subset, and for $\mathcal{P}(\mathcal{H})$ the orthocomplement of a closed subspace is given by the subspace orthogonal to this closed subspace.

2.5 The Fourth and Fifth Axiom: The Covering Law and Weak Modularity

The next two axioms are called the covering law and weak modularity. There is no obvious physical interpretation for them. They have been put forward mainly because they are satisfied in the lattice of closed subspaces of a complex Hilbert space. These two axioms are what we have called the ‘extra conditions’ when we talked about Piron’s representation theorem in the introduction of this section.

Axiom 4 (Covering Law) *The lattice \mathcal{L} of properties of the physical entity under study satisfies the covering law. This means that for $a, x \in \mathcal{L}$ and $p \in \Sigma$ we have:*

$$a < x < a \vee p \Rightarrow x = a \text{ or } x = a \vee p \quad (17)$$

Axiom 5 (Weak Modularity) *The orthocomplemented lattice \mathcal{L} of properties of the physical entity under study is weakly modular. This means that for $a, b \in \mathcal{L}$ we have:*

$$a < b \Rightarrow (b \wedge a') \vee a = b \quad (18)$$

These are the five axioms of standard quantum axiomatics. It can be shown that both axioms, the covering law and weak modularity, are satisfied for the two examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ [7, 8].

The two examples that we have mentioned show that both classical entities and quantum entities can be described by the common structure of a complete atomistic orthocomplemented lattice that satisfies the covering law and is weakly modular. Now we have to consider the converse, namely how this structure leads us to classical physics and to quantum physics.

3 The Representation Theorem

First we show how the classical and nonclassical parts can be extracted from the general structure, and second we show how the nonclassical parts can be represented by so-called generalized Hilbert spaces.

3.1 The Classical and Nonclassical Parts

Since both examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ satisfy the five axioms, it is clear that a theory where the five axioms are satisfied can give rise to a classical theory, as well as to a quantum theory. It is possible to filter out the classical part by introducing the notions of classical property and classical state.

Definition 1 (Classical Property) *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property system representing a physical entity, satisfying axioms 1, 2 and 3. We say that a property $a \in \mathcal{L}$ is a classical property if for all $p \in \Sigma$ we have*

$$p \in \kappa(a) \text{ or } p \in \kappa(a') \quad (19)$$

The set of all classical properties we denote by \mathcal{C} .

Again considering our two examples, it is easy to see that for the quantum case, hence for $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have no nontrivial classical properties. Indeed, for any closed subspace $A \in \mathcal{H}$, different from 0 and \mathcal{H} , we have rays of \mathcal{H} that are neither contained in A nor contained in A' . These are exactly the rays that correspond to states that are superposition states of states in A and states in A' . It is the superposition principle in standard quantum mechanics that makes that the only classical properties of a quantum entity are the trivial ones, represented by 0 and \mathcal{H} . It can also easily be seen that for the case of a classical entity, described by $\mathcal{P}(\Omega)$, all the properties are classical properties. Indeed, consider an arbitrary property $A \in \mathcal{P}(\Omega)$, then for any singleton $\{p\} \in \Sigma$ representing a state of the classical entity, we have $\{p\} \subset A$ or $\{p\} \subset A'$, since A' is the set theoretical complement of A .

Definition 2 (Classical State) *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property system of a physical entity satisfying axioms 1, 2 and 3. For $p \in \Sigma$ we introduce*

$$\omega(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{C}} a \quad (20)$$

$$\kappa_c(a) = \{\omega(p) \mid p \in \kappa(a)\} \quad (21)$$

and call $\omega(p)$ the classical state of the physical entity whenever it is in a state $p \in \Sigma$, and κ_c the classical Cartan map. The set of all classical states will be denoted by Ω .

Definition 3 (Classical State Property System) *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property system of a physical entity satisfying axioms 1, 2 and 3. The classical state property system corresponding with $(\Sigma, \mathcal{L}, \kappa)$ is $(\Omega, \mathcal{C}, \kappa_c)$.*

Let us look at our two examples. For the quantum case, with $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only two classical properties, namely 0 and \mathcal{H} . This means that there is only one classical state, namely \mathcal{H} . It is the classical state that corresponds to ‘considering the quantum entity under study’ and the state

does not specify anything more than that. For the classical case, every state is a classical state.

It can be proven that $\kappa_c : \mathcal{C} \rightarrow \mathcal{P}(\Omega)$ is an isomorphism[1, 11]. This means that if we filter out the classical part and limit the description of our general physical entity to its classical properties and classical states, the description becomes a standard classical physical description.

Let us filter out the nonclassical part.

Definition 4 (Nonclassical Part) *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property system of a physical entity satisfying axioms 1, 2 and 3. For $\omega \in \Omega$ we introduce*

$$\mathcal{L}_\omega = \{a \mid a < \omega, a \in \mathcal{L}\} \quad (22)$$

$$\Sigma_\omega = \{p \mid p \in \kappa(\omega), p \in \Sigma\} \quad (23)$$

$$\kappa_\omega(a) = \kappa(a) \text{ for } a \in \mathcal{L}_\omega \quad (24)$$

and we call $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ the nonclassical components of $(\Sigma, \mathcal{L}, \kappa)$.

For the quantum case, hence $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only one classical state \mathcal{H} , and obviously $\mathcal{L}_\mathcal{H} = \mathcal{L}$. Similarly we have $\Sigma_\mathcal{H} = \Sigma$. This means that the only nonclassical component is $(\Sigma, \mathcal{L}, \kappa)$ itself. For the classical case, since all properties are classical properties and all states are classical states, we have $\mathcal{L}_\omega = \{0, \omega\}$, which is the trivial lattice, containing only its minimal and maximal element, and $\Sigma_\omega = \{\omega\}$. This means that the nonclassical components are all trivial.

For the general situation of a physical entity described by $(\Sigma, \mathcal{L}, \kappa)$ it can be shown that \mathcal{L}_ω contains no classical properties with respect to Σ_ω except 0 and ω , the minimal and maximal element of \mathcal{L}_ω , and that if $(\Sigma, \mathcal{L}, \kappa)$ satisfies axioms 1, 2, 3, 4, and 5, then also $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega) \forall \omega \in \Omega$ satisfy axioms 1, 2, 3, 4 and 5 (see [1, 11]).

We remark that if axioms 1, 2 and 3 are satisfied we can identify a state $p \in \Sigma$ with the element of the lattice of properties \mathcal{L} given by:

$$s(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{L}} a \quad (25)$$

which is an atom of \mathcal{L} . More precisely, it is not difficult to verify that, under the assumption of axioms 1 and 2, $s : \Sigma \rightarrow \Sigma_\mathcal{L}$ is a well-defined mapping that is one-to-one and onto, $\Sigma_\mathcal{L}$ being the collection of all atoms in \mathcal{L} . Moreover, $p \in \kappa(a)$ iff $s(p) < a$. We can call $s(p)$ the property state corresponding to p and define

$$\Sigma' = \{s(p) \mid p \in \Sigma\} \quad (26)$$

the set of state properties. It is easy to verify that if we introduce

$$\kappa' : \mathcal{L} \rightarrow \mathcal{P}(\Sigma') \quad (27)$$

where

$$\kappa'(a) = \{s(p) \mid p \in \kappa(a)\} \quad (28)$$

that

$$(\Sigma', \mathcal{L}, \kappa') \cong (\Sigma, \mathcal{L}, \kappa) \quad (29)$$

when axioms 1, 2 and 3 are satisfied.

To see in more detail in which way the classical and nonclassical parts are structured within the lattice \mathcal{L} , we make use of this isomorphism and introduce the direct union of a set of complete, atomistic orthocomplemented lattices, making use of this identification.

Definition 5 (Direct Union) *Consider a set $\{\mathcal{L}_\omega \mid \omega \in \Omega\}$ of complete, atomistic orthocomplemented lattices. The direct union $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ of these lattices consists of the sequences $a = (a_\omega)_\omega$, such that*

$$(a_\omega)_\omega < (b_\omega)_\omega \Leftrightarrow a_\omega < b_\omega \quad \forall \omega \in \Omega \quad (30)$$

$$(a_\omega)_\omega \wedge (b_\omega)_\omega = (a_\omega \wedge b_\omega)_\omega \quad (31)$$

$$(a_\omega)_\omega \vee (b_\omega)_\omega = (a_\omega \vee b_\omega)_\omega \quad (32)$$

$$(a_\omega)'_\omega = (a'_\omega)_\omega \quad (33)$$

The atoms of $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ are of the form $(a_\omega)_\omega$ where $a_{\omega_1} = p$ for some ω_1 and $p \in \Sigma_{\omega_1}$, and $a_\omega = 0$ for $\omega \neq \omega_1$.

It can be proven that if \mathcal{L}_ω are complete, atomistic, orthocomplemented lattices, then also $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ is a complete, atomistic, orthocomplemented lattice (see [1, 11]). The structure of direct union of complete, atomistic, orthocomplemented lattices makes it possible to define the direct union of state property systems in the case axioms 1, 2, and 3 are satisfied.

Definition 6 (Direct Union of State Property Systems) *Consider a set of state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$, where \mathcal{L}_ω are complete, atomistic, orthocomplemented lattices and for each ω we have that Σ_ω is the set of atoms of \mathcal{L}_ω . The direct union $\bigoplus_{\omega} (\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ of these state property systems is the state property system $(\bigcup_{\omega} \Sigma_\omega, \bigoplus_{\omega} \mathcal{L}_\omega, \bigoplus_{\omega} \kappa_\omega)$, where $\bigcup_{\omega} \Sigma_\omega$ is the disjoint union of the sets Σ_ω , $\bigoplus_{\omega} \mathcal{L}_\omega$ is the direct union of the lattices \mathcal{L}_ω , and*

$$\bigoplus_{\omega} \kappa_\omega((a_\omega)_\omega) = \bigcup_{\omega} \kappa_\omega(a_\omega) \quad (34)$$

The first part of a fundamental representation theorem can now be stated. For this part it is sufficient that axioms 1, 2 and 3 are satisfied.

Theorem 1 (Representation Theorem: Part 1) *We consider a physical entity described by its state property system $(\Sigma, \mathcal{L}, \kappa)$. Suppose that axioms 1, 2 and 3 are satisfied. Then*

$$(\Sigma, \mathcal{L}, \kappa) \cong \bigoplus_{\omega \in \Omega} (\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \quad (35)$$

where Ω is the set of classical states of $(\Sigma, \mathcal{L}, \kappa)$ (see definition 2), Σ'_ω is the set of state properties, κ'_ω the corresponding Cartan map, (see (26) and (28)), and \mathcal{L}_ω the lattice of properties (see definition 4) of the nonclassical component $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$. If axioms 4 and 5 are satisfied for $(\Sigma, \mathcal{L}, \kappa)$, then they are also satisfied for $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ for all $\omega \in \Omega$.

Proof: see [1, 11]

3.2 Further Representation of the Nonclassical Components

From the previous section follows that if axioms 1, 2, 3, 4 and 5 are satisfied we can write the state property system $(\Sigma, \mathcal{L}, \kappa)$ of the physical entity under study as the direct union $\bigoplus_{\omega \in \Omega} (\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ over its classical state space Ω of its nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, and that each of these nonclassical components also satisfies axiom 1, 2, 3, 4 and 5. Additionally for each one of these nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ no classical properties except 0 and ω exist. It is for these nonclassical components that a further representation theorem can be proven such that a vector space structure emerges for each one of the nonclassical components. To do this we rely on the original representation theorem that Piron proved in [7].

Theorem 2 (Representation Theorem: Part 2) *Consider the same situation as in theorem 1, with additionally axiom 4 and 5 satisfied. For each nonclassical component $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, of which the lattice \mathcal{L}_ω has at least four orthogonal states¹, there exists a vector space V_ω , over a division ring K_ω , with an involution of K_ω , which means a function*

$$* : K_\omega \rightarrow K_\omega \quad (36)$$

such that for $k, l \in K_\omega$ we have:

$$(k^*)^* = k \quad (37)$$

$$(k \cdot l)^* = l^* \cdot k^* \quad (38)$$

¹Two states $p, q \in \Sigma_\omega$ are orthogonal if there exists $a \in \mathcal{L}_\omega$ such that $p < a$ and $q < a'$.

and an Hermitian product on V_ω , which means a function

$$\langle \cdot, \cdot \rangle : V_\omega \times V_\omega \rightarrow K_\omega \quad (39)$$

such that for $x, y, z \in V_\omega$ and $k \in K_\omega$ we have:

$$\langle x + ky, z \rangle = \langle x, z \rangle + k\langle y, z \rangle \quad (40)$$

$$\langle x, y \rangle^* = \langle y, x \rangle \quad (41)$$

$$\langle x, x \rangle = 0 \Leftrightarrow x = 0 \quad (42)$$

and such that for $M \subset V_\omega$ we have:

$$M^\perp + (M^\perp)^\perp = V_\omega \quad (43)$$

where $M^\perp = \{y \mid y \in V_\omega, \langle y, x \rangle = 0, \forall x \in M\}$. Such a vector space is called a generalized Hilbert space or an orthomodular vector space. And we have that:

$$(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \cong (\mathcal{R}(V), \mathcal{L}(V), \nu) \quad (44)$$

where $\mathcal{R}(V)$ is the set of rays of V , $\mathcal{L}(V)$ is the set of biorthogonally closed subspaces (subspaces that are equal to their biorthogonal) of V , and ν makes correspond with each such biorthogonal subspace the set of rays that are contained in it.

Proof: See [7, 8].

4 The Two Failing Axioms of Standard Quantum Mechanics

We have introduced all that is necessary to be able to put forward the theorem that has been proved regarding the failure of standard quantum mechanics for the description of the joint entity consisting of two separated quantum entities [1, 2]. Let us first explain what is meant by separated physical entities.

4.1 What Are Separated Physical Entities?

We consider the situation of a physical entity S that consists of two physical entities S_1 and S_2 . The definition of ‘separated’ that has been used in [1, 2] is the following. Suppose that we consider two experiments e_1 and e_2 that can be performed respectively on the entity S_1 and on the entity S_2 , such

that the joint experiments $e_1 \times e_2$ can be performed on the joint entity S consisting of S_1 and S_2 . We say that experiments e_1 and e_2 are separated experiments whenever for an arbitrary state p of S we have that (x_1, x_2) is a possible outcome for experiment $e_1 \times e_2$ if and only if x_1 is a possible outcome for e_1 and x_2 is a possible outcome for e_2 . We say that S_1 and S_2 are separated entities if and only if all the experiments e_1 on S_1 are separated from the experiments e_2 on S_2 .

Let us remark that S_1 and S_2 being separated does not mean that there is no interaction between S_1 and S_2 . Most entities in the macroscopic world are separated entities. Let us consider some examples to make this clear.

The earth and the moon, for example, are separated entities. Indeed, consider any experiment e_1 that can be performed on the physical entity earth (for example measuring its position), and any experiment e_2 that can be performed on the physical entity moon (for example measuring its velocity). The joint experiment $e_1 \times e_2$ consists of performing e_1 and e_2 together on the joint entity of earth and moon (measuring the position of the earth and the velocity of the moon at once). Obviously the requirement of separation is satisfied. The pair (x_1, x_2) (position of the earth and velocity of the moon) is a possible outcome for $e_1 \times e_2$ if and only if x_1 (position of the earth) is a possible outcome of e_1 and x_2 (velocity of the moon) is a possible outcome of e_2 . This is what we mean when we say that the earth has position x_1 and the moon velocity x_2 at once. Clearly this is independent of whether there is an interaction, the gravitational interaction in this case, between the earth and the moon.

It is not easy to find an example of two physical entities that are not separated in the macroscopic world, because usually nonseparated entities are described as one entity and not as two. In earlier work we have given examples of nonseparated macroscopic entities [17, 33, 19]. The example of connected vessels of water is a good example to give an intuitive idea of what nonseparation means.

Consider two vessels V_1 and V_2 each containing 10 liters of water. The vessels are connected by a tube, which means that they form a connected set of vessels. Also the tube contains some water, but this does not play any role for what we want to show. Experiment e_1 consists of taking out water of vessel V_1 by a siphon, and measuring the amount of water that comes out. We give the outcome x_1 if the amount of water coming out is greater than 10 liters. Experiment e_2 consists of doing exactly the same on vessel V_2 . We give outcome x_2 to e_2 if the amount of water coming out is greater than 10 liters. The joint experiment $e_1 \times e_2$ consists of performing e_1 and e_2 together on the joint entity of the two connected vessels of water. Because

of the connection, and the physical principles that govern connected vessels, for e_1 and for e_2 performed alone we find 20 liters of water coming out. This means that x_1 is a possible (even certain) outcome for e_1 and x_2 is a possible (also certain) outcome for e_2 . If we perform the joint experiment $e_1 \times e_2$ the following happens. If there is more than 10 liters coming out of vessel V_1 there is less than 10 liters coming out of vessel V_2 and if there is more than 10 liters coming out of vessel V_2 there is less than 10 liters coming out of vessel V_1 . This means that (x_1, x_2) is not a possible outcome for the joint experiment $e_1 \times e_2$. Hence e_1 and e_2 are nonseparated experiments and as a consequence V_1 and V_2 are nonseparated entities.

The nonseparated entities that we find in the macroscopic world are entities that are very similar to the connected vessels of water. There must be an ontological connection between the two entities, and that is also the reason that usually the joint entity will be treated as one entity again. A connection through dynamic interaction, as it is the case between the earth and the moon, interacting by gravitation, leaves the entities separated.

For quantum entities it can be shown that only when the joint entity of two quantum entities contains entangled states the entities are nonseparated quantum entities. It can be proven [17, 33, 19] that experiments are separated if and only if they do not violate Bell's inequalities. All this has been explored and investigated in many ways, and several papers have been published on the matter [17, 33, 19, 20, 21]. Interesting consequences for the Einstein Podolsky Rosen paradox and the violation of Bell's inequalities have been investigated [22, 23].

4.2 The Separated Quantum Entities Theorem

We are ready now to state the theorem about the impossibility for standard quantum mechanics to describe separated quantum entities [1, 2].

Theorem 3 (Separated Quantum Entities Theorem) *Suppose that S is a physical entity consisting of two separated physical entities S_1 and S_2 . Let us suppose that axiom 1, 2 and 3 are satisfied and call $(\Sigma, \mathcal{L}, \kappa)$ the state property system describing S , and $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ and $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ the state property systems describing S_1 and S_2 .*

If the fourth axiom is satisfied, namely the covering law, then one of the two entities S_1 or S_2 is a classical entity, in the sense that one of the two state property systems $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ or $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ contains only classical states and classical properties.

If the fifth axiom is satisfied, namely weak modularity, then one of the two

entities S_1 or S_2 is a classical entity, in the sense that one of the two state property systems $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ or $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ contains only classical states and classical properties.

Proof: see [1, 2]

The theorem proves that two separated quantum entities cannot be described by standard quantum mechanics. A classical entity that is separated from a quantum entity and two separated classical entities do not cause any problem, but two separated quantum entities need a structure where neither the covering law nor weak modularity are satisfied.

One of the possible ways out is that there would not exist separated quantum entities in nature. This would mean that all quantum entities are entangled in some way or another. If this is true, perhaps the standard formalism could be saved. Let us remark that even standard quantum mechanics presupposes the existence of separated quantum entities. Indeed, if we describe one quantum entity by means of the standard formalism, we take one Hilbert space to represent the states of this entity. In this sense we suppose the rest of the universe to be separated from this one quantum entity. If not, we would have to modify the description and consider two Hilbert spaces, one for the entity and one for the rest of the universe, and the states would be entangled states of the states of the entity and the states of the rest of the universe. But, this would mean that the one quantum entity that we considered is never in a well-defined state. It would mean that the only possibility that remains is to describe the whole universe at once by using one huge Hilbert space. It goes without saying that such an approach will lead to many other problems. For example, if this one Hilbert space has to describe the whole universe, will it also contain itself, as a description, because as a description, a human activity, it is part of the whole universe. Another, more down to earth problem is, that in this one Hilbert space of the whole universe also all classical macroscopical entities have to be described. But classical entities are not described by a Hilbert space, as we have seen in section 2. If the hypothesis that we can only describe the whole universe at once is correct, it would anyhow be more plausible that the theory that does deliver such a description would be the direct union structure of different Hilbert spaces. But if this is the case, we anyhow are already using a more general theory than standard quantum mechanics. So we can as well use the still slightly more general theory, where axioms 4 and 5 are not satisfied, and make the description of separated quantum entities possible.

All this convinces us that the shortcoming of standard quantum mechanics to be able to describe separated quantum entities is really a shortcoming of the mathematical formalism used by standard quantum mechanics, and more notably of the vector space structure of the Hilbert space used in standard quantum mechanics.

4.3 Operational Foundation of Quantum Axiomatics

To be able to explain the conceptual steps that are made to prove theorem 3 we have to explain how the concept of ‘separated’ is expressed in the quantum axiomatics that we introduced in section 2. Separated entities are defined by means of separated experiments. In the quantum axiomatics of section 2 we do not talk about experiments, which means that there is still a link that is missing. This link is made by what is called the operational foundation of the quantum axiomatic lattice formalism. Within this operational foundation a property of the entity under study is defined by the equivalence class of all experiments that test this property. We will not explain the details of this operational foundation, because some subtle matters are involved, and refer to [8, 1, 2] for these details. What we need to close the circle in this article is the fact that, making use of the operational foundations, it is possible to introduce ‘separated properties’ as properties that are defined by equivalence classes of separated experiments.

4.4 The Separated Quantum Entities Theorem Bis

Theorem 3 can then be reformulated completely in the language of the axiomatic quantum formalism that we introduced in section 2 in the following way:

Theorem 4 (Separated Quantum Entities Theorem Bis) *Suppose that we consider the compound entity S that consists of two physical entities S_1 and S_2 . Suppose that axioms 1, 2 and 3 are satisfied for S , S_1 and S_2 . Suppose that each property of S_1 is ‘separated’ from each property of S_2 . If axiom 4 is satisfied for S , then one of the two entities S_1 or S_2 contains only classical properties and classical states, and hence S_1 or S_2 is a classical entity. If axiom 5 is satisfied for S , then one of the two entities S_1 or S_2 contains only classical properties and classical states, and hence S_1 or S_2 is a classical entity.*

4.5 Linearity at Stake

If the covering law is not satisfied for the lattice \mathcal{L} that describes the compound entity consisting of two separated quantum entities, then this lattice cannot be represented into a vector space. This means that the superposition principle will not be valid for S . In standard quantum mechanics, situations have been encountered where the superposition principle is not valid, and one refers to these situations as ‘the presence of superselection rules’. For example the property ‘charge’ for a microparticle entails such a superselection rule. There are no superpositions of states with different charge. It has always been possible to incorporate superselection rules into the standard formalism by demanding that there should be no superpositions between different sectors of the Hilbert space. The reason that this could be done for superselection rules as the ones that arise from the property charge, is because the states that correspond to different values of a physical quantity are always orthogonal. This has made it possible to circumvent the problem by considering different orthogonal sectors of a common Hilbert space, and not allowing superpositions between states of different sectors. It can be shown that the superselection rules that arise from the situation of separated quantum entities correspond to states that are not orthogonal, which means that the traditional way of avoiding the problem cannot work. In [1, 2] explicit examples of states that are separated by a super selection rule are given. Also in [25] we give examples of such states.

4.6 Some Subtle Aspects of the Separated Quantum Entities Theorem

The ‘Separated Quantum Entities Theorem’ that was proved in [1, 2] was correctly criticized by Cattaneo and Nisticó [26]. As we mentioned already, the proof in [1, 2] is made by introducing separated experiments, where separated is defined as explained in section 4.1. Then separated properties are defined as properties that are tested by separated experiments, and once the property lattice of the joint entity is constructed in this way, the theorem can be proven. The whole construction in [1, 2] is built by starting with only yes/no-experiments, hence experiments that have only two possible outcomes. The reason that the construction in [1, 2] is made by means of yes/no-experiments has a purely historical origin. The version of operational quantum axiomatics elaborated in Geneva, where one of the authors was working when proving the separated quantum entities theorem, was a version where only yes/no-experiments are considered as basic opera-

tional concepts. There did exist at that time versions of operational quantum axiomatics that incorporated right from the start experiments with any number of possible outcomes as basic operational concepts, as for example the approach elaborated by Randall and Foulis [28, 29, 30]. Cattaneo and Nisticó proved in [26] that, by considering only yes/no-experiments as an operational basis for the construction of the property lattice of the compound entity consisting of separated entities, some of the possible experiments that can be performed on this compound entity are overlooked. It could well be that the experiments that had been overlooked in the construction of [1, 2] were exactly the ones that, once added, would give rise to additional properties and make the lattice of properties satisfy again axiom 4 and 5. That is the reason that Cattaneo and Nisticó state explicitly in their article [26] that they do not question the mathematical argument of the proof, but rather its operational basis. This was indeed a serious critique that had been pondered carefully. Although the author involved in this matter remembers clearly that he was convinced then that the lattice of properties would not change by means of the addition of the lacking experiments indicated by Cattaneo and Nisticó, and that his theorem remained valid, there did not seem an easy way to prove this. The only way out was to redo the construction but now starting with experiments with any number of outcomes as basic operational concepts. This is done in [27], and indeed, the separated quantum entities theorem can also be proved with this operational basis. This means that in [27] the critique of Cattaneo and Nistico has been answered, and the result is that the theorem remains valid. The construction presented in [27] is however much less transparent than the original one to be found in [1, 2]. That is the reason why it is interesting to analyze the most simple of all situations of such a compound entity, the one consisting of two separated spin 1/2 objects. This is exactly what we will do in [25]. On this simple example it is easy to go through the full construction of the lattice \mathcal{L} and its set of states Σ , such that we can see how fundamentally different it is from a structure that would entail a vector space type of linearity. Note that since the separated quantum entities theorem is a no-go theorem, also the simple example of [25] contains a proof of the no-go aspect of the original theorem.

5 Attempts and Perspectives for Solutions

In this section we mention briefly what are the attempts that have meanwhile taken place to find a solution to the problem that we have considered in this paper.

If we consider the aspect of the Separated Quantum Entities Theorem where an explicit construction of the lattice of properties and set of states of the compound entity consisting of separated subentities is made, then the theorem proves that this construction cannot be made within standard quantum mechanics, from which follows that standard quantum mechanics cannot describe separated quantum entities. Of course, in its profound logical form the Separated Quantum Entities Theorem is a no-go theorem, which means that also some of the other hypotheses that are used to prove the theorem can be false and hence also at the origin of the problem. Research, which partially took place even before the Separated Quantum Entities Theorem, and partially afterwards, gives us some valuable extra information about what are the possible directions that could be explored to ‘solve’ the problem connected with the Separated Quantum Entities Theorem.

5.1 Earlier Research on the Compound Entity Problem

At the end of the seventies, one of the authors studied the problem of the description of compound entities in quantum axiomatics, but this time staying within the quantum axiomatic framework where each considered entity is described by a complex Hilbert space, as in standard quantum mechanics [31, 32]. This means that the quantum axiomatic framework was only used to give an alternative but equivalent description of standard quantum mechanics, because even then the quantum axiomatic framework makes it possible to translate physical requirements in relation with the situation of a compound physical entity consisting of two quantum mechanical subentities. The main aim of this research on the problem was to find back the tensor product procedure of standard quantum mechanics for the description of the compound entity, but this time not as an *ad hoc* procedure, which it is in standard quantum mechanics, but from physically interpretable requirements. For these requirements, some so-called ‘coupling conditions’ were put forward.

Theorem 5 *We describe quantum entities S_1 , S_2 and S , respectively by their Hilbert space lattices (sets of closed subspaces of the Hilbert space), $\mathcal{L}(\mathcal{H}_1)$, $\mathcal{L}(\mathcal{H}_2)$ and $\mathcal{L}(\mathcal{H})$, and by their Hilbert space state spaces (sets of rays of the Hilbert spaces) $\Sigma(\mathcal{H}_1)$, $\Sigma(\mathcal{H}_2)$ and $\Sigma(\mathcal{H})$. Suppose that $\dim \mathcal{H}_1 > 2$ and $\dim \mathcal{H}_2 > 2$. Suppose that h_1, h_2 are functions:*

$$h_1 : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}) \tag{45}$$

$$h_2 : \mathcal{L}(\mathcal{H}_2) \rightarrow \mathcal{L}(\mathcal{H}) \tag{46}$$

such that for all $A_1, B_1, C_1, (A_1^i)_i \in \mathcal{L}(\mathcal{H}_1)$, $A_2, B_2, C_2, (A_2^j)_j \in \mathcal{L}(\mathcal{H}_2)$, $p_1 \in \Sigma(\mathcal{H}_1)$ and $p_2 \in \Sigma(\mathcal{H}_2)$ we have

$$A_1 \subset B_1 \Rightarrow h_1(A_1) \subset h_1(B_1) \quad (47)$$

$$A_2 \subset B_2 \Rightarrow h_2(A_2) \subset h_2(B_2) \quad (48)$$

$$h_1(\vee_i A_1^i) = \vee_i h_1(A_1^i) \quad (49)$$

$$h_2(\vee_j A_2^j) = \vee_j h_2(A_2^j) \quad (50)$$

$$h_1(\mathcal{H}_1) = h_2(\mathcal{H}_2) = \mathcal{H} \quad (51)$$

$$h_1(C_1) \leftrightarrow h_2(C_2) \quad (52)$$

$$h_1(p_1) \wedge h_2(p_2) \in \Sigma(\mathcal{H}) \quad (53)$$

where \leftrightarrow is the symbol for ‘compatible’, then $\mathcal{P}(\mathcal{H})$ is canonically isomorphic to $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ or to $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2^*)$.

Proof: see [31, 32]

The conditions (47), (48),(49),(50), (51),(52) and (53) are called the ‘coupling conditions’ in [31, 32]. The physical interpretation for the different conditions is quite straightforward. Conditions (47), (48),(49),(50) and (51) mean that the functions h_1 and h_2 are morphisms of the lattice structure. Hence they express that S_1 and S_2 can be recognized as subentities of S . Condition (52) expresses that properties of S_1 are compatible with properties of S_2 , and condition (53) expresses that when S_1 and S_2 are in certain states, then S is in a state uniquely determined by these states of S_1 and S_2 .

When the article [32] was written, the authors aimed at giving a physical justification for using the tensor product in standard quantum mechanics for the description of the compound entity consisting of two quantum entities. The theorem succeeds well in doing so. There is however one remarkable aspect of the theorem. Two possible solutions appear and they are not canonically isomorphic. This means that for the category of Hilbert space lattices and their morphisms none of the two solutions can be a categorical product, because than the two solutions should be canonically isomorphic. This is amazing, because one would expect that if one moves to the mathematical level that corresponds well with the physics, which should be the Hilbert space lattice rather than the Hilbert space itself, one would find one of the categorical products to correspond to what is needed for the description of the compound entity. Let us remark that the theorem shows that if the Hilbert spaces would all be real Hilbert spaces instead of complex, there is only one solution, in which case it could be a categorical product.

Of course, it is well known that the complex numbers play an essential role in quantum mechanics, such that the two solutions do represent different entities.

5.2 Investigating Further a Categorical Approach

Becoming aware of the fact that no categorical solution can be inferred from theorem 5, it became interesting to look straightforwardly for a categorical construction. This is what was done in [33]. A categorical product, more specifically a co-product, can be constructed, but it gives a structure that is very different from what one gets in standard quantum mechanics (the tensor product of Hilbert spaces), and from what one gets from theorem 5. A theorem that is very similar to the Separated Quantum Entities Theorem can be proven for the co-product. Again two of the axioms of traditional quantum axiomatics are never satisfied for the compound entity of two quantum entities if we would describe this compound entity by means of the co-product, except when one of the subentities is a trivial entity, with a lattice of properties containing only 0 and I [33]. One of the failing axioms is again the covering law, which means that also here, if we choose to use the co-product instead of the separated product to describe the compound entity, linearity is gone. We cannot go more into detail on this matter in this paper, but refer to [13, 14] where the situation of the three products, the separated product, the co-product and the Hilbert space tensor product, is studied in detail by one of the authors.

5.3 The Problem of Pure States and Mixed States

From what we have explained in the foregoing, the situation is such that (1) the compound entity consisting of two separated quantum entities cannot be described by standard quantum mechanics, and (2) there remains an unsolved problem in relation with the description of the compound entity of two (not necessarily separated) quantum entities, in the sense that traditional quantum mechanics only knows the tensor product of Hilbert spaces procedure, but this procedure cannot be fitted into an operational scheme at the axiomatic level. These results seem to indicate strongly that standard quantum mechanics must be generalized in the sense that a mathematical formalism should be worked out where the covering law is dropped, and hence linearity is lost. More recently however another possibility has come to the surface. Since also this possibility is relevant for the whole of the book where this article appears, we want to explain it briefly.

The Separated Quantum Entities Theorem of [2] and the Co-Product Theorem of [33] are in essence no-go theorems. And although both theorems give a very strong argument in favor of the view that standard quantum mechanics should be generalized by dropping the covering law and hence losing linearity, we have to be careful. The most profound conclusion that has to be drawn from any no-go theorem is that at least one of the hypotheses that is used to prove the theorem is false. This means that the situation may even be worse, namely that not only the covering law (and weak modularity) should be dropped, but that there is even another, perhaps more important, hypothesis false in standard quantum mechanics. Of course, normally one would start to elaborate a generalization by dropping the least possible number of hypotheses necessary. But our research shows that even if we drop the covering law (and weak modularity), and construct then the co-product, we still do not find a satisfactory way to describe the compound entity of two quantum entities. Moreover, as we mentioned, the co-product structure is so different from the tensor product of Hilbert spaces structure used in standard quantum mechanics in a more or less fruitful way, that it might well be that we are not looking at the right category. This type of reflections and other ones have led one of the authors to consider the following possibility: *perhaps we should reconsider the way in which pure states are described in quantum mechanics by means of rays of the Hilbert space. And more concretely, perhaps also density operators, that normally are interpreted as only describing mixed states, represent pure states as well as mixed states.* This idea has been considered and introduced in [9] and the physical and philosophical situation connected with it has been analyzed in [34]. The quantum formalism where one allows density operators to represent pure states has been called ‘extended quantum mechanics’ in [9].

It is clear that this conceptual change will not solve all of the problems, for example, the fact that separated quantum entities cannot be described by extended standard quantum mechanics is still true. This means that also extended quantum mechanics shall have to be formulated by means of a structure that is different from the complex Hilbert space that is used in standard quantum mechanics. But something is also gained that might make that the mathematical change that is needed under extended quantum mechanics is less drastic than the one that is needed under standard quantum mechanics. We can see this by noting that for extended quantum mechanics the state of a compound entity, whether it is a ray state or a density operator state, is always a product state of states of the subentities. This comes from the ‘mathematical’ fact that any density operator in the tensor product Hilbert space is a product of density operators in the compo-

nent Hilbert spaces. This means that there is more hope that a categorical construction for the lattice of properties and set of states of an extended quantum mechanics would give rise to a co-product that is closer to the tensor product of Hilbert spaces structure that is now used in standard quantum mechanics. We cannot say much more about this now, because we did not have the time to investigate sufficiently the operational and categorical structures that go along with extended quantum mechanics. We plan to engage in this investigation in the future. What we can see immediately is that if density operators also represent pure states, the axiom of atomicity will not be fulfilled for the pure states that arise from density operators.

References

- [1] D. Aerts, *The One and the Many: Towards a Unification of the Quantum and the Classical Description of One and Many Physical Entities*, Doctoral Thesis, Brussels Free University (1981).
- [2] D. Aerts, “Description of many physical entities without the paradoxes encountered in quantum mechanics”, *Found. Phys.*, **12**, 1131-1170 (1982).
- [3] E. Artin, *Geometric Algebra*, Interscience Publishers, Inc., New York (1957).
- [4] J. von Neumann, *Grundlagen der Quantenmechanik*, Springer-Verlag, Berlin, Heidelberg, New York (1932).
- [5] G. Birkhoff and J. von Neumann, “The logic of quantum mechanics”, *Ann. Math.*, **37**, 823-843 (1936).
- [6] G. Mackey, *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York (1963).
- [7] C. Piron, “Axiomatique quantique”, *Helv. Phys. Acta*, **37**, 439 (1964).
- [8] C. Piron, *Foundations of Quantum Physics*, Reading, Mass., W. A. Benjamin (1976).
- [9] D. Aerts, “Foundations of quantum physics: a general realistic and operational approach”, *Int. J. Theor. Phys.*, **38**, 289-358 (1999), lanl archive ref: *quant-ph/0105109*.

- [10] D. Aerts, E. Colebunders, A. Van der Voorde and B. Van Steirteghem, “State property systems and closure spaces: a study of categorical equivalence”, *Int. J. Theor. Phys.*, **38**, 359-385 (1999), lanl archive ref: *quant-ph/0105108*.
- [11] D. Aerts, “Classical theories and nonclassical theories as a special case of a more general theory”, *J. Math. Phys.*, **24**, 2441-2453 (1983).
- [12] C. Piron, *Mécanique Quantique: Bases et Applications*, Press Polytechnique de Lausanne (1990).
- [13] F. Valckenborgh, “Operational axiomatics and compound systems”, in *Current Research in Operational Quantum Logic: Algebras, Categories, Languages*, eds. Coecke, B., Moore, D. J. and Wilce, A., Kluwer Academic Publishers, Dordrecht, 219-244 (2000).
- [14] F. Valckenborgh, *Compound Systems in Quantum Axiomatics*, Doctoral Thesis, Brussels Free University (2001).
- [15] T. Durt and B. D’Hooghe, “The classical limit of the lattice-theoretical orthocomplementation in the framework of the hidden-measurement approach”, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore (2002).
- [16] D. Aerts, “Reality and probability: introducing a new type of probability calculus”, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore (2002).
- [17] D. Aerts, “Example of a macroscopical situation that violates Bell inequalities”, *Lett. Nuovo Cimento*, **34**, 107-111 (1982).
- [18] D. Aerts, “How do we have to change quantum mechanics in order to describe separated systems”, in *The Wave-Particle Dualism*, eds. Diner, S., et al., Kluwer Academic, Dordrecht, 419-431 (1984).
- [19] D. Aerts, “The description of separated systems and quantum mechanics and a possible explanation for the probabilities of quantum mechanics”, in *Micro-physical Reality and Quantum Formalism*, eds. van der Merwe, A., et al., Kluwer Academic Publishers, 97-115 (1988).

- [20] D. Aerts, “An attempt to imagine parts of the reality of the micro-world”, in the *Proceedings of the Conference 'Problems in Quantum Physics; Gdansk '89'*, World Scientific Publishing Company, Singapore, 3-25 (1990).
- [21] D. Aerts, S. Aerts, J. Broekaert and L. Gabora, “The violation of Bell inequalities in the macroworld”, *Found. Phys.*, **30**, 1387-1414, (2000).
- [22] D. Aerts, “The physical origin of the Einstein Podolsky Rosen paradox”, in *Open Questions in Quantum Physics: Invited Papers on the Foundations of Microphysics*, eds. Tarozzi, G. and van der Merwe, A., Kluwer Academic, Dordrecht, 33-50 (1985).
- [23] W. Christiaens, “Some notes on Aerts’ interpretation of the EPR-paradox and the violation of Bell-inequalities”, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore (2002).
- [24] D. Aerts, “Being and change: foundations of a realistic operational formalism”, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore (2002).
- [25] D. Aerts and F. Valckenborgh, “Linearity and compound physical systems: the case of two separated spin 1/2 entities”, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore (2002).
- [26] G. Cattaneo and G. Nisticó, “A note on Aerts’ description of separated entities”, *Found. Phys.*, **20**, 119 (1990).
- [27] D. Aerts, “Quantum structures, separated physical entities and probability”, *Found. Phys.*, **24**, 1227-1259 (1994).
- [28] C. Randall and D. Foulis, “A mathematical setting for inductive reasoning”, in *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science III*, ed. C. Hooker, Kluwer Academic, Dordrecht, 169 (1976).
- [29] C. Randall and D. Foulis, “The operational approach to quantum mechanics”, in *Physical Theories as Logico-Operational Structures*, ed. C. A. Hooker, Kluwer Academic, Dordrecht, 167 (1978).

- [30] D. Foulis and C. Randall, “What are quantum logics and what ought they to be?” in *Current Issues in Quantum Logic*, eds. E. Beltrametti and B. van Fraassen, Kluwer Academic, **35** (1981).
- [31] D. Aerts and I. Daubechies, “About the structure preserving maps of a quantum mechanical propositional system”, *Helv. Phys. Acta*, **51**, 637-660 (1978).
- [32] D. Aerts and I. Daubechies, “Physical justification for using the tensor product to describe two quantum systems as one joint system”, *Helv. Phys. Acta*, **51**, 661-675 (1978).
- [33] D. Aerts, “Construction of the tensor product for lattices of properties of physical entities”, *J. Math. Phys.*, **25**, 1434-1441 (1984).
- [34] D. Aerts, “The description of joint quantum entities and the formulation of a paradox”, *Int. J. Theor. Phys.*, **39**, 485-496 (2000).