# Failure of Standard Quantum Mechanics for the Description of Compound Quantum Entities*

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**Abstract**

We reformulate the 'separated quantum entities' theorem, i.e. the theorem that proves that *two separated quantum entities cannot be described by means of standard quantum mechanics*, within the fully elaborated operational Geneva-Brussels approach to quantum axiomatics, where the basic mathematical structure is that of a State Property System. We give arguments that show that the core of this result indicates a failure of standard quantum mechanics, and not just some peculiar shortcoming due to the axiomatic approach to quantum mechanics itself.

## 1 Introduction

We reformulate the theorem that has been proved by one of the authors [1, 2] that shows that standard quantum mechanics cannot described the situation of separated quantum systems with the operational Geneva-Brussels approach of State Property Systems. We also give arguments that show that the result of this theorem indices a failure of standard quantum mechanics, and not just a peculiarity of the axiomatic approach itself. To make the whole consistent and self contained we also reformulate in a precise way Piron's representation theorem in axiomatic quantum mechanics.

In standard quantum mechanics with each quantum entity corresponds a complex Hilbert space $\mathcal{H}$. A state $p$ (we will denote states by the symbols $p, q, r, \ldots$) of the quantum entity is described by a one-dimensional subspace (ray or unit vector) $v(p)$ of $\mathcal{H}$, and an observable by a self-adjoint operator on $\mathcal{H}$ [24]. In particular, a yes/no observable $\alpha$ (we will denote yes/no observables by the symbols $\alpha, \beta, \gamma, \ldots$) is represented by an orthogonal projector $P(\alpha)$ or by the closed subspace $A(\alpha)$ which is the range of this projector. The answer 'yes' occurs with certainty (probability equal to 1) for a yes/no observable $\alpha$, if and only if the state $p$ of the quantum entity is such that $v(p) \subset A(\alpha)$, whereas the answer 'no' occurs with certainty if and only if $v(p) \subset A(\alpha)^\perp$. Standard quantum mechanics focuses its description on the level of the mathematical structure of the Hilbert space. In 1936 Birkhoff and von Neumann introduced another level of description by focusing not on the structure of the Hilbert space itself, but on the structure of the set $\mathcal{P}(\mathcal{H})$ of closed subspaces of this Hilbert space, where each closed subspace $A(\alpha) \in \mathcal{P}(\mathcal{H})$ is interpreted as the 'logical' proposition related to the yes/no experiment $\alpha$. Birkhoff and von Neumann's aim was to point out that the mathematical structure of the set of quantum propositions $\mathcal{P}(\mathcal{H})$ is not that of a Boolean algebra, as it is the case for the set of propositions corresponding to a classical mechanics entity. This focus gave birth to the research field called 'quantum logic', as the study of the logical and mathematical properties of the set of propositions $\mathcal{P}(\mathcal{H})$. More specifically Birkhoff and von Neumann remarked that it is the distributive law between conjunction and disjunction which is not necessarily valid in quantum logic while it is obviously valid on classical logic.

Although the problems of 'quantum logic' in itself are very interesting, Birkhoff and von Neumann, by shifting the attention to $\mathcal{P}(\mathcal{H})$ instead of $\mathcal{H}$, introduced two other profound advantages. First of all the possibility to built an operational approach to quantum mechanics, because indeed the elements that give rise to $\mathcal{P}(\mathcal{H})$ are yes/no experiments, which are operational elements, compared to the elements that

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give rise to $\mathcal{H}$ itself. And secondly, equally important, the fact that on the level of $\mathcal{P}(\mathcal{H})$ quantum entities and classical entities can be described within one and the same mathematical category, which is not at all the case on the level of the state spaces ($\mathcal{H}$ in the case of quantum mechanics and the phase space in the case of classical mechanics). During the years that followed step by step it became clear what this for quantum entities and classical entities 'common' mathematical category was. A big step ahead was taken by the work of George Mackey immediately followed by the representation theorem of Constantin Piron. Mackey put forward a scheme where one starts from the set of yes/no experiments $Q$ and then formulates axioms on it to arrive at the standard mechanical case where $Q$ is related in the way explained to $\mathcal{P}(\mathcal{H})$, the set of closed subspaces of a complex Hilbert space $\mathcal{H}$ [18], and Piron puts forward a set of axioms that 'almost' do the job of bringing back the general framework to standard quantum mechanics [19, 20]. Piron's scheme was worked out over the years into a full operational approach [1, 2, 4, 5], and the mathematical category that carries this approach was identified in detail [10, 11, 14, 12], and called the category of 'state property systems and their morphisms', denoted as **SP**.

## 2 State Property Systems, Quantum and Classical Mechanics

In this section we present the way in which we arrive at the structure of a state property system, and how such a state property system, plus the necessary axioms, gives rise to the case of standard quantum mechanics, but also to the case of classical mechanics. For a detailed exposition we refer to [14] and [12].

### 2.1 State property systems

We start by considering two operationally defined sets, the set of states of the physical entity, denoted by $\Sigma = \{p, q, r, \ldots\}$, and the set of yes/no experiments on the physical entity, denoted by $Q = \{\alpha, \beta, \gamma, \ldots\}$, and a relation on $\Sigma \times Q$, denoted $p \lhd \alpha$, expressing the physical law: "the yes/no experiment $\alpha \in Q$ gives with certainty (probability equal to 1) the outcome 'yes' if the physical entity is in state $p \in \Sigma$". Two yes/no experiments $\alpha, \beta \in Q$ are said to be equivalent, and this is denoted as $\alpha \approx \beta$, iff $\forall p \in \Sigma : p \lhd \alpha \Leftrightarrow p \lhd \beta$. Then the operational concept of 'property' related to (or tested by) a yes/no experiment $\alpha$ is introduced as the equivalence class for the equivalence relation $\approx$ of all yes/no experiments that test this property $a$. That the yes/no experiments $\alpha$ tests the property $a$ is denoted by $\alpha \in a$, and obviously we have that if $\alpha \in a$ and $\beta \approx \alpha$, then $\beta \in a$, since $a$ is mathematically the class of yes/no experiments equivalent to $\alpha$. The set of properties of the physical entity under consideration is denoted by $\mathcal{L}$. The relation $\lhd$ on $\Sigma \times Q$ can be easily defined on $\Sigma \times \mathcal{L}$, and for $p \in \Sigma$ and $a \in \mathcal{L}$, $p \lhd a$ means now: "the property $a$ is actual if the physical entity is in state $p$". Hence with other words, a property $a$ is 'actual' iff each yes/no experiment of its equivalence class gives with certainty the outcome 'yes'. Additionally to $\Sigma$ and $\mathcal{L}$ we introduce a map $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$, called the Cartan map, such that for $a \in \mathcal{L}$ we have *$\kappa(a) = \{p \; | p \in \Sigma, p \lhd a\}$* +$\kappa(a) = \{p \in \Sigma \; | p \lhd a\}$+. Hence $\kappa(a)$ is the set of all states of the physical entity that make $a$ actual. This makes it possible to introduce a relation of 'implication', denoted $<$, on the set of properties defined as follows: for $a, b \in \mathcal{L}$ we have $a < b$ iff $\kappa(a) \subset \kappa(b)$. Hence $a < b$ means that whenever the property $a$ is actual for the physical entity then also the property $b$ is actual. It can easily be checked that $<$ is a partial order relation on $\mathcal{L}$. It can also be proven purely from the operational structure, hence without the necessity of any axioms, that the set of properties $\mathcal{L}$ of the physical entity is a complete lattice for this partial order relation (see for example [14]).

We have now all the material available to define what is the state property system related to the physical entity under consideration. The state property system is the triplet $(\Sigma, \mathcal{L}, \kappa)$, where $\Sigma$ is a set that plays the role of the set of states of the entity, $\mathcal{L}$ is a complete lattice that plays the role of the set of properties, and $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ is a one-to-one or injective function that plays the role of the Cartan map. We also have for $\{a_i\} \in \mathcal{L}$ and $I$ the maximal element and 0 the minimal element of the complete lattice $\mathcal{L}$ that:

$$\kappa(\wedge_i a_i) = \cap_i \kappa(a_i) \tag{1}$$

$$\kappa(I) = \Sigma \tag{2}$$

$$\kappa(0) = \emptyset \tag{3}$$

where $\wedge_i a_i$ is the meet of the set $\{a_i\}$ in the complete lattice $\mathcal{L}$.

Suppose we consider two state property systems $(\Sigma, \mathcal{L}, \kappa)$ and $(\Sigma', \mathcal{L}', \kappa')$. The morphisms of the category **SP** have been derived from a general covariance principle [10, 11, 14], *i.e.*, a morphism is a couple $(m, n)$, where $m$ is a map from $\Sigma'$ to $\Sigma$, and $n$ a map from $\mathcal{L}$ to $\mathcal{L}'$, such that:

$$m(p') \in \kappa(a) \Leftrightarrow p' \in \kappa'(n(a)) \tag{4}$$

Let us see how standard quantum mechanics as well as classical mechanics can both be fitted into this scheme . For standard quantum mechanics, $\Sigma$ is the set of rays of the Hilbert space $\mathcal{H}$ and $\mathcal{L}$ the set of closed subspaces $\mathcal{P}(\mathcal{H})$. The Cartan map maps each closed subspace on the set of rays that are contained in this closed subspace, and indeed (1), (2), and (3) are satisfied. For the classical case, $\Sigma$ is the phase space $\Omega$ and $\mathcal{L}$ is the set of subsets $\mathcal{P}(\Omega)$ of the phase space. The Cartan map is the identity.

We give now an account of how additional axioms can be formulated such that the general operational formalism of state property systems leads to the quantum mechanics and classical mechanics.

### 2.2 The axioms

A physical entity $S$ is described by its state-property system $(\Sigma, \mathcal{L}, \kappa)$, where $\Sigma$ is a set, its elements representing the states of $S$, $\mathcal{L}$ is a complete lattice, its elements representing the properties of $S$, and $\kappa$ is a one to one map from $\mathcal{L}$ to $\mathcal{P}(\Sigma)$, satisfying (1), (2), and (3), and expressing the physical situation: "The property $a \in \mathcal{L}$ is actual if the entity $S$ is in state $p \in \Sigma$" by $p \in \kappa(a)$. This is the structure that we derive from only operational aspects of the approach. The first axiom that we introduce consists in demanding that a state is determined by the set of properties that are actual in this state.

**Axiom 1 (State Determination).** *For $p, q \in \Sigma$ such that*

$$\bigwedge_{p \in \kappa(a)} a = \bigwedge_{q \in \kappa(b)} b \tag{5}$$

*we have $p = q$.*

We remark that in [19], [20], [1], [2], [4] and [5] this axiom is considered to be satisfied *a priori*. It was only later that we became aware of the fact that 'state determination' demands an axiom and cannot be derived from the operational content of the theory. The second axiom consists in demanding that the states can be considered as atoms of the property lattice, where an atom of a lattice is a smallest element of this lattice different from the minimal element 0.

**Axiom 2 (Atomisticity).** *For $p \in \Sigma$ we have that*

$$s(p) = \bigwedge_{p \in \kappa(a)} a \tag{6}$$

*is an atom of $\mathcal{L}$.*

Obviously these two axioms are satisfied for the two examples, classical mechanics $(\Omega, \mathcal{P}(\Omega), \kappa)$, and quantum mechanics $(\Sigma, \mathcal{P}(\mathcal{H}), \kappa)$, that we considered. From axioms 1 and 2, it follows that $\mathcal{L}$ is moreover atomistic and that $s$ is a bijection from $\Sigma$ to the set of atoms of $\mathcal{L}$.

For the third axiom it is already very difficult to give a complete physical interpretation. This third axiom introduces the structure of an orthocomplementation for the lattice of properties. At first sight the orthocomplementation could be seen as a structure that plays a similar role for properties as the negation in logic plays for propositions. But that is not a very careful way of looking at things. We cannot go into the details of the attempts that have been made to interpret the orthocomplementation in a physical way, and refer to [20, 21, 1, 2, 4] for those that are interested in this problem. Also in [22], [23], [17], [13] and [15] this problem is considered in depth.

**Axiom 3 (Orthocomplementation).** *The lattice $\mathcal{L}$ of properties of the physical entity under study is orthocomplemented. This means that there exists a function $' : \mathcal{L} \to \mathcal{L}$ such that for $a, b \in \mathcal{L}$ we have:*

$$(a')' = a \tag{7}$$

$$a < b \Rightarrow b' < a' \tag{8}$$

$$a \wedge a' = 0 \quad \text{and} \quad a \vee a' = I \tag{9}$$

For $\mathcal{P}(\Omega)$ the orthocomplement of a subset is given by the complement of this subset, and for $\mathcal{P}(\mathcal{H})$ the orthocomplement of a closed subspace is given by the subspace orthogonal to this closed subspace.

The next two axioms are called the covering law and orthomodularity. There is no obvious physical interpretation for them. They have been put forward mainly because they are satisfied in the lattice of closed subspaces of a complex Hilbert space.

**Axiom 4 (Covering Law).** *The lattice $\mathcal{L}$ of properties of the physical entity under study satisfies the covering law. This means that for $a, x \in \mathcal{L}$ and $p \in \Sigma$ we have:*

$$a < x < a \vee s(p) \Rightarrow x = a \text{ or } x = a \vee s(p) \tag{10}$$

**Axiom 5 (Orthomodularity).** *The orthocomplemented lattice $\mathcal{L}$ of properties of the physical entity under study is orthomodular. This means that for $a, b \in \mathcal{L}$ we have:*

$$a < b \Rightarrow (b \wedge a') \vee a = b \tag{11}$$

These are the five axioms of standard quantum axiomatic. It can be shown that both axioms, the covering law and orthomodularity, are satisfied for the two examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ [19, 20].

The two examples that we have mentioned show that both classical entities and quantum entities can be described by the common structure of a state property system satisfying axiom 1, 2, 3, 4 and 5. Now we have to consider the converse, namely how this structure leads us to classical physics and to quantum physics.

### 2.3 The representation theorem

First we show how the classical and nonclassical parts can be extracted from the general structure, and second we show how the nonclassical parts can be represented by so-called generalized Hilbert spaces.

Since both examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ satisfy the five axioms, it is clear that a theory where the five axioms are satisfied can give rise to a classical theory, as well as to a quantum theory. It is possible to filter out the classical part by introducing the notions of classical property and classical state. Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property system representing a physical entity, satisfying axioms 1, 2 and 3. We say that a property $a \in \mathcal{L}$ is a classical property if for all $p \in \Sigma$ we have

$$p \in \kappa(a) \text{ or } p \in \kappa(a') \tag{12}$$

The set of all classical properties we denote by $\mathcal{C}$. For $p \in \Sigma$ we introduce

$$\omega(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{C}} a \tag{13}$$

$$\kappa_c(a) = \{\omega(p) \mid p \in \kappa(a)\} \tag{14}$$

and call $\omega(p)$ the classical state of the physical entity whenever it is in a state $p \in \Sigma$, and $\kappa_c$ the classical Cartan map. The set of all classical states will be denoted by $\Omega$. The classical state property system corresponding with $(\Sigma, \mathcal{L}, \kappa)$ is $(\Omega, \mathcal{C}, \kappa_c)$.

Again considering our two examples, it is easy to see that for the quantum case, hence for $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have no nontrivial classical properties. Indeed, for any closed subspace $A \in \mathcal{H}$, different from 0 and $\mathcal{H}$, we have rays of $\mathcal{H}$ that are neither contained in $A$ nor contained in $A'$. These are exactly the rays that correspond to states that are superposition states of states in $A$ and states in $A'$. It is the superposition principle in standard quantum mechanics that makes that the only classical properties of a quantum entity are the trivial ones, represented by 0 and $\mathcal{H}$. It can also easily be seen that for the case of a classical entity, described by $\mathcal{P}(\Omega)$, all the properties are classical properties. Indeed, consider an arbitrary property $A \in \mathcal{P}(\Omega)$, then for any singleton $\{p\} \subset \Omega$ representing a state of the classical entity, we have $\{p\} \subset A$ or $\{p\} \subset A'$, since $A'$ is the set theoretical complement of $A$. Since for the quantum case we have only two classical properties, namely 0 and $\mathcal{H}$, it means that there is only one classical state, namely $\mathcal{H}$. It is the classical state that corresponds to 'considering the quantum entity under study' and the state does not specify anything more than that. For the classical case, every state is a classical state. It can be proven that $\kappa_c : \mathcal{C} \to \mathcal{P}(\Omega)$ is an isomorphism [1, 4]. This means that if we filter out the classical part and limit the description of our general physical entity to its classical properties and classical states, the description becomes a standard classical physical description.

Let us filter out the nonclassical part. For $\omega \in \Omega$ we introduce

$$\mathcal{L}_\omega = \{a \in \mathcal{L} \mid a < \omega\} \tag{15}$$
$$\Sigma_\omega = \{p \in \Sigma \mid p \in \kappa(\omega)\} \tag{16}$$
$$\kappa_\omega(a) = \kappa(a) \text{ for } a \in \mathcal{L}_\omega \tag{17}$$

and we call $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ the nonclassical components of $(\Sigma, \mathcal{L}, \kappa)$.

For the quantum case, hence $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only one classical state $\mathcal{H}$, and obviously $\mathcal{L}_\mathcal{H} = \mathcal{L}$. Similarly we have $\Sigma_\mathcal{H} = \Sigma$. This means that the only nonclassical component is $(\Sigma, \mathcal{L}, \kappa)$ itself. For the classical case, since all properties are classical properties and all states are classical states, we have $\mathcal{L}_\omega = \{0, \omega\}$, which is the trivial lattice, containing only its minimal and maximal element, and $\Sigma_\omega = \{\omega\}$. This means that the nonclassical components are all trivial.

For the general situation of a physical entity described by $(\Sigma, \mathcal{L}, \kappa)$ it can be shown that $\mathcal{L}_\omega$ contains no classical properties with respect to $\Sigma_\omega$ except 0 and $\omega$, the minimal and maximal element of $\mathcal{L}_\omega$, and that if $(\Sigma, \mathcal{L}, \kappa)$ satisfies axioms 1, 2, 3, 4, and 5, then also $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ $\forall \omega \in \Omega$ satisfy axioms 1, 2, 3, 4 and 5 (see [1] and [4]).

It is not difficult to verify that, under the assumption of axioms 1 and 2, $s : \Sigma \to \Sigma_\mathcal{L}$ (as defined in (6)) is a well-defined mapping that is one-to-one and onto, $\Sigma_\mathcal{L}$ being the collection of all atoms in $\mathcal{L}$. Moreover, $p \in \kappa(a)$ iff $s(p) < a$. We can call $s(p)$ the property state corresponding to $p$ and define

$$\Sigma' = \{s(p) \mid p \in \Sigma\} \tag{18}$$

the set of property states. It is easy to verify that if we introduce

$$\kappa' : \mathcal{L} \to \mathcal{P}(\Sigma') \tag{19}$$

where

$$\kappa'(a) = \{s(p) \mid p \in \kappa(a)\} \tag{20}$$

that

$$(\Sigma', \mathcal{L}, \kappa') \cong (\Sigma, \mathcal{L}, \kappa) \tag{21}$$

when axioms 1 and 2 are satisfied.

To see in more detail in which way the classical and nonclassical parts are structured within the lattice $\mathcal{L}$, we make use of this isomorphism and introduce the direct union of a set of complete, atomistic orthocomplemented lattices, making use of this identification. Consider a set $\{\mathcal{L}_\omega \mid \omega \in \Omega\}$ of complete, atomistic orthocomplemented lattices. The direct union $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}\mathcal{L}_\omega$ of these lattices consists of the sequences $a = (a_\omega)_\omega$, such that

$$(a_\omega)_\omega < (b_\omega)_\omega \Leftrightarrow a_\omega < b_\omega \ \forall \omega \in \Omega \tag{22}$$
$$(a_\omega)_\omega \wedge (b_\omega)_\omega = (a_\omega \wedge b_\omega)_\omega \tag{23}$$
$$(a_\omega)_\omega \vee (b_\omega)_\omega = (a_\omega \vee b_\omega)_\omega \tag{24}$$
$$(a_\omega)'_\omega = (a'_\omega)_\omega \tag{25}$$

The atoms of $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}\mathcal{L}_\omega$ are of the form $(a_\omega)_\omega$ where $a_{\omega_1} = p$ for some $\omega_1$ and $p \in \Sigma_{\omega_1}$, and $a_\omega = 0$ for $\omega \neq \omega_1$. It can be shown easily that if $\mathcal{L}_\omega$ are complete, atomistic, orthocomplemented lattices, then also $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}\mathcal{L}_\omega$ is a complete, atomistic, orthocomplemented lattice (see for instance [1] and [4]).

The structure of direct union of complete, atomistic, orthocomplemented lattices makes it possible to define the direct union of state property systems in the case axioms 1, 2, and 3 are satisfied. Consider a set of state property systems $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$, where $\mathcal{L}_\omega$ are complete, atomistic, orthocomplemented lattices and for each $\omega$ we have that $\Sigma_\omega$ is the set of atoms of $\mathcal{L}_\omega$. The direct union $\bigcirc\!\!\!\!\vee_\omega(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ of these state property systems is the state property system $(\cup_\omega\Sigma_\omega, \bigcirc\!\!\!\!\vee_\omega\mathcal{L}_\omega, \bigcirc\!\!\!\!\vee_\omega\kappa_\omega)$, where $\cup_\omega\Sigma_\omega$ is the disjoint union of the sets $\Sigma_\omega$, $\bigcirc\!\!\!\!\vee_\omega\mathcal{L}_\omega$ is the direct union of the lattices $\mathcal{L}_\omega$, and

$$\bigcirc\!\!\!\!\vee_\omega\kappa_\omega((a_\omega)_\omega) = \cup_\omega\kappa_\omega(a_\omega) \tag{26}$$

The first part of a fundamental representation theorem can now be stated. For this part it is sufficient that axioms 1, 2 and 3 are satisfied.

**Theorem 1 (Representation Theorem: Part 1).** *We consider a physical entity described by its state property system* $(\Sigma, \mathcal{L}, \kappa)$*. Suppose that axioms 1, 2 and 3 are satisfied. Then*

$$(\Sigma, \mathcal{L}, \kappa) \cong \bigovee_{\omega \in \Omega} (\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \tag{27}$$

*where* $\Omega$ *is the set of classical states of* $(\Sigma, \mathcal{L}, \kappa)$*,* $\Sigma'_\omega$ *is the set of state properties,* $\kappa'_\omega$ *the corresponding Cartan map, (see (18) and (20)), and* $\mathcal{L}_\omega$ *the lattice of properties of the nonclassical component* $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$*. If axioms 4 and 5 are satisfied for* $(\Sigma, \mathcal{L}, \kappa)$*, then they are also satisfied for* $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ *for all* $\omega \in \Omega$*.*

Proof: see [1] and [4].

From the previous section follows that if axioms 1, 2, 3, 4 and 5 are satisfied we can write the state property system $(\Sigma, \mathcal{L}, \kappa)$ of the physical entity under study as the direct union $\bigovee_{\omega \in \Omega}(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ over its classical state space $\Omega$ of its nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, and that each of these nonclassical components also satisfies axiom 1, 2, 3, 4 and 5. Additionally for each one of these nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ no classical properties except 0 and $\omega$ exist. It is for these nonclassical components that a further representation theorem can be proven such that a vector space structure emerges for each one of the nonclassical components. To do this we rely on the original representation theorem proved in [19].

**Theorem 2 (Representation Theorem: Part 2).** *Consider the same situation as in theorem 1, with additionally axiom 4 and 5 satisfied. For each nonclassical component* $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$*, of which the lattice* $\mathcal{L}_\omega$ *is of rank greater than or equal to four, there exists a generalized Hilbert space, that is a vector space* $V_\omega$*, over a division ring* $K_\omega$*, with an involution of* $K_\omega$*, which means a function*

$$^* : K_\omega \to K_\omega \tag{28}$$

*such that for* $k, l \in K_\omega$ *we have:*

$$\begin{aligned} (k^*)^* &= k && (29) \\ (k \cdot l)^* &= l^* \cdot k^* && (30) \end{aligned}$$

*and an Hermitian product on* $V_\omega$*, which means a function*

$$\langle \ , \ \rangle : V_\omega \times V_\omega \to K_\omega \tag{31}$$

*such that for* $x, y, z \in V_\omega$ *and* $k \in K_\omega$ *we have:*

$$\begin{aligned} \langle x + ky, z \rangle &= \langle x, z \rangle + k\langle x, y \rangle && (32) \\ \langle x, y \rangle^* &= \langle y, x \rangle && (33) \\ \langle x, x \rangle = 0 &\Leftrightarrow x = 0 && (34) \end{aligned}$$

*and such that for* $M \subset V_\omega$ *we have:*

$$M^\perp + (M^\perp)^\perp = V_\omega \tag{35}$$

*where* $M^\perp = \{y \ |y \in V_\omega, \langle y, x \rangle = 0, \forall x \in M\}$ *and *Such a vector space is called a generalized Hilbert space or an orthomodular vector space. And we have that:**

$$(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \cong (\mathcal{R}(V), \mathcal{P}(V), \nu) \tag{36}$$

*where* $\mathcal{R}(V)$ *is the set of rays of* $V$*,* $\mathcal{P}(V)$ *is the set of biorthogonally closed subspaces (subspaces that are equal to their biorthogonal) of* $V$*, and* $\nu$ *makes correspond with each such biorthogonal subspace the set of rays that are contained in it.*

Proof: See [19] and [20].

The name 'generalized Hilbert space' was introduced, because it can be shown that if the division ring $K_\omega$ is taken to be the real or complex numbers, or the quaternions, then the generalized Hilbert space becomes a Hilbert space.

# 3 The Failing Axioms of Standard Quantum Mechanics

We have introduced all that is necessary to be able to put forward the theorem that has been proved regarding the failing mathematical structure of standard quantum mechanics for the description of the joint entity consisting of two separated quantum entities [1, 2]. Let us first explain what is meant by separated physical entities.

### 3.1 What Are Separated Physical Entities?

We consider the situation of a physical entity $S$ that consists of two physical entities $S_1$ and $S_2$. The definition of 'separated' that has been used in [1] and [2] is the following. Suppose that we consider two experiments $e_1$ and $e_2$ that can be performed respectively on the entity $S_1$ and on the entity $S_2$, such that the joint experiments $e_1 \times e_2$ can be performed on the joint entity $S$ consisting of $S_1$ and $S_2$. We say that experiments $e_1$ and $e_2$ are separated experiments whenever for an arbitrary state $p$ of $S$ we have that $(x_1, x_2)$ is a possible outcome for experiment $e_1 \times e_2$ if and only if $x_1$ is a possible outcome for $e_1$ and $x_2$ is a possible outcome for $e_2$. We say that $S_1$ and $S_2$ are separated entities if and only if all the experiments $e_1$ on $S_1$ are separated from the experiments $e_2$ on $S_2$.

Let us remark that $S_1$ and $S_2$ being separated does not mean that there is no interaction between $S_1$ and $S_2$. Most entities in the macroscopic world are separated entities. Let us consider some examples to make this clear. The earth and the moon, for example, are separated entities. Indeed, consider any experiment $e_1$ that can be performed on the physical entity earth (for example measuring its position), and any experiment $e_2$ that can be performed on the physical entity moon (for example measuring its velocity). The joint experiment $e_1 \times e_2$ consists of performing $e_1$ and $e_2$ together on the joint entity of earth and moon (measuring the position of the earth and the velocity of the moon at once). Obviously the requirement of separation is satisfied. The pair $(x_1, x_2)$ (position of the earth and velocity of the moon) is a possible outcome for $e_1 \times e_2$ if and only if $x_1$ (position of the earth) is a possible outcome of $e_1$ and $x_2$ (velocity of the moon) is a possible outcome of $e_2$. This is what we mean when we say that the earth has position $x_1$ and the moon velocity $x_2$ at once. Clearly this is independent of whether there is an interaction, the gravitational interaction in this case, between the earth and the moon.

It is not easy to find an example of two physical entities that are not separated in the macroscopic world, because usually nonseparated entities are described as one entity and not as two. In earlier work we have given examples of nonseparated macroscopic entities [3, 6, 8]. The example of connected vessels of water is a good example to give an intuitive idea of what nonseparation means. Consider two vessels $V_1$ and $V_2$ each containing 10 liters of water. The vessels are connected by a tube, which means that they form a connected set of vessels. Also the tube contains some water, but this does not play any role for what we want to show. Experiment $e_1$ consists of taking out water of vessel $V_1$ by a siphon, and measuring the amount of water that comes out. We give the outcome $x_1$ if the amount of water coming out is greater than 10 liters. Experiment $e_2$ consists of doing exactly the same on vessel $V_2$. We give outcome $x_2$ to $e_2$ if the amount of water coming out is greater than 10 liters. The joint experiment $e_1 \times e_2$ consists of performing $e_1$ and $e_2$ together on the joint entity of the two connected vessels of water. Because of the connection, and the physical principles that govern connected vessels, for $e_1$ and for $e_2$ performed alone we find 20 liters of water coming out. This means that $x_1$ is a possible (even certain) outcome for $e_1$ and $x_2$ is a possible (also certain) outcome for $e_2$. If we perform the joint experiment $e_1 \times e_2$ the following happens. If there is more than 10 liters coming out of vessel $V_1$ there is less than 10 liters coming out of vessel $V_2$ and if there is more than 10 liters coming out of vessel $V_2$ there is less than 10 liters coming out of vessel $V_1$. This means that $(x_1, x_2)$ is not a possible outcome for the joint experiment $e_1 \times e_2$. Hence $e_1$ and $e_2$ are nonseparated experiments and as a consequence $V_1$ and $V_2$ are nonseparated entities.

The nonseparated entities that we find in the macroscopic world are entities that are very similar to the connected vessels of water. There must be an ontological connection between the two entities, and that is also the reason that usually the joint entity will be treated as one entity again. A connection through dynamic interaction, as it is the case between the earth and the moon, interacting by gravitation, leaves the entities separated. For quantum entities it can be shown that only when the joint entity of two quantum entities contains entangled states the entities are nonseparated quantum entities. It can be proven [3, 6, 8] that experiments are separated if and only if they do not violate Bell's inequalities. All this has been explored and investigated in many ways, and several papers have been published on the matter [3, 6, 8, 9, 7, 7, 16].

### 3.2 The Separated Quantum Entities Theorem

We are ready now to state the theorem about the impossibility for standard quantum mechanics to describe separated quantum entities [1, 2].

**Theorem 3 (Separated Quantum Entities Theorem).** *Suppose that $S$ is a physical entity consisting of two separated physical entities $S_1$ and $S_2$. Let us suppose that axiom 1, 2 and 3 are satisfied and call $(\Sigma, \mathcal{L}, \kappa)$ the state property system describing $S$, and $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ and $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ the state property systems describing $S_1$ and $S_2$.*
*If the fourth axiom is satisfied, namely the covering law, then one of the two entities $S_1$ or $S_2$ is a classical entity, in the sense that one of the two state property systems $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ or $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ contains only classical states and classical properties.*
*If the fifth axiom is satisfied, namely weak modularity, then one of the two entities $S_1$ or $S_2$ is a classical entity, in the sense that one of the two state property systems $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ or $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ contains only classical states and classical properties.*

Proof: see [1, 2]

The theorem proves that two separated quantum entities cannot be described by standard quantum mechanics.

A classical entity that is separated from a quantum entity and two separated classical entities do not cause any problem, but two separated quantum entities need a structure where neither the covering law nor weak modularity are satisfied.

One of the possible ways out is that there would not exist separated quantum entities in nature. This would mean that all quantum entities are entangled in some way or another. If this is true, perhaps the standard formalism could be saved. Let us remark that even standard quantum mechanics presupposes the existence of separated quantum entities. Indeed, if we describe one quantum entity by means of the standard formalism, we take one Hilbert space to represent the states of this entity. In this sense we suppose the rest of the universe to be separated from this one quantum entity. If not, we would have to modify the description and consider two Hilbert spaces, one for the entity and one for the rest of the universe, and the states would be entangled states of the states of the entity and the states of the rest of the universe. But, this would mean that the one quantum entity that we considered is never in a well-defined state.

It would mean that the only possibility that remains is to describe the whole universe at once by using one huge Hilbert space. It goes without saying that such an approach will lead to many other problems. Another, more down to earth problem is, that in this one Hilbert space of the whole universe also all classical macroscopical entities have to be described. But classical entities are not described by a Hilbert space. If the hypothesis that we can only describe the whole universe at once is correct, it would anyhow be more plausible that the theory that does deliver such a description would be the direct union structure of different Hilbert spaces. But if this is the case, we anyhow are already using a more general theory than standard quantum mechanics. So we can as well use the still slightly more general theory, where axioms 4 and 5 are not satisfied, and make the description of separated quantum entities possible.

All this convinces us that the shortcoming of standard quantum mechanics to be able to describe separated quantum entities is really a shortcoming of the mathematical formalism used by standard quantum mechanics, and more notably of the vector space structure of the Hilbert space used in standard quantum mechanics.

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