# Linearity and compound physical systems: the case of two separated spin 1/2 entities*

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**Abstract**

We illustrate some problems that are related to the existence of an underlying linear structure at the level of the property lattice associated with a physical system, for the particular case of two explicitly separated spin 1/2 objects that are considered, and mathematically described, as one compound system. It is shown that the separated product of the property lattices corresponding with the two spin 1/2 objects does not have an underlying linear structure, although the property lattices associated with the subobjects in isolation manifestly do. This is related at a fundamental level to the fact that separated products do not behave well with respect to the covering law (and orthomodularity) of elementary lattice theory. In addition, we discuss the orthogonality relation associated with the separated product in general and consider the related problem of the behavior of the corresponding Sasaki projections as partial state space mappings.

## 1 Introduction

In another contribution in this volume [1], we have given an overview of a general mathematical framework, known under several names, that can

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be used for the description of physical systems in general and compound physical systems in particular. This framework was developed in its most important aspects in Geneva and Brussels [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. One of the characteristics of this approach is the fact that the basic, primitive elements of the formalism have a sound realistic and operational interpretation. Indeed, a physical entity is described by means of its states, and the experimental projects which can be performed on samples of this system. Additional structure is gradually introduced as a series of physical postulates or mathematical axioms, ranging from the physically very plausible to axioms of an admittedly more technical nature, the latter introduced with the aim of bringing the structure closer to standard classical and quantum physics. We want to emphasize the generality of such an axiomatic approach and the fact that the results are valid in general, independently of the particularities of the formalism.

It has been shown that two of the more technical of these axioms — that are definitely satisfied for standard quantum systems — are not valid in the mathematical model that results from these general prescriptions for a compound physical system that consists of two operationally separated quantum objects [6, 7, 10]. One of the two failing axioms is equivalent with the linearity of the set of states for a quantum entity, hence with the superposition principle.

One of the themes of this book is to investigate how the failure of this "linearity" axiom is related to other perspectives on the problem of a "non-linear" quantum mechanics. In this paper we want to apply our axiomatic approach to the particular case of two separated spin 1/2 objects that are described as a whole. According to standard quantum physics, an isolated spin 1/2 system can be mathematically represented by the complex Hilbert space $\mathbb{C}^2$. More precisely, its set of possible states corresponds with the collection of all one-dimensional subspaces (rays) in this space, and observables with (some of the) self-adjoint operators on $\mathbb{C}^2$. The advantage is that for this relatively simple situation we can not only explicitly construct a mathematical model, but also keep an eye on the physical meaning of the mathematical objects and understand why the linearity axiom of standard quantum mechanics fails, at least in this case.

Let us give a brief overview of the basic ideas of the approach. In the next section, these ideas will become more clear, when we apply them to a particular example, the spin part of a single spin 1/2 object, *in extenso*. According to the prescriptions of the axiomatic approach, one should first construct the property lattice $\mathcal{L}$ and set of (pure) states $\Sigma$ associated with the physical system under investigation, reflecting an underlying program of realism that is pursued [4]. In general, the state space is an orthogonality space,[1] while the property lattice, which is constructed from a class of *yes/no-experiments*, is always a complete atomistic lattice, usually taken to be orthocomplemented as well [8]. The connection between both structures is given by the Cartan map

$$\kappa : \mathcal{L} \to \mathcal{P}(\Sigma) : a \mapsto \{p \in \Sigma \mid p \lhd a\} \tag{1}$$

where $\lhd$ implements the physical idea of actuality of $a$, if the physical system is in a state $p$. The Cartan map is always a meet-preserving unital injection, hence $\mathcal{L} \cong \kappa[\mathcal{L}] \subseteq \mathcal{P}(\Sigma)$, leading to a state space representation of the property lattice. In addition, denoting the collection of all atoms in $\mathcal{L}$ by $\Sigma_{\mathcal{L}}$, we have $\kappa[\Sigma_{\mathcal{L}}] = \{\{p\} \mid p \in \Sigma\} \cong \Sigma$, hence we can identify these two sets, which we will often do. From a physical perspective, this relation reflects the fact that a physical state should embody a maximal amount of information at the level of the property lattice $\mathcal{L}$, even for individual samples of the physical system. In the axiomatic approach, a prominent role is played by the collection of biorthogonally closed subsets $\mathcal{F}(\Sigma) = \{A \subseteq \Sigma \mid A = A^{\perp\perp}\}$ of $\Sigma$. Indeed, the orthocomplementation can be introduced under the form of two axioms, which imply that $\kappa[\mathcal{L}] \subseteq \mathcal{F}(\Sigma)$ and $\kappa[\mathcal{L}] \supseteq \mathcal{F}(\Sigma)$, respectively. This *state-property duality* lies at the heart of the axiomatic approach [10, 11].

Using this general framework, one of the basic aims is to establish a set of additional specific axioms, free from any probabilistic notions at its most basic level, to recover the formalism of standard quantum physics. Therefore, this approach is a theory of individual physical systems, rather than statistical ensembles. In doing so, a general theory is developed not only for quantal systems, but that also incorporates classical physical systems. The classical parts of a physical system are mathematically reflected in a decomposition of the property lattice in irreducible components [5, 6, 7, 12]. For a genuine quantum system then, that satisfies all the requirements put forward in [5] and [6, 7], the celebrated representation theorem of Piron states that these property lattices can be represented in a suitable generalized Hilbert (or orthomodular) space. More precisely, he showed that every irreducible complete atomistic orthocomplemented lattice $\mathcal{L}$ of length $\geq 4$ that is orthomodular and satisfies the covering law (sometimes called a Piron lattice), can be represented as the collection of all closed subspaces $\mathcal{L}(\mathcal{H})$ of

[1] An orthogonality space consists of a set $\Sigma$ and an orthogonality relation $\perp$, that is, a relation that is anti-reflexive and symmetric. One writes $A^{\perp} = \{q \in \Sigma \mid q \perp p \text{ for all } p \in A\}$, for $A \subseteq \Sigma$.

an appropriate orthomodular space $\mathcal{H}$ [2]. Mathematically speaking, there then exists a c-isomorphism $\mathcal{L} \cong \mathcal{L}(\mathcal{H})$.[2] The physical motivation for this particular lattice structure comes mainly from realistic and operational considerations. At first sight, the mathematical demands of orthomodularity and covering law look rather technical. They are usually justified by taking a more active (and ideal) point of view with respect to the physical meaning of the elements in the property lattice (for an overview, see [13]).

## 2 A Single Spin 1/2 System

To illustrate the physical meaning of these mathematical considerations, we shall treat some relatively simple particular cases *in extenso*. First, we illustrate the construction of the property lattice and state space for the spin part of a single spin 1/2 physical system. Denote the collection of possible states or, alternatively, preparations, for such a physical system by $\Sigma$. As we have seen, empirical access to the physical system is formalized by a set of yes/no-experiments $\mathcal{Q}$, and we proceed with an investigation of $\mathcal{Q}$, which will correspond with Stern-Gerlach experiments.

More precisely, for each spatial direction, a non-trivial definite experimental project is associated with a Stern-Gerlach experiment in that direction, relative to some reference direction; $\alpha_{\theta,\phi}$ denotes the experimental project associated with such an experiment in the direction given by $(\theta, \phi)$, with the following prescription for the attribution of results, if the experiment is properly conducted on a particular sample of the physical system:

> Attribute the positive result (outcome "yes") if the spin 1/2 object is detected at the upper position; otherwise, attribute a negative result (outcome "no").

The collection of all yes/no-experiments will be denoted by $Q$. Consequently, at this point

$$Q \supseteq \{\alpha_{\theta,\phi} \mid 0 \leq \theta < \pi,\ 0 \leq \phi < 2\pi\} \tag{2}$$

The states of the spin 1/2 particle are the spin states $p(\theta, \phi)$ in the different spatial directions:

$$\Sigma = \{p(\theta, \phi) \mid \ 0 \leq \theta < \pi,\ 0 \leq \phi < 2\pi\} \tag{3}$$

One of the fundamental ingredients of any physical theory is linked with the following somewhat imprecise statement:

[2] A unital c-morphism between two complete ortholattices is a mapping that preserves arbitrary joins and orthocomplements.

The yes/no-experiment $\alpha$ gives with certainty the outcome "yes" whenever the sample object happens to be in a state $p$.

This statement will be expressed symbolically by a binary relation between the set of states and the class of yes/no-experiments. More precisely, the connection between the experimental access to the physical system and physical reality itself can be formalized by a binary relation $\triangleleft \subseteq \Sigma \times Q$. This relation symbolizes the following idea: $p \triangleleft \alpha$ means that if the physical system is (prepared) in a state $p$, the positive result for $\alpha$ would be obtained, should one execute the yes/no-experiment. In this case, the yes/no-experiment is said to be *true* for the object, if it is in the state $p$. It is conceptually important to note the counterfactual locution. Indeed, this formulation will allow us to attribute many properties to a particular sample of a physical system. The binary relation induces in a natural way a map, which is intimately related to the Cartan map:

$$S_T : Q \to \mathcal{P}(\Sigma) : \alpha \mapsto \{p \in \Sigma \mid p \triangleleft \alpha\} \tag{4}$$

For the spin 1/2 particle it is an experimental fact that $p(\theta, \phi) \triangleleft \alpha_{\theta', \phi'}$ iff $(\theta, \phi) = (\theta', \phi')$. There is no relation $\triangleleft$ between $p(\theta, \phi)$ and $\alpha_{\theta', \phi'}$ when $(\theta, \phi) \neq (\theta', \phi')$.

$Q$ is naturally equipped with an inversion relation

$$\widetilde{\ } : Q \to Q : \alpha \mapsto \widetilde{\alpha} \tag{5}$$

the yes/no-experiment $\widetilde{\alpha}$ has by definition the same experimental set-up as $\alpha$, but the positive and negative alternatives are interchanged. This means that $p \triangleleft \widetilde{\alpha}$ if the yes/no-experiment $\alpha$ gives with certainty the outcome "no" whenever the state of the physical entity is $p$. One then has the induction of a natural, physically motivated pre-order structure on $Q$:

$$\alpha < \beta \text{ iff } S_T(\alpha) \subseteq S_T(\beta) \tag{6}$$

which is used to generate the property lattice. Indeed, it is natural to call two yes/no-experiments equivalent if they cannot be distinguished experimentally, that is, $\alpha \approx \beta$ iff $S_T(\alpha) = S_T(\beta)$ iff $p \triangleleft \alpha \Leftrightarrow p \triangleleft \beta$. For a quantum spin 1/2 particle, it is well known that, according to experiment, one has

$$\widetilde{\alpha}_{\theta,\phi} \approx \alpha_{\pi-\theta,\phi+\pi} \tag{7}$$

At this moment, we have made $Q$ into a pre-ordered class, with some sort of an inversion relation. There is a fundamental operation which associates with any collection of yes/no-questions a new yes/no-experiment. Thus, the set of yes/no-experiments should be closed under products. More formally, we have an operation

$$\Pi : \mathcal{P}(Q) \to Q : \{\alpha_j \mid j \in J\} \mapsto \Pi\{\alpha_j \mid j \in J\} \tag{8}$$

The experimental procedure for this yes/no-experiment consists in choosing randomly one of the $\alpha_i$ and executing the associated experiment. With this specification, we obviously have

$$\widetilde{\Pi}\{\alpha_j \mid j \in J\} = \Pi\{\widetilde{\alpha}_j \mid j \in J\} \tag{9}$$

which appears somewhat strange at first, and its misunderstanding has been a point of some dispute in the past. In fact, this clever definition of product experiments allows us to attribute unambiguously various different properties to (some preparation of) a particular physical system, without having to explicitly test for all properties on the same object [6]. According to our prescriptions the binary relation $\triangleleft$ should satisfy $p \triangleleft \Pi\{\alpha_j \mid j \in J\} \Leftrightarrow p \triangleleft \alpha_j$ for all $j \in J$, or equivalently

$$S_T(\Pi\{\alpha_j \mid j \in J\}) = \bigcap\{S_T(\alpha_j) \mid j \in J\} \tag{10}$$

For example, for a spin 1/2 particle it is experimentally known that

$$\Pi(\{\alpha_{\theta,\phi}, \alpha_{\theta',\phi'}\}) \approx \widetilde{\tau} \tag{11}$$

unless $(\theta, \phi)$ and $(\theta', \phi')$ represent the same spatial directions.

Finally, there exist *trivial* yes/no-experiments $\tau$ and $\widetilde{\tau}$. A possible experimental procedure for $\tau$ would consist in doing nothing with the physical system under consideration and always give the positive result. Both yes/no-experiments are in some sense ideal elements, and can be viewed as being added for technical reasons.

The equivalence relation $\approx$ on $Q$ partitions $Q$ in the collection of equivalence classes, according to a standard argument. Moreover, the pre-ordered structure on $Q$ collapses into a partial order on $\mathcal{L} := Q/\approx = \{[\alpha] \mid \alpha \in Q\}$, with $[\alpha]$ denoting the equivalence class of $\alpha$; $S_T$ lifts to the Cartan map $\kappa$, and $\mathcal{L}$ can be mentally put in a one-to-one correspondence with the set of properties or elements of reality of the physical system, in a sense derived from that of Einstein, Podolsky and Rosen [14]. Moreover, it is not very difficult to show that $\mathcal{L}$ becomes a complete lattice [5], with

$$\bigwedge\{[\alpha_j] \mid j \in J\} = [\Pi\{\alpha_j \mid j \in J\}] \tag{12}$$

Given the physical meaning of the equivalence relation, one can unambiguously state that a property is *actual* if one of its corresponding yes/no-experiments is true. It is the lattice $\mathcal{L}$ that we use to describe the properties of a physical system. Note that $\mathcal{L}$, like any complete lattice, always contains a maximal element $I = [\tau]$ and a minimal element $0 = [\widetilde{\tau}]$.

From now on, we will denote equivalence classes $[\alpha]$ by $a$. For our particular example, we put $a(\theta, \phi) := [\alpha_{\theta,\phi}]$, to make the distinction very clear.

Also the binary relation $\triangleleft$ lifts to the level of the property lattice $\mathcal{L}$. With some abuse of notation, we then have $p \in \kappa(a) \Leftrightarrow p \triangleleft a$. The physical interpretation is the following: $p \in \kappa(a)$ stands for "The property $a$ is actual if the physical system is in a state $p$". For our example, we have $\kappa(a(\theta, \phi)) = \{p(\theta, \phi)\}$.

In a complete lattice, any subcollection of elements has a join or supremum. For a set of properties $([\alpha_i])_i \in \mathcal{L}$ the join can be defined in a purely mathematical way as follows:

$$\bigvee\{[\alpha_j] \mid j \in J\} = \bigwedge\{[\beta] \mid \alpha_j < \beta \text{ for all } j \in J\} \tag{13}$$

It is for this reason that the join of a collection of properties has no obvious physical interpretation.[3]

Let us reconsider our example. Identifying, with some abuse of terminology, the properties with their corresponding equivalence classes $a(\theta, \phi) = [\alpha_{\theta,\phi}]$, it is true that

$$a(\theta, \phi) \wedge a(\theta', \phi') = 0 \tag{14}$$

if $(\theta, \phi)$ and $(\theta', \phi')$ represent different spatial directions, since $\Pi\{\alpha_{\theta,\phi}, \alpha_{\theta',\phi'}\} \approx \widetilde{\tau}$. Indeed, there are no preparations for a spin 1/2 system in which both properties can be actual at the same time.

For our example, it is an experimental fact that if we consider two yes/no-experiments $\alpha_{\theta,\phi}$ and $\alpha_{\theta',\phi'}$ where $(\theta, \phi)$ and $(\theta', \phi')$ represent different spatial directions, there is no third yes/no-experiment of the type $\alpha_{\theta'',\phi''}$ such that $\alpha_{\theta,\phi} < \alpha_{\theta'',\phi''}$ and $\alpha_{\theta',\phi'} < \alpha_{\theta'',\phi''}$. This proves that the only yes/no-experiment $\beta$ such that $\alpha_{\theta,\phi} < \beta$ and $\alpha_{\theta',\phi'} < \beta$ is $\tau$. Hence for $(\theta, \phi) \neq (\theta', \phi')$ we have:

$$a(\theta, \phi) \vee a(\theta', \phi') = I \tag{15}$$

[3]We remark that the meet operation is equivalent to logical conjunction. The "join" operation is however in general not equivalent to the "or" operation of logic. For classical physical entities the "join" operation is equivalent to logical disjunction, but this is not the case for quantum entities. This fact is at the origin of the common use of the word "quantum logic" for the lattice structure that arises in this way.

At this moment, we have found from operational considerations all the structural ingredients to define the basic mathematical structure attributed to the compound system that consists of two (operationally) separated spin 1/2 particles. This structure consists in a triple $(\Sigma, \mathcal{L}, \kappa)$ or $(\Sigma, \mathcal{L}, \triangleleft)$, that we have called a state-property system elsewhere [15, 16]. The elements of $\Sigma$ are the states attributed to the physical system under investigation, the elements of $\mathcal{L}$ correspond with its possible properties, and the connection between both sets is given by a Cartan map or, equivalently, a suitable binary relation, as we have seen.

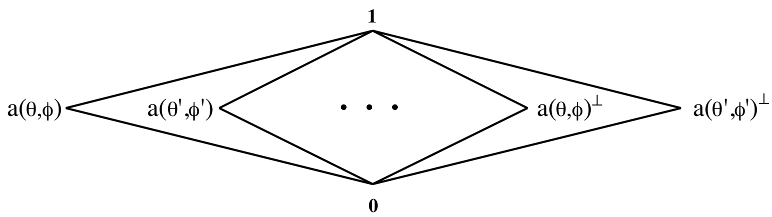
$$\Sigma = \{p(\theta, \phi) \mid 0 \leq \theta < \pi,\ 0 \leq \phi < 2\pi\} \tag{16}$$

$$\mathcal{L} = \{a(\theta, \phi) \mid 0 \leq \theta < \pi,\ 0 \leq \phi < 2\pi\} \cup \{0\} \cup \{I\} \tag{17}$$

$$\kappa(0) = \emptyset, \quad \kappa(a(\theta, \phi)) = \{p(\theta, \phi)\}, \text{ and } \kappa(I) = \Sigma \tag{18}$$

In particular, note that $\kappa$ maps atoms in $\mathcal{L}$ to singletons in $\Sigma$, and that $\kappa$ gives a state space representation of $\mathcal{L}$.

The (infinite) property lattice $\mathcal{L}$ for a single spin 1/2 object can be visually displayed by giving its Hasse diagram:




In the axiomatic approach, two states are defined to be *orthogonal* if there exists a yes/no-experiment $\alpha \in Q$ such that $p \triangleleft \alpha$ and $q \triangleleft \widetilde{\alpha}$. For our example, only one state will be orthogonal to a given state $p(\theta, \phi)$, being the state $p(\pi - \theta, \phi + \pi)$. In this way, $\Sigma$ becomes an orthogonality space.

It is an experimental fact that $a(\theta, \phi)$ is never a *classical* property.[4] Indeed, if we prepare a spin 1/2 object in a state that corresponds to a direction orthogonal to $(\theta, \phi)$, then neither $\alpha_{\theta,\phi}$ nor $\widetilde{\alpha}_{\theta,\phi}$ is true.

For the sake of illustration, let us also consider a second measurement scheme that would be deemed equivalent with $\alpha_{\theta,\phi}$ according to the axiomatic approach. Let $\beta_{\theta,\phi}$ have the same experimental arrangement, except for the fact that the bottom channel is blocked by a suitable absorbing

[4] A "property" $a = [\alpha]$ is said to be classical, if for any state of the physical system either $\alpha$ or $\widetilde{\alpha}$ is true.

device, in order to prevent an object to be localized below. It is experimentally known that $S_T(\alpha_{\theta,\phi}) = S_T(\beta_{\theta,\phi})$, hence $\alpha_{\theta,\phi} \approx \beta_{\theta,\phi}$. Observe also that $\Pi\{\alpha_{\theta,\phi}, I\} \approx \alpha_{\theta,\phi}$, but $\widetilde{\Pi}\{\alpha_{\theta,\phi}, I\} \not\approx \widetilde{\alpha}_{\theta,\phi}$.

The connection between the axiomatic approach and the standard quantum mechanical description of the spin part of a single spin 1/2 particle, is given by the well known (and easy to see) fact that this lattice can be represented as the collection of all closed subspaces of $\mathbb{C}^2$, that is, $\mathcal{L}(\mathbb{C}^2)$, with

$$a(\theta, \phi) \mapsto [(\exp(-i\frac{\phi}{2})\cos\frac{\theta}{2}, \exp(i\frac{\phi}{2})\sin\frac{\theta}{2})] \tag{19}$$

Note that this mapping indeed preserves the orthogonality relation.

For a single spin 1/2 object, we thus have a relatively simple property lattice, in which all non-trivial elements are also representatives of (pure) states. Denoting the collection of one-dimensional subspaces of the Hilbert space $\mathcal{H} = \mathbb{C}^2$ by $\Sigma_\mathcal{H}$, we can also put $\Sigma \cong \Sigma_\mathcal{H} \cong \mathbb{C}P^1$, this last set being complex projective 1-space.

Once one has arrived at the basic structure of a state-property system, the axiomatic approach proceeds by introducing further axioms on this structure, with the aim of bringing the structure closer to standard quantum mechanics. It is an easy task to verify that all the axioms, as stated in [1], are satisfied for the property lattice displayed above. In the next section, however, we will give an explicit example in which the axioms of orthomodularity and the covering law both fail.

# 3 The Separated Product of Two Spin 1/2 Systems

One of the easiest compound physical systems that intuitively and conceptually presents itself, is the case of two separated spin 1/2 objects that are described as one whole. Consider two such systems, respectively represented by property lattices $\mathcal{L}_i(\mathbb{C}^2)$, for $i = 1, 2$, and suppose that we want to give a mathematical description for this situation. In this section, we will explicitly construct the property lattice and state space that corresponds to this physical situation.

In general, the separated product — the mathematical description of this situation — $\mathcal{L}_1 \circledwedge \mathcal{L}_2$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ can be constructed in two different ways. First, one can give an explicit construction from the bottom up, starting from the collection of yes/no-experiments for this system. This construction has the advantage that every property corresponds to an equivalence class of experimental projects, so in principle one has at one's disposal an experimental procedure that tests for any property. Second, the separated product can be mathematically generated through a biorthocomplementation procedure, starting from the orthogonality space $(\Sigma_1 \times \Sigma_2, \perp)$, with the orthogonality relation given by

$$(p_1, p_2) \perp (q_1, q_2) \text{ iff } (p_1 \perp_1 q_1 \text{ or } p_2 \perp_2 q_2) \tag{20}$$

This construction is more convenient from a mathematical point of view, but has the drawback that it is a purely formal construction, which needs an *a posteriori* physical interpretation. Here, we will give an overview of the first approach, at least for the particular case that is the main subject of this paper. For a more detailed exposition of the general case, we refer to [6, 7, 10].

First, we should be slightly more specific about what we mean with two objects being separated. Intuitively speaking, a necessary operational condition should be the following: it should be possible to devise an experimental procedure, say $e_1 \times e_2$, with outcome set $O_{e_1} \times O_{e_2}$, on the compound system as a whole for every pair of experiments $(e_1, e_2)$, with $e_1$ an experiment with outcome set $O_{e_1}$ on the first object and similarly for $e_2$. Moreover, whatever experiment we decide to perform on one of the objects, should yield a result that is independent of the state of the other object and *vice versa*. That is, if the compound system is in a state such that $(x_1, x_2)$ is a possible outcome for the experiment $e_1 \times e_2$, then the first object is in a state such that $x_1$ is a possible result for the experiment $e_1$, and similarly for the second object. In addition, any experiment corresponding to one of the subobjects, can be executed independent of the presence or absence of the other subobject. Moreover, if an outcome is possible for an experiment $e_1$ to be performed on the first object, then this outcome can be obtained irrespective of the presence or absence of the other object. Note that this operational idea of separation is closely related to the notion presented by Einstein, Podolsky and Rosen [14]. Also, note that there is a big conceptual difference between the physical notions of *separation* and *interaction*, the latter notion being related to the causal structure of physical reality.

As before, we will mainly restrict ourselves to spin measurements on a spin 1/2 object, because in this case any experiment on one of the subobjects has only two possible results. On the other hand, an arbitrary experiment of the form $\alpha_1(\theta_1, \phi_1) \times \alpha_2(\theta_2, \phi_2)$ on the compound physical system has 4 possible outcomes: $(y, y), (y, n), (n, y)$ and $(n, n)$, where for notational reasons we have slightly adapted our notation. According to the prescriptions of the axiomatic approach, we have to construct the collection of yes/no-experiments associated with all these product experiments. First, observe that product experiments which have at least one component equivalent with a trivial experiment on the corresponding subobject, are equivalent with either a trivial experiment on the compound system, or an experiment which only involves one of the subobjects. For example, $\alpha_1(\theta_1, \phi_1) \times \tau_2$ is true iff $\alpha_1(\theta_1, \phi_1)$ is true, with respect to the first subobject. To make the distinction, we will put $C\alpha_1$ to be the experimental project on the compound system that consists in performing the experiment corresponding to $\alpha_1$ on the first subobject, and a similar convention for the second subobject.

Thus, let us consider a product experiment of the form $\alpha_1(\theta_1, \phi_1) \times \alpha_2(\theta_2, \phi_2)$, which has 4 possible outcomes. With this product experiment, one can *a priori* associate $2^4$ different yes/no-experiments, corresponding to all subsets of the outcome set. The two trivial subsets are equivalent with trivial yes/no-experiments on the compound system, hence will be left out of the rest of the discussion. Temporarily abbreviating $\alpha_i(\theta_i, \phi_i)$ by $\alpha_i$ and $\alpha_i(\pi - \theta_i, \phi_i + \pi)$ by $\widetilde{\alpha}_i$ for a particular direction $(\theta_i, \phi_i)$, we will use the following conventional notations for yes/no-experiments associated with subsets of a particular form, displayed in a well-organized table form below. At the same time, we indicate the corresponding inverse yes/no-experiments:

| Notation | Outcome set | Inverse yes/no-experiment |
|---|---|---|
| $C\alpha_1$ | (y,y), (y,n) | $C\widetilde{\alpha}_1$ |
| $C\alpha_2$ | (y,y), (n,y) | $C\widetilde{\alpha}_2$ |
| $\alpha_1 \triangle \alpha_2$ | (y,y) | $\widetilde{\alpha}_1 \nabla \widetilde{\alpha}_2$ |
| $\alpha_1 \nabla \alpha_2$ | (y,y), (y,n), (n,y) | $\widetilde{\alpha}_1 \triangle \widetilde{\alpha}_2$ |
| $\alpha_1 \Theta \alpha_2$ | (y,y),(n,n) | $\widetilde{\alpha}_1 \Theta \alpha_2 \approx \alpha_1 \Theta \widetilde{\alpha}_2$ |

Observe that the notation $C\widetilde{\alpha}_j$ is unambiguous, in the sense that we have $(C\widetilde{\alpha}_j) \approx C(\widetilde{\alpha}_j)$. Considering yes/no-experiments of this general form, we can generate all yes/no-experiments associated with the product experiment $\alpha_1(\theta_1, \phi_1) \times \alpha_2(\theta_2, \phi_2)$. For example, a yes/no-experiment that tests for the result $(n, n)$ could be constructed as $\alpha_1(\pi - \theta_1, \pi + \phi_1) \triangle \alpha_2(\pi - \theta_2, \pi + \phi_2)$. Indeed, this yes/no-experiment would be true if the compound system is in a state such that $\alpha_1(\pi - \theta_1, \pi + \phi_1)$ is true with respect to the first subobject, and $\alpha_2(\pi - \theta_2, \pi + \phi_2)$ is true with respect to the second subobject.

In this way, we can obtain all properties for the compound system that consists of two separated spin 1/2 objects, and we can direct our attention towards the construction of the property lattice. Because of the demand that both subobjects are separated, we have

$$p \lhd C\alpha_1(\theta_1, \phi_1) \text{ iff } p_1 \lhd_1 \alpha_1(\theta_1, \phi_1) \tag{21}$$

$$p \lhd C\alpha_2(\theta_2, \phi_2) \text{ iff } p_2 \lhd_2 \alpha_2(\theta_2, \phi_2) \tag{22}$$

$$p \lhd \alpha_1(\theta_1, \phi_1)\triangle\alpha_2(\theta_2, \phi_2) \text{ iff } p_1 \lhd_1 \alpha_1(\theta_1, \phi_1), \text{ and } p_2 \lhd_2 \alpha_2(\theta_2, \phi_2) \tag{23}$$

$$p \lhd \alpha_1(\theta_1, \phi_1)\nabla\alpha_2(\theta_2, \phi_2) \text{ iff } p_1 \lhd_1 \alpha_1(\theta_1, \phi_1) \text{ or } p_2 \lhd_2 \alpha_2(\theta_2, \phi_2) \tag{24}$$

$$\begin{aligned} p \lhd \alpha_1(\theta_1, \phi_1)\Theta\alpha_2(\theta_2, \phi_2) \text{ iff either } & p_1 \lhd_1 \alpha_1(\theta_1, \phi_1) \text{ and } p_2 \lhd_2 \alpha_2(\theta_2, \phi_2), \\ \text{or } & p_1 \lhd_1 \alpha_1(\pi - \theta_1, \pi + \phi_1) \text{ and } p_2 \lhd_2 \alpha_2(\pi - \theta_2, \pi + \phi_2) \end{aligned} \tag{25}$$

It is not very difficult to see from these prescriptions that the state of the global system is completely known whenever one knows the states of the two separated spin 1/2 objects that make up the compound system. Consequently, the set of states can be taken as $\Sigma_1 \times \Sigma_2$, with

$$\Sigma_1 = \{p_1(\theta_1, \phi_1) \mid 0 \leq \theta_1 < \pi,\ 0 \leq \phi_1 < 2\pi\} \tag{26}$$

$$\Sigma_2 = \{p_2(\theta_2, \phi_2) \mid 0 \leq \theta_2 < \pi,\ 0 \leq \phi_2 < 2\pi\} \tag{27}$$

However, for notational reasons we shall often use abbreviations of the form $p_j$ for a general element of $\Sigma_j$.

Consequently, we can represent the property lattice corresponding to this situation as a subcollection of $\mathcal{P}(\Sigma_1 \times \Sigma_2)$. Properties of the first kind would be represented by singletons $(p_1(\theta_1, \phi_1), p_2(\theta_2, \phi_2))$; properties of the second kind consist of all sets of the general form $\{p_1(\theta_1, \phi_1)\} \times \Sigma_2 \cup \Sigma_1 \times \{p_2(\theta_2, \phi_2)\}$; finally, properties of the third kind are two-element sets of the general form $\{(p_1(\theta_1, \phi_1), p_2(\theta_2, \phi_2)), (p_1(\pi - \theta_1, \pi + \phi_1), p_2(\pi - \theta_2, \pi + \phi_2))\}$. The full property lattice is then generated by taking arbitrary products of corresponding yes/no-experiments, which amounts to taking intersections of the corresponding images of the Cartan map, as we have seen before, because the Cartan map is one-to-one and preserves intersections, due to the corresponding property of the map $S_T$. Denoting $S^2 := [0, \pi) \times [0, 2\pi)$, we then obtain for the full property lattice corresponding to this physical situation the intersection system $\mathcal{I}(\Omega)$ generated by the collection of all these sets $\Omega$, and it is geometrically clear, by considering the set $\Sigma_1 \times \Sigma_2$, that the second equation also holds:

$$\begin{aligned} \mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2 &\cong \mathcal{I}(\mathcal{T} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{S} \cup \mathcal{U} \cup \mathcal{B}) \quad (28) \\ &\cong \mathcal{T} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{S} \cup \mathcal{U} \cup \mathcal{P} \quad (29) \end{aligned}$$

with

$$\mathcal{T} = \{\emptyset, \Sigma_1 \times \Sigma_2\}$$

$$\mathcal{A}_1 = \{\{p_1(\theta_1, \phi_1)\} \times \Sigma_2 \mid (\theta_1, \phi_1) \in S^2\}$$

$$\mathcal{A}_2 = \{\Sigma_1 \times \{p_2(\theta_2, \phi_2)\} \mid (\theta_2, \phi_2) \in S^2\}$$

$$\mathcal{S} = \{(p_1(\theta_1, \phi_1), p_2(\theta_2, \phi_2)) \mid (\theta_i, \phi_i) \in S^2\}$$

$$\mathcal{U} = \{\{p_1(\theta_1, \phi_1)\} \times \Sigma_2 \cup \Sigma_1 \times \{p_2(\theta_2, \phi_2)\} \mid (\theta_i, \phi_i) \in S^2\}$$

$$\mathcal{B} = \{\{(p_1(\theta_1, \phi_1), p_2(\theta_2, \phi_2)), (p_1(\pi - \theta_1, \pi + \phi_1), p_2(\pi - \theta_2, \pi + \phi_2))\} \mid (\theta_i, \phi_i) \in S^2\}$$

$$\mathcal{P} = \{(p_1(\theta_1, \phi_1), p_2(\theta_2, \phi_2)), (p_1(\theta_1', \phi_1'), p_2(\theta_2', \phi_2')) \mid \theta_i \neq \theta_i', \phi_j \neq \phi_j'\}$$

Note that $\mathcal{B} \subset \mathcal{P}$. In this way, we have obtained a collection of properties that together make up the property lattice that describes a compound system consisting of two separated spin 1/2 objects, the infimum of a collection of properties being their intersection.

Some of these properties appear familiar, given the fact that the global system consists of two subsystems of which the mathematical description is known. On the other hand, there are also some properties, notably in $\mathcal{P}$, which have a more classical appearance, in the sense that they consist of a set theoretical union of two states, without any new superpositions that do arise in this particular way. It follows from geometrical considerations that these elements arise from intersections of elements in $\mathcal{A}_1$ and $\mathcal{A}_2$. On the other hand, taking two different elements of $\Sigma_1 \times \Sigma_2$ such that one of the coordinates coincides, the property $a$ generated by these two elements has $\pi_j(a) = \Sigma_j$, with $\pi_j$ the canonical projection on the other coordinate, hence contains plenty of other elements of $\Sigma_1 \times \Sigma_2$ than the two generating states. This situation is reminiscent to the notion of a superselection rule in standard quantum mechanics, to which we come back later. Observe that all these properties, even the more enigmatic ones, have a clear operational meaning, in the sense that there exists a corresponding experimental procedure that can test for this property.

What about the orthogonality relation? Suppose that $(p_1, p_2), (q_1, q_2) \in \Sigma_1 \times \Sigma_2$. If $p_1 \perp_1 q_1$, we have seen that there is some direction $(\theta_1, \phi_1)$ such that $p_1$ is represented by $a_1(\theta_1, \phi_1)$ and $q_1$ by $a_1(\pi - \theta_1, \phi_1 + \pi)$. Because the corresponding experiments $C\alpha_1(\theta_1, \phi_1)$ and $C\alpha_1(\pi - \theta_1, \phi_1 + \pi)$ can also be performed on the compound system, it follows that $(p_1, p_2) \perp (q_1, q_2)$. Conversely, if $(p_1, p_2) \perp (q_1, q_2)$, there exists a yes/no-experiment $\alpha \in Q$ such that $(p_1, p_2) \lhd \alpha$ and $(q_1, q_2) \lhd \widetilde{\alpha}$. Recall that $(p_1, p_2) \lhd \alpha$ formalizes the physical idea that the execution of the experiment $\alpha$ should yield a positive result with certainty, would the experiment be properly performed on a particular sample of the compound physical system that happens to be in a state given by $(p_1, p_2)$.

There are several basic forms to be considered for $\alpha$. Suppose first that $\alpha = C\alpha_1$ for some $\alpha_1 \in Q_1$; we have seen that $\widetilde{\alpha} = C\widetilde{\alpha}_1$, hence $p_1 \perp_1 q_1$. The same type of argument works if $\alpha = C\alpha_2$ for some $\alpha_2 \in Q_2$ to conclude that $p_2 \perp_2 q_2$. Next, suppose that $\alpha = \alpha_1 \triangle \alpha_2$, then $\widetilde{\alpha} = \widetilde{\alpha}_1 \triangledown \widetilde{\alpha}_2$ . By the prescription of the experimental procedure and by the separation of the two objects, formally encoded in (20) - (24), we have on the one hand, $p_1 \lhd_1 \alpha_1$ and $p_2 \lhd_2 \alpha_2$, and on the other hand $q_1 \lhd_1 \widetilde{\alpha}_1$ or $q_2 \lhd_2 \widetilde{\alpha}_2$, which implies that $p_1 \perp_1 q_1$ or $p_2 \perp_2 q_2$. All other cases being similar, we conclude that

$$(p_1, p_2) \perp (q_1, q_2) \text{ iff } p_1 \perp_1 q_1 \text{ or } p_2 \perp_2 q_2 \tag{30}$$

The same type of argument also shows that the property lattice is orthocomplemented. The reader can verify that orthocomplements for the elements of various forms in $\mathcal{L}_1(\mathbb{C}^2) \textcircled{\small$\wedge$} \mathcal{L}_2(\mathbb{C}^2)$ are those that are given in the following convenient table. In this table, the $p_i$ and $q_j$ stand for points in the state spaces that correspond to the two different subobjects. In addition, we demand that $p_1 \neq q_1$ and $p_2 \neq q_2$.

| Element | Orthocomplement |
|---|---|
| $\{(p_1, p_2)\}$ | $\{p_1\}^\perp \times \Sigma_2 \cup \Sigma_1 \times \{p_2\}^\perp$ |
| $\{(p_1, p_2), (q_1, q_2)\}$ | $\{(p_1^\perp, q_2^\perp), (q_1^\perp, q_2^\perp)\}$ |
| $\{p_1\} \times \Sigma_2$ | $\{p_1\}^\perp \times \Sigma_2$ |
| $\Sigma_1 \times \{p_2\}$ | $\Sigma_1 \times \{p_2\}^\perp$ |
| $\{p_1\} \times \Sigma_2 \cup \Sigma_1 \times \{p_2\}$ | $\{(p_1^\perp, p_2^\perp)\}$ |

Next, we will show that both the orthomodularity and the covering law fail, taking the mathematical representation for our compound physical system to be $\mathcal{L}_1(\mathbb{C}^2) \textcircled{\wedge} \mathcal{L}_2(\mathbb{C}^2)$, which can be taken, as we have seen, as the property lattice for our compound system. We start with orthomodularity. Denoting by $[x]$ the one-dimensional subspace spanned by an element $x \in \mathbb{C}^2$, let $a = \{([\psi_1], [\psi_2])\}$ and $b = \{([\psi_1], [\psi_2]), ([\phi_1], [\phi_2])\}$, with $[\psi_1] \not\perp_1 [\phi_1]$ and $[\psi_2] \not\perp [\phi_2]$. Then $a^\perp = \{[\psi_1]\}^\perp \times \Sigma_2 \cup \Sigma_1 \times \{[\psi_2]\}^\perp$. Consequently, meets corresponding to intersections, we have $b \wedge a^\perp = \emptyset$, hence $a = a \vee (b \wedge a^\perp) < b$ (a strict inequality!). If orthomodularity were valid, we would have obtained $a \vee (b \wedge a^\perp) = b$, which proves our assertion.

It is also easy to show that the covering law cannot be valid for this particular example, too. To see this, take a lattice element of $\mathcal{P}$, say $\{([\psi_1], [\psi_2]), ([\phi_1], [\phi_2])\}$. It is always possible to choose a third element $([\xi_1], [\xi_2])$, such that $\xi_1$ and $\psi_1$ are two linearly independent elements, and also $\xi_2$ and $\phi_2$. Then

$$\{([\psi_1], [\psi_2]), ([\phi_1], [\phi_2])\} \wedge \{([\xi_1], [\xi_2])\} = \emptyset \tag{31}$$

$$\{([\psi_1], [\psi_2]), ([\phi_1], [\phi_2])\} \vee \{([\xi_1], [\xi_2])\} = \Sigma_1 \times \Sigma_2 \tag{32}$$

This element should cover $\{([\psi_1], [\psi_2]), ([\phi_1], [\phi_2])\}$ if the covering law were valid. However, the element $\{[\psi_1]\} \times \Sigma_2 \cup \Sigma_1 \times \{[\phi_2]\}$ belongs to $\mathcal{L}_1(\mathbb{C}^2) \textcircled{\wedge} \mathcal{L}_2(\mathbb{C}^2)$ and

$$\{([\psi_1], [\psi_2]), ([\phi_1], [\phi_2])\} \subset \{[\psi_1]\} \times \Sigma_2 \cup \Sigma_1 \times \{[\phi_2]\} \subset \Sigma_1 \times \Sigma_2 \tag{33}$$

which is a contradiction, because these are strict inclusions.

We can then safely conclude that the property lattice $\mathcal{L}_1(\mathbb{C}^2) \textcircled{\wedge} \mathcal{L}_2(\mathbb{C}^2)$ is *not* isomorphic to a Piron lattice (associated with an orthomodular space), due to the fact that orthomodularity and the covering law fail. Consequently, an underlying linear structure such that $\mathcal{L}_1(\mathbb{C}^2) \textcircled{\wedge} \mathcal{L}_2(\mathbb{C}^2)$ would correspond to the complete lattice of all closed subspaces is out of the question: one cannot construct an underlying Hilbert space for which the collection of all closed subspaces would correspond with the property lattice associated with this physical situation.

## 4 The Orthogonality Relation

In this section, we want to take a closer look at the orthogonality relation on a general $\Sigma_1 \times \Sigma_2$ that generates the property lattice corresponding to the separated product. It will be convenient to demonstrate some general results, the first for a general orthogonality space, the second valid for the particular orthogonality relation given by (19).

**Lemma 1.** *In an arbitrary orthogonality space* $(\Sigma, \perp)$*, we have*

$$( \bigcup_{j \in J} M_j )^{\perp} = \bigcap_{j \in J} M_j^{\perp} \tag{34}$$

Proof: Recall that an orthogonality relation is by definition irreflexive and symmetric. If $A \subseteq B$, then $B^{\perp} \subseteq A^{\perp}$, hence $( \cup_{j \in J} M_j )^{\perp} \subseteq M_k^{\perp}$, for each $k \in J$. Observe also that $A \subseteq A^{\perp\perp}$ for any $A \subseteq \Sigma$, by symmetry. Consequently, if $F$ is any subset of $\Sigma$, we obtain $F \subseteq \cap_{j \in J} M_j^{\perp}$ iff $F \subseteq M_j^{\perp}$ for each $j \in J$ iff $M_j \subseteq F^{\perp}$ for each $j \in J$ iff $\cup_{j \in J} M_j \subseteq F^{\perp}$ iff $F \subseteq ( \cup_{j \in J} M_j )^{\perp}$, which proves the other inclusion. □

**Proposition 1.** *Suppose that* $\Sigma_1 \times \Sigma_2$ *is an orthogonality space, equipped with the orthogonality relation (19). Let* $M_j \subseteq \Sigma_j$*,* $j = 1, 2$ *and* $(p_1, p_2) \in \Sigma_1 \times \Sigma_2$*. Then*

$$\{(p_1, p_2)\}^{\perp} = (\{p_1\}^{\perp} \times \Sigma_2) \cup (\Sigma_1 \times \{p_2\}^{\perp}) \tag{35}$$

$$(\{p_1\} \times M_2)^{\perp} = (\{p_1\}^{\perp} \times \Sigma_2) \cup (\Sigma_1 \times M_2^{\perp}) \tag{36}$$

$$(M_1 \times M_2)^{\perp} = (M_1^{\perp} \times \Sigma_2) \cup (\Sigma_1 \times M_2^{\perp}) \tag{37}$$

Proof: First, $(r_1, r_2) \perp (p_1, p_2)$ iff $r_1 \perp_1 p_1$ or $r_2 \perp_2 p_2$ iff $(r_1, r_2) \in \{p_1\}^{\perp} \times \Sigma_2$ or $(r_1, r_2) \in \Sigma_1 \times \{p_2\}^{\perp}$. Second, if $r_1 \in \{p_1\}^{\perp}$ or $r_2 \in M_2^{\perp}$, then $(r_1, r_2) \in (\{p_1\} \times M_2)^{\perp}$; conversely, let $(r_1, r_2) \in (\{p_1\} \times M_2)^{\perp}$; if $r_1 \in \{p_1\}^{\perp}$, there is nothing to prove; if not, take an arbitrary $m_2 \in M_2$; because $(r_1, r_2) \perp (p_1, m_2)$ and $r_1 \not\perp_1 p_1$, it follows that $r_2 \perp_2 m_2$, hence $r_2 \in M_2^{\perp}$. The final equation follows from the next calculation, using some of the previous results:

$$\begin{aligned}
(M_1 \times M_2)^{\perp} &= ( \bigcup_{r_1 \in M_1} (\{r_1\} \times M_2) )^{\perp} \\
&= \bigcap_{r_1 \in M_1} ( (\{r_1\} \times M_2)^{\perp} ) \\
&= \bigcap_{r_1 \in M_1} ( (\{r_1\}^{\perp} \times \Sigma_2) \cup (\Sigma_1 \times M_2^{\perp}) ) \\
&= ( \bigcap_{r_1 \in M_1} (\{r_1\}^{\perp} \times \Sigma_2) ) \cup (\Sigma_1 \times M_2^{\perp}) \\
&= ( \bigcup_{r_1 \in M_1} \{r_1\} )^{\perp} \times \Sigma_2 \cup (\Sigma_1 \times M_2^{\perp}) \\
&= (M_1^{\perp} \times \Sigma_2) \cup (\Sigma_1 \times M_2^{\perp})
\end{aligned}$$

what was to be proved. □

Let $(\Sigma_j, \perp_j)$, $j = 1, 2$, be two $T_1$ orthogonality spaces, that is, we additionally demand that $\forall p_j \in \Sigma_j : \{p_j\}^{\perp_j \perp_j} = \{p_j\}$. Suppose that there exist $p_1, q_1 \in \Sigma_1$ such that $p_1 \neq q_1$, and similarly for $\Sigma_2$. The following straightforward calculation shows that the two-element set $\{(p_1, p_2), (q_1, q_2)\}$ is always a closed subspace of the orthogonality space $(\Sigma_1 \times \Sigma_2, \perp)$, with the orthogonality given by (19):

$$\begin{aligned}
\{(p_1,p_2),(q_1,q_2)\}^{\perp\perp} &= (\{(p_1,p_2)\}^{\perp} \cap \{(q_1,q_2)\}^{\perp})^{\perp} \\
&= ((\{p_1\}^{\perp} \times \Sigma_2) \cup (\Sigma_1 \times \{p_2\}^{\perp}) \cap \\
&\qquad \cap\, (\{q_1\}_1^{\perp} \times \Sigma_2) \cup\ (\Sigma_1 \times \{q_2\}^{\perp}))^{\perp} \\
&= (((\{p_1\}^{\perp} \cap \{q_1\}^{\perp}) \times \Sigma_2) \cup (\{p_1\}^{\perp} \times \{q_2\}^{\perp}) \cup \\
&\qquad \cup\, (\Sigma_1 \times (\{p_2\}^{\perp} \cap \{q_2\}^{\perp})) \cup (\{q_1\}^{\perp} \times \{p_2\}^{\perp}))^{\perp} \\
&= ((\{p_1\}^{\perp} \cap \{q_1\}^{\perp}) \times \Sigma_2)^{\perp} \cap (\{p_1\}^{\perp} \times \{q_2\}^{\perp})^{\perp} \cap \\
&\qquad \cap\, (\Sigma_1 \times (\{p_2\}^{\perp} \cap \{q_2\}^{\perp}))^{\perp} \cap (\{q_1\}^{\perp} \times \{p_2\}^{\perp})^{\perp} \\
&= (\{p_1,q_1\}^{\perp\perp} \times \Sigma_2) \cap (\{p_1\} \times \Sigma_2 \cup \Sigma_1 \times \{q_2\}) \cap \\
&\qquad \cap\, (\Sigma_1 \times \{p_2,q_2\}^{\perp\perp}) \cap (\{q_1\} \times \Sigma_2 \cup \Sigma_1 \times \{p_2\}) \\
&= (\{p_1\} \times \Sigma_2 \cup \{p_1,q_1\}^{\perp\perp} \times \{q_2\}) \cap \\
&\qquad \cap\, (\{q_1\} \times \{p_2,q_2\}^{\perp\perp} \cup\ \Sigma_1 \times \{p_2\}) \\
&= \{(p_1,p_2)\} \cup \{(q_1,q_2)\} \\
&= \{(p_1,p_2),(q_1,q_2)\}
\end{aligned}$$

Consequently, these two elements do not generate an irreducible projective plane. So in general there exist in the property lattice corresponding to the separated product, a host of two-element sets that form closed subspaces, relative to this orthogonality relation, a situation that is unheard off in standard quantum physics.

In a usage derived from that of standard quantum physics, one can say that two properties $a$ and $b$ in a property lattice are separated by a *superselection rule* whenever $p \lhd a \vee b$ implies either $p \lhd a$ or $p \lhd b$. In standard quantum physics, all known superselection rules can be accommodated for by restricting some global Hilbert space, attributed to the physical system under investigation, to a suitable collection of mutually orthogonal subspaces, not allowing states that do not belong to one of these orthogonal components. However, observe that for the separated product of two spin 1/2 objects there do even exist *non-orthogonal* states that are separated by a superselection rule, in particular pairs of states that constitute many of the properties in $\mathcal{P}$.

## 5 Sasaki Regularity

Yet another characterization of the covering law can be formulated for (complete) atomistic orthomodular lattices, using the projections in a suitable involution semigroup of mappings associated with the property lattice. Let $\mathcal{L}$ be a complete atomistic orthomodular lattice, with orthocomplementation $a \mapsto a^\perp$, then $\mathcal{L}$ satisfies the covering law iff each so-called Sasaki projection

$$\phi_a : \mathcal{L} \to \mathcal{L} : x \mapsto (x \vee a^\perp) \wedge a \tag{38}$$

maps any atom not smaller than $a^\perp$ to an atom, that is, for any $a \in \mathcal{L}$ the restriction and corestriction

$$\phi_a : \Sigma \setminus \{p \in \Sigma \mid p < a^\perp\} \to \Sigma : p \mapsto (p \vee a^\perp) \wedge a \tag{39}$$

is well-defined [5]. In general, it is convenient to give a special name to all Sasaki projections that satisfy this last condition. We will call them *regular* Sasaki projections.

Because $\mathcal{L}$ is isomorphic with the orthomodular lattice of all Sasaki projections under some suitable conditions [13], and the Sasaki projections can be interpreted as representing state transitions corresponding to a positive response for idealized measurement procedures associated with the properties, this procedure refers to a more active point of view on physical systems. Indeed, one assumes the existence of an ideal class of measurement procedures, such that the state before such a measurement becomes a well-defined state after the experiment, whenever one has obtained the positive result. In view of this interpretation, it seems indeed more natural to consider Sasaki projections as partially defined state space mappings. Given the fact that $\kappa[\mathcal{L}] = \mathcal{F}(\Sigma)$ under the usual orthocomplementation axioms of the axiomatic approach, we then have to consider a family of mappings

$$\phi_M : \mathcal{D}(\phi_M) \to \Sigma : p \mapsto (\{p\} \cup M^\perp)^{\perp\perp} \cap M \tag{40}$$

with $\mathcal{D}(\phi_M) \subseteq \Sigma$, and $M = \kappa(a)$ for some $a \in \mathcal{L}$. The latter condition arises because it is exactly subsets of this form that represent properties attributed to the physical system. As we have seen, $\phi_M$ is regular iff $\mathcal{D}(\phi_M) = \{p \in \Sigma \mid p \notin M^\perp\}$.

Given the role of the covering law in the representation theorems and its interpretation, it is then of considerable interest to investigate the presence of any aberrant Sasaki projections for operationally separated objects that are described as one compound physical system, both in general and with respect to our example. Because of their putative interpretation as state transitions, we consider the Sasaki projections as partial state space mappings, and investigate them at the level of the state space description $(\Sigma_1 \times \Sigma_2, \perp)$.

Consequently, let $(\Sigma_1, \perp_1)$ and $(\Sigma_2, \perp_2)$ be two Sasaki regular $T_1$ orthogonality spaces, in the sense that Sasaki projections associated with biorthogonally closed sets are regular, and take $(p_1, p_2) \in \Sigma_1 \times \Sigma_2$ such that $(p_1, p_2) \not\perp M_1 \times M_2$, with $M_1 = M_1^{\perp\perp}$ and $M_2 = M_2^{\perp\perp}$. According to our previous results, this implies that $p_1 \not\perp_1 M_1$ and $p_2 \not\perp_2 M_2$. After some calculation efforts, one obtains

$$\begin{aligned}
\phi_{M_1 \times M_2}(p_1, p_2) &= (\{(p_1, p_2)\} \cup (M_1 \times M_2)^\perp)^{\perp\perp} \cap (M_1 \times M_2) \\
&= (\{(p_1, p_2)\} \cup (M_1^\perp \times \Sigma_2) \cup (\Sigma_1 \times M_2^\perp))^{\perp\perp} \cap (M_1 \times M_2) \\
&= ((\{p_1\}^\perp \times \Sigma_2 \cup \Sigma_1 \times \{p_2\}^\perp) \cap (M_1 \times M_2))^\perp \cap (M_1 \times M_2) \\
&= ((\{p_1\}^\perp \cap M_1) \times M_2 \cup M_1 \times (\{p_2\}^\perp \cap M_2))^\perp \cap (M_1 \times M_2) \\
&= ((\{p_1\}^\perp \cap M_1)^\perp \times \Sigma_2 \ \cup\ \Sigma_1 \times M_2^\perp) \cap \\
&\quad \cap\ (M_1^\perp \times \Sigma_2 \ \cup\ \Sigma_1 \times (\{p_2\}^\perp \cap M_2)^\perp)\ \cap\ (M_1 \times M_2) \\
&= ((\{p_1\}^\perp \cap M_1)^\perp \times \Sigma_2 \ \cup\ \Sigma_1 \times M_2^\perp) \cap \\
&\quad \cap\ M_1 \times ((\{p_2\} \cup M_2^\perp)^{\perp\perp} \cap\ M_2) \\
&= ((\{p_1\} \cup M_1^\perp)^{\perp\perp} \cap M_1) \times ((\{p_2\} \cup M_2^\perp)^{\perp\perp} \cap M_2)
\end{aligned}$$

and the right hand side belongs to $\Sigma_1 \times \Sigma_2$, by assumption. In particular, with some abuse of notation

$$\phi_{(q_1, q_2)}(p_1, p_2) = (q_1, q_2) \tag{41}$$

$$\phi_{\{q_1\} \times \Sigma_2}(p_1, p_2) = (q_1, p_2) \tag{42}$$

Consequently, regularity is preserved for all Sasaki projections corresponding to biorthogonally closed subsets of the general form $M_1 \times M_2$. Therefore, we have to screen for other candidates that could violate our regularity condition. Luckily, we don't have to look too far. Indeed, consider one of the peculiar biorthogonally closed sets of the form $M = \{(q_1, q_2), (r_1, r_2)\}$, which, as we have seen, can be found in any property lattice corresponding to a separated product. We will show that, for $(p_1, p_2) \notin M^\perp$:

$$\phi_M(p_1, p_2) = M = \{(q_1, q_2), (r_1, r_2)\} \tag{43}$$

whenever $p_1 \not\perp_1 q_1, p_1 \not\perp_1 r_1, p_2 \not\perp_2 q_2, p_2 \not\perp_2 r_2$. Indeed, if $(p_1, p_2) \notin M^\perp$, then either (1) $p_1 \not\perp_1 q_1$ and $p_2 \not\perp_2 q_2$, or (2) $p_1 \not\perp_1 r_1$ and $p_2 \not\perp_2 r_2$, or (3) both. Consequently,

$$(\{(p_1, p_2)\} \cup M^\perp)^{\perp\perp} = (\{(p_1, p_2)\}^\perp \cap\ M)^\perp$$

$$= (\{(q_1, q_2), (r_1, r_2)\} \cap (\{p_1\}^\perp \times \Sigma_2 \cup \Sigma_1 \times \{p_2\}^\perp))^\perp$$

$$= \begin{cases} \Sigma_1 \times \Sigma_2 & \text{if (1) and (2) are valid} \\ \{q_1\}^\perp \times \Sigma_2 \cup \Sigma_1 \times \{q_2\}^\perp & \text{if only (2) is valid} \\ \{r_1\}^\perp \times \Sigma_2 \cup \Sigma_1 \times \{r_2\}^\perp & \text{if only (1) is valid} \end{cases}$$

from which (42) easily follows. Because $M$ is not an atom, $\phi_M$ is not regular, and $\phi_M$ can no longer be interpreted as a state transition resulting from a positive response for the yes/no-experiment that corresponds with $M$. This is apparently due to the construction of the yes/no-experiments and the properties associated with product experiments, and is a manifestation of the symmetry with respect to the possible state transitions the two separated constituents can undergo for one and the same positive outcome, attributed to the corresponding yes/no-experiment.

In summary, if both $\Sigma_1$ and $\Sigma_2$ contain at least two different states and if in both state spaces there exists a state that is not orthogonal to both these states, that is, for all non-trivial orthogonality spaces, the previous argument is valid and we have demonstrated the following

**Theorem 1.** *If* $(\Sigma_1, \perp_1)$ *and* $(\Sigma_2, \perp_2)$ *are two Sasaki regular* $T_1$ *orthogonality spaces, the orthogonality space* $(\Sigma_1 \times \Sigma_2, \perp)$*, with the orthogonality given by (19), is not Sasaki regular, whenever* $\perp_1$ *and* $\perp_2$ *are non-trivial.*

Because the property lattice attributed to a classical physical system usually corresponds with the collection of all subsets of some set $\Sigma$, one easily sees that the orthogonality relation in this case becomes trivial: every pair of distinct states is orthogonal. Consequently, the theorem is not valid whenever at least one of the two orthogonality spaces represents a classical physical system.

Of course, it then also follows for the same reason that in our particular example Sasaki regularity is not preserved.

# 6 Discussion

Several combinatorial mathematical constructions have been proposed for the description of compound physical systems, starting from the representations of the hypothetical subobjects in those compound physical systems. Two of them were studied by one of the authors: the so-called separated product, that constructs the property lattice for the compound system that consists of two explicitly separated physical objects [6, 7]; the coproduct, that generates the property lattice of two separated physical systems, for which only experimental projects on one of the subobjects at a time, chosen at random, are taken into account [17, 18, 19]. The name of the latter comes from the fact that one can show that it corresponds with (the underlying object of) the mathematical coproduct in an appropriate categorical sense. In addition, in [20], the property lattice associated with the more traditional Hilbert tensor product space representation for compound physical systems has been studied.

In this paper, we have examined in some detail the problem of the mathematical description of the conceptually important physical situation that consists in two separated spin 1/2 objects that are considered as one compound physical system. In particular, we have shown that a representation as a collection of closed subspaces of a linear space is impossible in general, due to the fact that the covering law fails. Thus, there seems to be some relation between the notion of a *compound* physical system and the mathematical property of *linearity*. We have also spent some time on studying another perspective which is intimately related to the covering law: the regularity of the corresponding collection of Sasaki projections. It is tempting to speculate how the possible development of a generalized "non-linear" quantum physics could eventually put in a different light the problems that quantum mechanics experiences to describe (operationally) separated quantum objects.

For atomistic lattices, one can show that the covering law is equivalent with the so-called *exchange property*, which states that for each $x \in \mathcal{L}$ and for any pair of atoms $p, q \in \Sigma_{\mathcal{L}}$, the following condition is valid: $p \wedge x = 0$ and $p < q \vee x$ together imply $q < p \vee x$ [21, 22]. This condition is reminiscent of the superposition principle of quantum mechanics (and is trivially satisfied for the property lattice of a classical physical system, which is usually taken to be the collection of all subsets of state space). Consequently, the covering law seems deeply related to the possibility of attributing a linear structure to the state space of an arbitrary physical system.

The fact that it is mainly the covering law that is responsible for a linear representation of the property lattice, also follows from the following theorem [21]: For any irreducible complete atomistic orthocomplemented lattice of length $\geq 4$ that satisfies the covering law, there exists a division ring $K$ with an involution $\lambda \mapsto \lambda^*$, and a vector space $V$ over $K$ with a hermitian form $f : V \times V \to K$, such that $\mathcal{L}$ is ortho-isomorphic to the lattice of all closed subspaces of $V$, relative to $f$. Consequently, the stronger condition of orthomodularity is not necessary to obtain a linear structure.[5]

[5]Actually, there exists an even weaker representation theorem, which states that a

In our opinion, the mathematical description of this physical situation has at least some relevance with respect to the enigmatic classical limit. At the very least, this approach yields another perspective on the problematic associated with the one and the many, as it was aptly called by one of the authors [6]. Indeed, the construction in this particular case seems to be empirically and operationally adequate in that it incorporates all experiments one can possibly perform on two separated spin 1/2 objects separately and as a whole, and therefore seems to evade the critique of Cattaneo and Nisticó [23].

Of course, if two physical objects are separated in the operational sense that was used in this paper, one usually does not bother about representing the properties that explicitly account for the separation. From this point of view, the so-called coproduct property lattice arises if one considers the collection of properties $\mathcal{L}_1 \cup \mathcal{L}_2$, together with all possible products (in the sense that we have explained), as empirically adequate for the description of the compound physical system. In other words, the fact that one describes two physical systems as a whole does not lead one to consider global experiments on both objects at once. One can object that a description of a compound physical system that takes only into account the possible properties of the subobjects and not of the compound system as a whole is necessarily incomplete.

The underlying set of the coproduct for our example would be isomorphic with the collection of all ordered pairs of non-zero (closed) subspaces of $\mathbb{C}^2$, with an additional global least element pasted at the bottom:

$$\mathcal{L}_1(\mathbb{C}^2) \coprod \mathcal{L}_2(\mathbb{C}^2) = \{(M_1, M_2) \mid M_i \in \mathcal{L}_i^0(\mathbb{C}^2),\ i = 1, 2\} \uplus \{0\} \tag{44}$$

For a more profound study of the properties of this structure, we refer to [17, 18, 19]. In general, also in this case the covering law fails, although all corresponding Sasaki projections seem to behave regularly, which is possible because orthomodularity is in general not valid.

The standard quantum physical prescriptions for the construction of a mathematical representation of a physical system that is conceived as being made up of several components, require one to construct the Hilbert tensor product of the Hilbert spaces corresponding to the putative subobjects, and the possible selection of an appropriate closed subspace, to account for the fermionic or bosonic nature of these constituents. It is clear from our analysis that this procedure is not possible for the case of two separated spin 1/2

linear representation holds for irreducible complete lattices $\mathcal{L}$ of length $\geq 4$, if both $\mathcal{L}$ and its opposite lattice $\mathcal{L}^*$ are atomistic and satisfy the covering law [21].

particles. On the other hand, the tensor product procedure can be justified in the axiomatic approach, given the fact that the putative compound system satisfies the standard prescriptions of the axiomatic approach [18, 20, 24].

Last but not least, we think that the standard notion of so-called "identity of elementary physical objects in a compound system", which is so problematic at a fundamental conceptual level, is not particularly problematic in our approach. Indeed, there may not be such thing as a physical system consisting of two identical subobjects. Indeed, such a system may have to be considered as one global physical system, that may even manifest itself at spatially separated regions, a problem that would more properly be related to our *a priori*, possibly macroscopically biased, ideas on localization in space. Indeed, experimental evidence suggests that the property of being localized in space, is in general not a classical property (see [8] and references therein). In that case, the putative compoundness would be a mental construction that is ascribed in retrospect to the physical system before it actually interacted with a suitable measuring device.

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