Probing the Structure of Quantum Mechanics*

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We believe that in the decades to come quantum theory will play an increasingly important role for many different fields. One of the reasons is that technology aims at manipulating and controlling information and energy in ever smaller regions of space and windows of time. As a consequence the behavior of the entities to manipulate and control will become more and more quantum. This means not only spectacular advances of new techniques and outlooks on revolutionary applications, but also a constant stress and attention on the theory of quantum mechanics itself.

It is well known that quantum mechanics has been scrutinized in all kind of ways during the past decades, but that still a lot of conceptual problems remain. The problems of standard quantum mechanics are however not only of a conceptual nature. Also the formal mathematical structure of quantum mechanics has been investigated with the aim to make the theory more operational and to found the basic concepts in direct correspondence with what happens in the laboratory. Such an operationally founded quantum mechanics may soon become of great value because of the technological

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advances, that will demand a more straightforward connection between the theory and the type of manipulations and control to be executed in the laboratory.

Although operational quantum mechanics is in full development, we must admit that the time has not yet come for it to function as a 'better to apply and more easy to use' theory for experimentation. The reason is that the operational quantum structures that have been elaborated, while carefully but boldly aiming at physical clearness and transparency, stumble upon a lot of problems of purely technical mathematical nature. Quantum mechanics is not only conceptually a very difficult theory, it entails also a very sophisticated mathematical apparatus.

It becomes even more and more clear that both dimensions of difficulty, the conceptual one and the mathematical structural one, are linked in a profound way. It has been shown that some of the deep conceptual problems of quantum mechanics – the so called quantum paradoxes – that make it impossible for standard quantum mechanics to be a straightforward operational theory, are due to structural shortcomings of the mathematical apparatus of standard quantum mechanics. Let us make clear by an analogy what we mean by the last statement. Suppose that history had evolved differently for classical mechanics and that Hamiltonian mechanics had been formulated without Newton mechanics. If physicists then had to apply Hamiltonian mechanics to the whole of the domain of reality where classical physics is used, strange conceptual problems would certainly have arrived due to the too limited mathematical structure of Hamiltonian mechanics to cope with all type of situations encountered in the macroscopic physical world. For example, the simple situation of a sphere rolling on a plane, which entails a nonholonomic constraint, would not have been possible to describe by the classical physicists only being able to make use of Hamiltonian theory, because Hamiltonian physics cannot describe nonholonomic constraints. If then, in a stubborn way, and because no other theory was available, the classical physicist would have persisted in his or her attempt to deliver a Hamiltonian description for the sphere rolling on a plane, conceptual paradoxes would probably have appeared as a consequence. Certainly if we push the analogy a litter further and also suppose that the classical physicist would only have access to the situation of the rolling sphere by means of sophisticated experiments, and not with his or her eyes and human intuition.

Let us sketch briefly the problem of standard quantum mechanics that we refer to in the analogy of the foregoing paragraph – the deficiency of the mathematical apparatus of standard quantum mechanics – and that part of articles of this book focus on, and where the four mentioned concepts of the subtitle of the book, *nonlinearity*, *nonlocality*, *computation* and *axiomatics*, play a role.

The first approach that is put forward in the book, and that reveals aspects of the mentioned deficiency of standard quantum mechanics, comes from a line of research that is active for some decades, and where it has been shown that the limitations of the mathematical structure of standard quantum mechanics are in great part at the origin of the problems related to the situation of compound quantum entities, hence the Einstein Podolsky Rosen (EPR) type of situations.

The research that we refer to in this first approach is undertaken within what is called the axiomatic approach to standard quantum mechanics. That is why we will call it the 'axiomatic approach'. Standard quantum mechanics is retrieved in this approach by a set of five axioms formulated on the very general structure of a lattice. First some general problems that seemed to be mostly of a technical nature, and with no clear physical significance, were discovered in the attempt to retrieve the standard tensor product procedure to describe the compound entity consisting of two sub entities from a coupling procedure on the level of the axiomatic approach [1, 2, 3, 4, 5]. A full blow was given to the standard quantum mechanics formalism when it was proven that one of the most simple of all situations, namely the situation of a compound entity that consists of two 'separated' quantum entities, cannot be described by standard axiomatics. And more specifically it was shown that two of the five axioms that lead to standard quantum mechanics are not satisfied for the situation of a compound entity consisting of two separated quantum entities [6, 7, 8, 9]. Moreover one of these failing axioms is equivalent to the linearity of the state space of the physical entity under consideration. This means that if a nonlinear generalization of standard quantum mechanics would be elaborated, a completely different approach for the Einstein Podolsky Rosen paradox like situations could be worked out. Rapidly other results in axiomatic quantum mechanics confirmed this finding. All indicated a fundamental difficulty with the 'linearity axiom' in relation with the description of the compound entity consisting of two quantum entities [10, 11, 12].

The second approach that is treated in the book, although from a completely different direction, hits upon the same problem as the one we mentioned in the foregoing paragraphs. This approach comes from a direct attempt to built a nonlinear quantum mechanics, and therefore we will call it the 'nonlinearity approach'. Thinking of quantum mechanics as a limiting case of a more fundamental nonlinear theory one encounters difficulties which are both conceptual and technical. The conceptual problems were from the very beginning deeply related to the question of how to treat separated entities and how to discuss nonlinear dynamics of entangled states.

It seems that the link between separability conditions and the possible forms of nonlinearities was noticed for the first time in [13], the same year as [1] where the problem was identified in the axiomatic approach. The assumption that a nonlinear correction to the Schrödinger equation should be additive on product states led the authors of [13] to the conclusion that only the logarithmic term is acceptable. One of the drawbacks of the analysis given in [13] was that the discussion was limited to product states. The question of entangled states appeared in this context for the first time in [14] with the conclusion that difficulties may be fundamental and hard to overcome. The point was further elaborated, in a rather general setting, in [15]. The explicit definition of nonlinear evolution of entangled states proposed in [16] was quickly shown to lead to unphysical influences between separated systems [17, 18, 19, 20] (for a recent discussion cf. [21]).

However, once the difficulty was formulated for concrete and explicit models it became clear that the problem is more subtle and that some implicit assumptions may be of crucial importance. A part of the way out was suggested in the important paper [22]. From the perspective of the past decade we can say that the first step of the solution is to correctly identify the one-particle space of pure states. The difficulty is always present if one insists on representation of pure states in terms of rays or vectors in a Hilbert space. This appears justified if one works at the level of Schrödinger equations. Still, we know that the Schrödinger equation can be replaced by the von Neumann equation for one dimensional projectors. The advantage of such a viewpoint is that the von Neumann equation can describe evolution of entangled subsystems whereas the same cannot be said of the Schrödinger equation. The solution proposed in [22] was to start with nonlinear evolution equations appropriately defined for density matrices and recover Schrödinger-type evolutions by restricting the dynamics to projectors. One can say that reduced density matrices obtained via partial tracing from projectors on entangled states have to be treated as pure states. Such states are 'pure' in the sense of being 'fully quantum', a point of view which is in a striking agreement with the quantum axiomatic results discussed in the first few papers of this book. The opinion that in nonlinear quantum mechanics one has to distinguish between two types of 'mixtures' was expressed already in 1991 in [24]. The density-matrix perspective was further elaborated by Jordan in [23] who explicitly constructed the dynamics in terms of nonlinear von Neumann equations. The originality of [23] was not in the very form of the evolution equations, which were discussed in the context of generalizations of quantum mechanics earlier in [24], but in the link of such equations to the separability problems for entangled states. Further analysis showed that a consistent application of Polchinski-type multi-particle extensions leads to equations which *look* nonlocal in configuration space but remain physically local in the physical space [25]. Explicit solutions of such physically local equations allowed one to understand various subtle interplays between tensor product structures, nonlinearity, and locality on one hand, and complete positivity of nonlinear maps on the other [26]. Finally, quite recently the proposal from [22] was generalized in [27] to multiple-time correlation experiments of the type discussed in [15, 19]. It seems that even though many questions in nonlinear quantum mechanics may remain open, the nonlocality — if appropriately treated — is not a true difficulty.

Let us return to the axiomatic approach, and show that the research there evolved in a way that is parallel and at the same time complementary to what happened in the nonlinearity approach.

Different types of products under slightly different coupling conditions were tried out, but always the structure that was found on the more general axiomatic level, where the failing axioms had been dropped, showed out to be very different from the tensor product structure used in the coupling in standard quantum mechanics [12, 28, 29]. Meanwhile however also some simple situations of coupled spins had been studied, and there it was revealed that the tensor product structure used in the coupling of standard quantum mechanics could be completely regained if a rigid coupling was introduced representing the entanglement [30, 31, 32]. In these models not only the rays of the considered Hilbert spaces, but also the density operators appeared as pure states, which at first sight was considered to be a weak point of the models. After reflecting more on these models it became clear however that a generalization of standard quantum mechanics could be built in this way, where the rays as well as the density operators represent pure states, and the density operators also, at the same time, represent mixed states [33]. The fact that from an experimental point of view, by limiting oneself to one quantum entity, it is not possible to make a difference between the pure state and the mixed state represented by the same density operator, is due to the linearity of standard quantum mechanics. The linear structure in some way hides the difference between the pure state and the mixture represented by the same density operator. We have called the quantum mechanics where also density operators represent pure states 'completed quantum mechanics' in [33]. That the problem was revealed by studying the situation of the compound entity of two quantum entities is due to the fact that in this situation nonlinearity shows up at the ontological level.

We do not present a solution in this book, *i.e.* the elaboration of a generalized non linear quantum mechanics. We merely present the material needed to see the way that one could eventually go for the development of such a theory. Future research shall have to make clear whether our analysis of the situation is correct, and hence a generalized nonlinear quantum mechanics can be built, resolving the problems with standard quantum mechanics that we have mentioned.

In the first two articles of this book, 'D. Aerts and F. Valckenborgh, *The linearity of quantum mechanics at stake: the description of separated quantum entities*' and 'D. Aerts and F. Valckenborgh, *Linearity and compound physical systems: the case of two spin 1/2 entities*', the deficiency of the mathematical structure of standard quantum mechanics that we mentioned is analyzed in detail within the axiomatic approach. In the first article a clear account of traditional quantum axiomatics is put forward and it is shown how the last two axioms are at the origin of the impossibility to deliver a model for separated quantum entities. It is also shown how one of these axioms is equivalent with the linearity of the state space and hence with the superposition principle. In the second article the description of two separated spins 1/2 is worked out in detail, such that it can be seen, for this simple case, how the mathematical structure that arises is very different from the standard quantum mechanics description of two spins 1/2 in a complex Hilbert space. Here it can be pointed out concretely where linearity fails, for example no superposition states exists for two couples of states, where both states of one of the spins are different from both states of the other spin, while superpositions do exist for couples of states where one of the states of both spins is equal. Translated into standard quantum mechanical language one could say that super selection rules of a new nature show up, between states that are not orthogonal, such that they cannot be treated as traditional super selection rules, by avoiding superpositions between different orthogonal subspaces of the Hilbert space.

If density operators can also represent pure states of a quantum entity, another one of the five axioms of traditional quantum axiomatics has to be abandoned. In the third article of the book 'D. Aerts, *Being and change: foundations of a realistic operational formalism*, an operational axiomatic approach to quantum mechanics is developed in all its generality. Also the axiom that avoids pure states to be described by density operators is omitted in the formalism proposed here. The article refers to some of the earlier developments, but also introduces the newest advances within this approach. It is a continuation of [33, 34], but now more attention is paid to the development of the dynamical aspects of operational axiomatics. The change of state under influence of a measurement and the dynamical change of state are integrated into a 'general change of state under influence of a context', such that 'dynamics' and 'measurement influence' appear as two aspects of a more general type of change. The formalism is also prepared explicitly for wider applications than just an application to quantum mechanics in its description of the microworld. For example, the formalism is made sufficiently general to allow also an influence of the context (measurement or dynamical) on the state, which is not the case, neither for classical entities nor for quantum entities, but which is often the case for applications to other fields where contextual influence is present, *e.g.* biology and cognition. Classical and quantum physical situations are retrieved as special cases where the state of the entity under study does not influence the context (dynamical or measurement), and where in both cases dynamical context influences the state, and in the case of a quantum physical situation also measurement context influences the state.

We mentioned already how the structure of standard quantum mechanics falls short when it comes to the description of compound entities. Some years after the discovery of this shortcoming another shortcoming of the structure of standard quantum mechanics of a similar nature was revealed. By studying the classical limit in a simple quantum model for the spin of a spin 1/2 quantum entity – but a quantum model that is defined in the larger structural context of axiomatic quantum mechanics than standard Hilbert space quantum mechanics – it could be proven that again the last two of the traditional axioms of quantum axiomatics are not satisfied in the region 'between quantum and classical' [35, 36, 37, 56]. This means again that it is the linearity of standard quantum mechanics that makes it impossible to describe a continuous transition from quantum to classical, something that can be done within a generalized nonlinear quantum formalism, as the one used in [35, 36, 37, 56]. The 'between quantum and classical situation' was studied more in detail in [39, 40, 41, 42], and meanwhile it had become clear that there is also a problem with one of the other axioms of quantum axiomatics, the axiom related to the existence of an orthogonality relation on the set of states of the physical entity under consideration.

The fourth article of this book, 'T. Durt and B. D'Hooghe, *The classical limit of the lattice-theoretical orthocomplementation in the framework of the hidden-measurement approach*', investigates the classical limit in this perspective. By looking to different types of orthogonality relations it is proven that determinism is not enough for an entity to entail a full classical structure.

Within traditional quantum axiomatics the classical part and the quantum part of a physical entity can be filtered out, such that a general physical entity can have classical properties and quantum properties and also a mixture of both [43]. In the fifth article of this book, 'D. Aerts and D. Deses, *State property systems and closure spaces: extracting the classical and nonclassical parts*', is investigated in which way this classical and quantum parts can still be filtered out, even if the two last axioms and also the axiom that causes problems with the orthogonality are not satisfied. The categorical equivalence between state property systems, the structures that in quantum axiomatics describe a physical entity by means of its states and its properties, and closure spaces, a mathematical generalization of topologies, that was derived in earlier work [44, 45, 46, 47], is used to derive a decomposition theorem that is a generalization of the original decomposition theorem as presented in [7]. This decomposition theorem translates through the equivalence of categories to a decomposition theorem of closure spaces into their connected components.

The operational axiomatic approaches to quantum mechanics that we have considered have traditionally concentrated on the description of a physical entity by means of its states and properties. In a certain sense one could say that the probabilistic aspects of quantum theory have been neglected in these approaches. In the foregoing sections we concentrated on the advances of a structural nature that have been made in quantum axiomatics, related to the study of the compound entity of separated entities and the investigation of the classical limit within a formalism that is more general than standard quantum mechanics. There also has been an important step ahead on the conceptual level in relation with quantum probability. It was shown that the structure of the quantum probability model could be derived from a hypothesis about the physical origin of quantum probability that is the following: quantum probability is due to the presence of fluctuations on the interaction between measurement apparatus and physical entity under study [48, 49]. The approach that introduces the quantum probabilities in this way has meanwhile been called the 'hidden measurement approach', and different aspects of it have been studied [50, 51, 52]. In the sixth article of this book, 'S. Aerts, *Hidden measurements from contextual axiomatics*, the hidden measurement approach is investigated, and three simple requirements are put forward that make it possible to uniquely recover the structure of hidden measurements.

It is worth noting that the ontology proposed in hidden variables theories differs from the ontology proposed in other interpretations. Therefore, it is possible in principle to conceive crucial experiments during which the validity of a particular interpretation could be tested. This is what occurred for instance in the numerous EPR-Bell experiments that were realized during the last three decades [53]. Such experiments are crucial experiments during which it is possible to discriminate between local-realistic ontologies and the other ontologies (non-local realistic ones à la Bohm [54] or non-realistic ones à la Bohr [55]). Similarly, it is possible in principle to discriminate between the hidden measurement interpretation and the standard interpretation provided the fluctuations of the hidden state of the apparatus are not instantaneous which means that the detector remembers its hidden state for a while. Then non-standard correlations ought to appear between successive outcomes obtained from the same apparatus [56]. The seventh article of this book 'T. Durt, J. Baudon, R. Mathevet, J. Robert and B. Viaris de Lesegno, *Memory effects in atomic interferometry: a negative result*' describes an attempt to detect such correlations. This experiment was negative in the sense that standard predictions were confirmed. An upper bound could be found for the value of hypothetical hidden measurement memory times. If these times are too small however, they cannot be detected, and the hidden measurement approach gives an ad hoc description of quantum phenomena. Similarly, it is possible, by making use of the inefficiency of presently available detectors (this is the so-called efficiency loophole) to simulate the results of present EPR-Bell experiments by ad hoc local realistic models.

Loose of the hidden measurement approach, the structure of probability, whether it is classical probability or quantum probability, poses another problem of a conceptual nature. As we mentioned already, the generalized axiomatic approaches have been developed focusing on the description of the states and the properties of the physical entity under consideration. A property is linked to a test with 'certain' outcome. But 'certainty' is a concept that cannot easily be recovered by a probability theory that is founded on traditional measure theory. The reason is that events with probability equal to 1 are not completely certain events. This problem is investigated in the eighth article of this book, 'D. Aerts, *Reality and probability: introducing a new type of probability calculus.* It is proven that 'certainty' can be recovered from a probabilistic approach if a new type of probability theory is introduced, called 'subset probability', where the probability is evaluated by a subset of the interval $[0, 1]$ instead of an element of $[0, 1]$, as it is the case in traditional probability theory founded on measure theory. The subset probability is a generalization of traditional probability theory that is retrieved when all subsets are singletons of the interval $[0, 1]$. Not only 'certainty' can be modelled in a natural way by a subset probability, but also situations 'near to certainty' can be described in a way that avoids the problems encountered with traditional probability theory. The structure of a state property system, that has been studied extensively in the axiomatic approach [44, 45, 46, 47], is recovered as corresponding to the description of 'certainty' in the subset probabilistic approach.

Quantum computation constitutes another promising and fascinating contemporary field of research. The basic idea is that quantum systems do not behave as deterministic systems, but exhibit a flexibility that has no classical counterpart. For instance, quantum bits (qubits) can be superposed and teleported, and it can be shown that in principle a processor based on qubits works in certain circumstances (when the quantum entanglement and the superposition principle are optimally exploited) exponentially faster than its classical counterpart. The hidden measurement approach [48, 49, 50, 51, 52], combined with the results on separated and nonseparated physical entities in quantum axiomatics [6, 7, 8, 9, 10, 11], invites us to consider the quantum computation process from a more general perspective. Traditional quantum computation considers different steps in its computational process. The data of a problem are encoded into the quantum state of N entangled spin $\frac{1}{2}$. The quantum computation process, described by a unitary evolution, brings then this state into another state of the N entangled spin $\frac{1}{2}$, such that the outcome of the calculation, and the resolution of the problem, can be extracted from this state. Two of the basic quantum aspects that are at play in this quantum calculation process are entanglement and indeterminism. The models that have been developed within the hidden measurement approach [30, 48, 49, 35, 36, 37, 56, 39, 40, 41, 42], and more specifically the $\epsilon\rho$-model [57], make it possible to parametrize the amount of indeterminism and entanglement by means of two variables $\epsilon, \rho \in [0, 1]$, such that for $\epsilon = \rho = 0$ we get the classical situation of a Turing calculation process with no entanglement and no indeterminism, while for $\epsilon = \rho = 1$ we get a pure quantum calculation process. For intermediate values of ϵ and ρ an intermediate 'between quantum and classical' calculation process can be modelled. This makes it possible to investigate the influence of the two aspects 'entanglement' and 'indeterminism' in the quantum computation process. In the ninth article of the book, 'D. Aerts and B. D'Hooghe, *Quantum computation: towards the construction of a 'between quantum and classical' computer*, this perspective is considered. The nonlinearity problem that we mentioned already also appears here, since it can be shown that the 'between quantum and classical' situations gives rise to a structure that does not satisfy the linearity axioms of traditional quantum axiomatics.

Also connections between quantum axiomatics and fuzzy set theory have been studied. A quantum axiomatic system defined by a set of experimentally verifiable propositions can be represented by a suitably chosen family of fuzzy sets over the set of states such that conjunction and disjunction are given by Giles'[62] fuzzy set intersection and union[63, 64, 65]. Each proposition is represented by a fuzzy set whose membership function value in a point is given by the probability of the experimental proposition if the system is in the corresponding state. Although Giles' operations satisfy the law of contradiction and excluded middle, they do not satisfy the law of idempotency. Also, there is an infinite number of possible fuzzy set connectives and hence an infinite number of possible definitions for conjunction and disjunction of two fuzzy sets representing experimental propositions. For instance, the fuzzy set connectives introduced by Zadeh in his historic paper[66] are idempotent but violate the laws of contradiction and excluded middle. However, these and other fuzzy set operations considered usually in the literature are defined pointwise, i.e., the membership function value of the conjunction (disjunction) of two fuzzy sets is completely defined by the membership function values of the fuzzy sets in that point only. As a result, the pointwise defined fuzzy set connectives do not make a distinction when genuinely different fuzzy sets have the same membership function value in a certain point. Amongst others, such fuzzy set connectives can not be both idempotent and satisfy the laws of excluded middle and contradiction at the same time. As such, these fuzzy set connectives are not completely satisfactory to define conjunction and disjunction of fuzzy sets representing propositions of a quantum entity, since meet and join are idempotent and do satisfy the law of excluded middle and contradiction. In an attempt to solve this problem for fuzzy sets representing propositions of classical systems, Buckley and Siler [58, 59, 60] proposed fuzzy set connectives (*i.e.*, conjunction and disjunction) parametrized by a correlation coefficient between the two fuzzy sets such that the law of contradiction, excluded middle and idempotency hold. In the tenth article of the book, 'B. D'Hooghe and J. Pykacz, *Buckley-Siler connectives for quantum logics of fuzzy sets*, the Buckley-Siler approach is generalized to the case of a quantum entity and illustrated on a fuzzy set representation of the spin properties of a spin-1/2 particle [61].

The eleventh article of the book, 'W. Christiaens, *Some notes on Aerts' interpretation of the EPR-paradox and the violation of Bell-inequalities*' studies the Einstein Podolsky Rosen type of paradoxes [67] in the light of the approaches that we have exposed in the foregoing sections. Cartwright's model for the violation of Bell's inequalities [68] is investigated and compared with Aerts's model [69]. It is shown that a causal view can be advanced for a situation of nonlocality in quantum mechanics if one of the basic assumptions about reality, namely that a physical entity is always present inside space, is relaxed. Also the creation discovery view [69], where it is taken for granted that an experiment on a physical entity contains two fundamental aspects, the discovery of an existing part of reality and the creation of new part of reality, is investigated from a philosophical point of view.

This brings us to the next point, quantum cryptography. Quantum cryptography is the most efficient application of the fundamental and experimental current of research centered around the interpretational problems of quantum mechanics that was developed during the last decades. It is a combination of deep physical insight, new technology, and ingenious reflection. It is highly representative of the influence that could have quantum mechanics on tomorrow's technology and ... last but not least, it works [70]!

An essential difference between classical theories and the quantum theory is the fact that in the latter the influence of the observer (of the apparatus, of the whole measurement context) cannot be neglected, and, moreover, can have dramatic consequences on the properties of the system under study, an idea that is central in the hidden measurement approach as well as in the Copenhagen interpretation. The complementary principle is a direct illustration of this non-classical feature: when observables do not commute (such as position and momentum for instance), it is often impossible to measure them simultaneously without dispersion, and, in general, the dispersions of the outcomes obtained during their individual measurements obey uncertainty relations. Complementarity is exploited in quantum cryptography [71] where, instead of considering uncertainty relations as a negative limitation for the users (traditionally called Alice and Bob), they appear to be useful because they allow Alice and Bob to reveal the presence of a third party (Eve) that would eavesdrop the signal exchanged between them and/or to limit her knowledge of this signal. In order to do so, it is necessary that the signal is encoded in complementary variables (non-commuting bases). In the twelfth article of the book, 'T. Durt and B. Nagler, *Quantum cryptographic encryption in three complementary bases through a Mach-Zehnder set up*', a protocol for quantum key distribution is described in which it is shown that the wave-particle complementarity plays a fundamental role; this complementarity is also related to the complementarity between position and momentum if we consider position to be a corpuscular property and momentum (which is related to de Broglie wave-length) to be an undulatory property.

It is worth noting that, although it is possible in principle to prepare and to measure one photon in a given position (or to emit and to detect one photon in a given temporal window), technologically, the problem is not solved yet. This is due to the fact that at our scale, when large amount of photons are present, they often behave as (classical) waves, and that it is not so easy to reveal their corpuscular properties. Detectors based on the photo-electric effect exploit such properties, but they are not very efficient when few photons are present (this is related to the aforementioned efficiency loophole). Similarly, we are not able yet, today, to produce on request a single photon state. These limitations suggested a semi-classical (or semi-quantum) protocol for key distribution in which the information is encoded in corpuscular properties and in which technological limitations play the same role as quantum uncertainties in quantum cryptography. This protocol is described in the thirteenth article of the book, 'T. Durt, *Quantum cryptography without quantum uncertainties*'. It is a direct illustration of the hidden measurement approach in which unavoidable fluctuations characterize the interaction between the observer and the physical world. The stochasticity of these contextual fluctuations can be exploited in order to send a secure cryptographic key. The semi-classical protocol fills the gap between quantum cryptography and classical techniques in which the message is hidden among a huge random noise. Considered so it is a mesoscopic protocol, that belongs to a framework more general than the standard quantum one, that contains also the classical framework as a limiting case. It is even, more than an illustration, an application of the hidden measurement approach.

If linear quantum mechanics is a special case of a nonlinear theory then there must be some freedom in what is nowadays understood as *canonical quantization*. The papers that follow are related to the problem of quantization. One of the problems that remains unsolved in great part is how to classify equations which are physically admissible. Here different criteria may be introduced, based on locality, positivity, or integrability. A class of candidate equations selected on the basis of positivity, locality, and probabilistic requirements was introduced in [72]. All these equations were of the form $i\dot{\rho} = [H, f(\rho)]$ where $[f(\rho), \rho] = 0$. Their physically nontrivial solutions were found in [73] for $f(\rho) = \rho^2$ and recently generalized to other f's in [74]. A link of such general f's with nonextensive statistical mechanics was described in [75]. More general classes of physically interesting von

Neumann-type equations derived from multiple Nambu-type brackets were introduced in [76].

The question of integrability is always a difficult one. One of the reasons is that it is not completely clear what should be actually meant by this notion. The definition which is very useful is that integrability means practical integrability by means of soliton techniques. The following articles contain new results on integrability of generalized von Neumann equations. The fourteenth article of the book, 'J. L. Cieśliński, *How to construct Darboux-invariant equations of von Neumann type*', generalizes earlier results of [77] to a large class of Darboux transformations. Essentially the same family of equations is treated in the fifteenth article of the book, 'M. Czachor, N. V. Ustinov, *Darboux-integrable equations with non-Abelian nonlinearities*' along the lines of [77]. The two articles use different mathematical constructions showing two different aspects of Darboux-covariance of the same set of equations. Strictly speaking one has to admit that it is not fully clear to what extent the two approaches are equivalent but all the explicit examples given in the papers can be formulated in either of the two ways.

The sixteenth article of the book, 'S. Leble, *Dressing chain equations associated with difference soliton systems*', employs still another variant of the Darboux transformation: The chain equations. The elements which are common in all the three papers are: The use of Darboux-covariant Lax pairs, representation of evolution equations by compatibility conditions, the presence of non-Abelian nonlinearities, and the well known Nahm system as a particular example. The latter shows also that the notion of a generalized von Neumann-type equation covers here a very large class of nonlinear evolution equations extending far beyond the standard formalism of linear quantum mechanics.

The collection of articles on aspects of generalized quantization is completed by the seventeenth article of the book, 'M. Kuna, J. Naudts, *Covariance approach to the free photon field*'. The authors start with the notion of a generalized covariance system and a generalized GNS construction, an approach which follows their earlier results published in [78, 79]. The generality inherently present in their formalism is here purposefully restricted in order to rederive the standard Fock space representation of free electromagnetic fields. However, the results of [78, 79] show that the formalism they propose is flexible enough to incorporate also various non-canonical systems such as those based on non-commutative spacetime [80] or non-canonical vacua [81].

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