Towards a general operational and realistic framework for quantum mechanics and relativity theory*

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Abstract

We propose a general operational and realistic framework that aims at a generalization of quantum mechanics and relativity theory, such that both appear as special cases of this new theory. Our framework is operational, in the sense that all aspects are introduced with specific reference to events to be experienced, and realistic, in the sense that the hypothesis of an independent existing reality is taken seriously. To come to this framework we present a detailed study of standard quantum mechanics within the axiomatic approach to quantum mechanics, more specifically the Geneva-Brussels approach, identifying two of the traditional 6 axioms as 'failing axioms'. We prove that these two failing axioms are at the origin of the impossibility for standard quantum mechanics to describe a continuous change from quantum to classical and hence its inability to describe macroscopic physical reality. Moreover we show that the same two axioms are also at the origin of the impossibility for standard quantum mechanics to deliver a model for the compound entity of two 'separated' quantum entities. We put forward that it is necessary to replace these two axioms in order to proceed to the more general theory. Next we analyze the nature of the quantum probability model and show that it can be interpreted as the consequence of the presence of a lack of knowledge on the interaction between the measurement apparatus and the physical entity under consideration. These two insights, the failing axioms and the nature of quantum probability, give rise to a very specific view on the quantum phenomenon of nonlocality. Nonlocality should be interpreted as nonspatiality. This means that an entity in a nonlocal state, like for example the typical EPR state, is 'not inside space'. As a consequence, space is no longer the all embracing theatre of reality, but a structure that has emerged together with the macroscopic material entities that have emerged from the microworld. This clarifies why general relativity theory cannot be used as a basis for the elaboration of the new generalized theory, since in general relativity theory the set of events is taken *a priori* to be the time-space continuum. Hence in general relativity theory time-space is a basic structure considered to capture all of reality. In our framework we introduce 'happenings' and the 'set of happenings' as constituting reality. A happening is however not identified with a point of time-space, as this is the case for an events of general relativity theory. We analyze different aspects of the new framework, and list the most important problems to be investigated for an elaboration of this framework into a workable and as complete as possible theory.

1 Introduction

Quantum mechanics, even after so many years of reflection of the brightest scientists of our époque, still confronts us with really fundamental problems. While microscopic effects predicted by it have been experimentally verified, they remain irreconcilable with macroscopic reality. Moreover some of these effects are incompatible with modern physics' other major theory, namely, relativity, thereby introducing a deep schism at the very basis of mainstream science. In this article we focus on the following question: "What kind of theory can be envisaged that will replace quantum mechanics as a better description of physical reality?"

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We believe that very fundamental changes will take place from a theoretical point of view in the decades to come, and that what is now referred to as standard quantum mechanics[1] will be looked upon as a special case of a more general theory still to be developed. When this new theory exists we will then also understand why standard quantum mechanics gave such good predictions for experiments performed with quantum entities. Of course, general relativity should also appear in some way as a special cases of the new theory to be developed.

One of the reasons why it is so difficult to develop the new theory is because general relativity and quantum mechanics are fundamentally different theories in respect of how they were constructed. General relativity is a masterpiece of conscious construction starting from deep physical principles and simple but very general ideas, such as the equivalence of gravitational and inertial mass, and mainly in its foundations the work of art of one person, Albert Einstein. Quantum mechanics however has grown out of a complex configuration of problems, guided by strange metaphors, such as the wave-particle duality, and abstract mathematics, such as matrix calculus and the Hilbert space formalism. Hence it is no wonder that the theories are basically very different and very difficult to integrate.

The approach to quantum mechanics that we have been elaborating, and that will also be the inspiration for our ideas on the future of quantum mechanics and the nature of the new theory, is not the most commonly known approach. There are several reasons for this, but one of them is certainly that it is very different from the approaches that start more straightforwardly from quantum mechanics and/or relativity theory as they are formulated in their standard form. In the sections to follow we will shortly describe our approach and also point out why we think that it contains a great potential to help generating the framework for the new theory that could integrate relativity theory and quantum mechanics.

2 Operational Axiomatic Quantum Mechanics

The framework that we develop has roots in work of John von Neumann, in collaboration with Garett Birkhoff, that is almost as old as the standard formulation of quantum mechanics itself [1]. Indeed already during the beginning years of quantum mechanics, the formalism that is now referred to as standard quantum mechanics [2], was thought to be too specific by the founding fathers themselves. One of the questions that obviously was at the origin of this early dissatisfaction is: "Why would a complex Hilbert space deliver 'the' unique mathematical structure for a complete description of the microworld? Would that not be amazing? What is so special about a complex Hilbert space that its mathematical structure would play such a fundamental role?"

Let us turn for a moment to general relativity to raise a suspicion towards the fundamental role of the complex Hilbert space for quantum mechanics. General relativity is founded on the mathematical structure of Riemann geometry. In this case however it is much more plausible that indeed the right mathematical structure has been taken. Riemann developed his theory as a synthesis of the work of Gauss, Lobaskjevski and Bolay on nonEuclidean geometry, and his aim was to work out a theory for the description of the geometrical structure of the world in all its generality. Hence Einstein took recourse to the work of Riemann to express his ideas and intuitions on space time and its geometry and this lead to general relativity. General relativity could be called in this respect 'the geometrization of a part of the world including gravitation'.

There is, of course, a definite reason why von Neumann used the mathematical structure of a complex Hilbert space for the formalization of quantum mechanics, but this reason is much less profound than it is for Riemann geometry and general relativity. The reason is that Heisenberg's matrix mechanics and Schrödinger's wave mechanics turned out to be equivalent, the first being a formalization of the new mechanics making use of l_2, the set of all square summable complex sequences, and the second making use of $L_2(\mathbb{R}^3)$, the set of all square integrable complex functions of three real variables. The two spaces l_2 and $L_2(\mathbb{R}^3)$ are canonical examples of a complex Hilbert space. This means that Heisenberg and Schrödinger were working already in a complex Hilbert space, when they formulated matrix mechanics and wave mechanics, without being aware of it. This made it a straightforward choice for von Neumann to propose a formulation of quantum mechanics in an abstract complex Hilbert space, reducing matrix mechanics and wave mechanics to two specific cases.

One problem with the Hilbert space representation was known from the start. A (pure) state of a quantum entity is represented by a unit vector or ray of the complex Hilbert space, and not by a vector.

[1] We call the theory formulated by John von Neumann in 1934 'standard quantum mechanics' [2].

Indeed vectors contained in the same ray represent the same state or one has to renormalize the vector that represents the state after it has been changed in one way or another. It is well known that if rays of a vector space are called points and two dimensional subspaces of this vector space are called lines, the set of points and lines corresponding in this way to a vector space, form a projective geometry. What we just remarked about the unit vector or ray representing the state of the quantum entity means that in some way the projective geometry corresponding to the complex Hilbert space represents more intrinsically the physics of the quantum world as does the Hilbert space itself. This state of affairs is revealed explicitly in the dynamics of quantum entities, that is built by using group representations, and one has to consider projective representations, which are representations in the corresponding projective geometry, and not vector representations [3].

The title of the article by John von Neumann and Garett Birkhoff [1] that we mentioned as the founding article for our approach is 'The logic of quantum mechanics'. Let us explain shortly what Birkhoff and von Neumann do in this article. First of all they remark that an operational proposition of a quantum entity is represented in the standard quantum formalism by an orthogonal projection operator or by the corresponding closed subspace of the Hilbert space $\mathcal{H}$. Let us denote the set of all closed subspaces of $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$. Next Birkhoff and von Neumann show that the structure of $\mathcal{P}(\mathcal{H})$ is not that of a Boolean algebra, the archetypical structure of the set of propositions in classical logic. More specifically it is the distributive law between conjunction and disjunction

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \tag{1}$$

that is not necessarily valid for the case of quantum propositions $a, b, c \in \mathcal{P}(\mathcal{H})$. A whole line of research, called quantum logic, was born as a consequence of the Birkhoff and von Neumann article. The underlying philosophical idea is that, in the same manner as general relativity has introduced nonEuclidean geometry into the reality of the physical world, quantum mechanics introduces nonBoolean logic. The quantum paradoxes would be due to the fact that we reason with Boolean logic about situations with quantum entities, while these situations should be reasoned about with nonBoolean logic.

Although fascinating as an idea and worth taking seriously [4], it is not this idea that is at the origin of our approach. A more important aspect of what Birkhoff and von Neumann did in their article, not to be found in the title, is that they shifted the attention on the mathematical structure of the set of operational propositions $\mathcal{P}(\mathcal{H})$ instead of the Hilbert space $\mathcal{H}$ itself. In this sense it is important to pay attention to the fact that $\mathcal{P}(\mathcal{H})$ is the set of all operational propositions, *i.e.* the set of yes/no experiments on a quantum entity. They opened a way to connect abstract mathematical concepts of the quantum formalism, namely the orthogonal projection operators or closed subspaces of the Hilbert space, directly with physical operations in the laboratory, namely the yes/no experiments.

George Mackey followed in on this idea when he wrote his book on the mathematical foundations of quantum mechanics [5]. He starts the other way around and considers as a basis the set $\mathcal{L}$ of all operational propositions, meaning propositions being testable by yes/no experiments on a physical entity. Then he introduces as an axiom that this set $\mathcal{L}$ has to have a structure isomorphic to the set of all closed subspaces $\mathcal{P}(\mathcal{H})$ of a complex Hilbert space in the case of a quantum entity. He states that it would be interesting to invent a set of axioms on $\mathcal{L}$ that gradually would make $\mathcal{L}$ more and more alike to $\mathcal{P}(\mathcal{H})$ to finally arrive at an isomorphism when all the axioms are satisfied. While Mackey wrote his book results as such were underway. A year later Constantin Piron proved a fundamental representation theorem. Starting from the set $\mathcal{L}$ of all operational propositions of a physical entity and introducing five axioms on $\mathcal{L}$ he proved that $\mathcal{L}$ is isomorphic to the set of closed subspaces $\mathcal{P}(V)$ of a generalized Hilbert space V whenever these five axioms are satisfied [6]. Although we do not want to get too technical, we have to elaborate on some of the aspects of this representation theorem to be able to explain our framework.

We mentioned already that Birkhoff and von Neumann had noticed that the set of closed subspaces $\mathcal{P}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ is not a Boolean algebra, because distributivity between conjunction and disjunction is not satisfied (see equation (1)). The set of closed subspaces of a complex Hilbert space forms however a lattice, which is a more general mathematical structure than a Boolean algebra, moreover, a lattice where the distributivity rule (equation (1)) is satisfied is a Boolean algebra, which indicates that the lattice structure is the one to consider for the quantum mechanical situation. As we will see more in detail later, and to make again a reference to general relativity, the lattice structure is indeed to a Boolean algebra what general Riemann geometry is to Euclidean geometry. And moreover, meanwhile we have understood why the structure of operational propositions of the world is not a Boolean algebra but a lattice. This is strictly due to the fact that measurements can have an uncontrollable influence on the state of the physical entity under consideration. We will explain this insight in detail later on, but mention it already now, such that it is clear that the intuition of Birkhoff and von Neumann, and later Mackey, Piron and others, although only mathematical intuition at that time, was correct.

When Piron proved his representation theorem in 1964, he concentrated on the lattice structure for the formulation of the five axioms. Meanwhile much more research has been done, both physically motivated in an attempt to make the approach more operational, as well as mathematically, trying to get axiomatically closer to the complex Hilbert space. In the presentation of our framework, we give the most recent update of it, and hence neglect somewhat the original formulation, for example when we explain the representation theorem of Piron.

But before outlining our approach, we want to explain why we think that this approach holds the potential to generate the framework for the new theory to be developed generalizing quantum mechanics and relativity theory. General relativity is a theory that brings part of the world that in earlier Newtonian mechanics was classified within dynamics to the geometrical realm of reality, and more specifically confronting us with the pre-scientific and naive realistic vision on space, time, matter and gravitation. It teaches us in a deep and new way, compared to Newtonian physics, 'what are the things that exists and how they exist and are related and how they influence each other'. But there is one deep lack in relativity theory: it does not take into account the influence of the observer, the effect that the measuring apparatus has on the thing observed. It does not confront the subject-object problem and its influence on how reality is. It cannot do this because its mathematical apparatus is based on the Riemann geometry of time-space, hence prejudicing that time-space is there, filled up with fields and matter, that are also there, independent of the observer. There is no fundamental role for the creation of 'new' within relativity theory, everything just 'is' and we are only there to 'detect' how this everything 'is'. That is also the reason why general relativity can easily be interpreted as delivering a model for the whole universe, whatever this would mean. We know that quantum mechanics takes into account in an essential way the effect of the observer through the measuring apparatus on the state of the physical entity under study. In the new to be developed theory this effect should certainly also appear in a fundamental way. We believe that general relativity has explored to great depth the question "how can things **be** in the world". The theory that we develop explores in great depth the question "how can be **acted** in the world". And it does explore this question of 'action in the world' in a very similar manner as general relativity theory does with its question of 'being of the world'. This means that our approach can be seen as the development of a general theory of 'actions in the world' in the same manner that Riemann's approach can be seen as a general theory of 'geometrical forms existing in the world'. Of course Riemann is not equivalent to general relativity, a lot of detailed physics had to be known to apply Riemann resulting in general relativity. This is the same with our approach, it has the potential to deliver the framework for the new theory, in a similar way as Riemann's geometry had the potential to deliver the framework for general relativity.

We want to remark that in principle a theory that described the possible actions in the world, and a theory that delivers a model for the whole universe, should not be incompatible. It should even be so that the theory that delivers a model of the whole universe should incorporate the theory of actions in the world, which would mean for the situation that exists now, general relativity should contain quantum mechanics, if it really delivers a model for the whole universe. That is why we believe that Einstein's attitude, trying to incorporate the other forces and interactions within general relativity, contrary to common believe, was the right one, globally speaking. What Einstein did not know at that time was 'the reality of nonlocality in the micro-world'. From our approach follows that nonlocality should be interpreted as nonspatiality, which means that the reality of the micro-world, and hence the reality of the universe as a whole, is not time-space like. Time-space is not the global theatre of reality, but rather a cristalization and structuration of the macro-world. Time-space has come into existence together with the macroscopic material entities, and hence it is 'their' time and space, but it is not the theatre of the microscopic quantum entities. This fact is the fundamental reason why general relativity, built on the mathematical geometrical Riemannian structure of time-space, cannot be the canvas for the new theory to be developed. A way to express this technically would be to say that the set of events cannot be identified with the set of time-space points as is done in relativity theory. Recourse will have to be taken to a theory that describes reality as a kind of pre-geometry, and where the geometrical structure arises as a consequence of interactions that collapse into the time-space context. We think that the approach that we develop can deliver the framework as well as the methodology to construct and elaborate such a theory. This is in our opinion the most fundamental role that quantum mechanics, or better the generalizations of quantum mechanics in the spirit of our approach, because we believe that standard quantum mechanics is mathematically too specific and too constrained to play this role, will play in the decades to come. In the next section we introduce the basic objects of our approach.

3 State Property Spaces

Mackey and Piron introduced the set of yes/no experiments but then immediately shifted to an attempt to axiomatize mathematically the lattice of (operational) propositions of a quantum entity, Mackey postulating right away an isomorphism with $\mathcal{P}(\mathcal{H})$ and Piron giving five axioms to come as close as possible to $\mathcal{P}(\mathcal{H})$. Also Piron's axioms are however mostly motivated by mimicking mathematically the structure of $\mathcal{P}(\mathcal{H})$. In later work Piron made a stronger attempt to found operationally part of the axioms [7], and this attempt was worked out further in [8, 9, 10], to arrive at a full operational foundation only recently [11, 12, 13, 14].

Also mathematically the circle was closed only recently. At the time when Piron gave his five axioms that lead to the representation within a generalized Hilbert space, there only existed three examples of generalized Hilbert spaces that fitted all the axioms, namely real, complex and quaternionic Hilbert space, also referred to as the three standard Hilbert spaces[2]. Years later Hans Keller constructed the first counterexample, more specifically an example of an infinite dimensional generalized Hilbert space that is not isomorphic to one of the three standard Hilbert spaces [15]. The study of generalized Hilbert spaces, nowadays also called orthomodular spaces, developed into a research subject of its own, and recently Maria Pia Solèr proved a groundbreaking theorem in this field. She proved that an infinite dimensional generalized Hilbert space that contains an orthonormal base is isomorphic with one of the three standard Hilbert spaces [16]. It has meanwhile also been possible to formulate an operational axiom, called 'plane transitivity' on the set of operational propositions that implies Solèr's condition [17], which completes the axiomatics for standard quantum mechanics by means of six axioms, the original five axioms of Piron and plane transitivity as sixth axiom.

Let us explain now the operational axiomatic approach to quantum mechanics that we develop in its most recent version. Operational propositions have meanwhile been called properties. Hence the basic things to consider for a physical entity S are (1) its set of states Σ, we denote states by symbols $p, q, r, \ldots$, and (2) its set of properties $\mathcal{L}$, we denote properties by symbols $a, b, c, \ldots$, and (3) a relation of 'actuality' between the states and properties that expresses the basic statement: "the property $a \in \mathcal{L}$ is actual if the entity is in state $p \in \Sigma$". This we do by introducing a function $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ such that $\kappa(a)$ is the set of all states of the entity that make property a actual, and κ is called the Cartan map[3]. The basic relation between states and properties "the property $a \in \mathcal{L}$ is actual if the entity is in state $p \in \Sigma$" is then equivalent with the mathematical expression "$p \in \kappa(a)$". The triple $(\Sigma, \mathcal{L}, \kappa)$ is called a state property space[12].

Definition 1 (state property space). *The triple $(\Sigma, \mathcal{L}, \kappa)$, called a state property space, consists of two sets Σ and $\mathcal{L}$, where Σ is the set of states of a physical entity S, and $\mathcal{L}$ its set of properties, and a function $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$, called the Cartan map, such that for $a \in \mathcal{L}$, we have that $\kappa(a)$ is the set of states that make a actual.*

The state property space will be the basic mathematical structure that we start with. It can easily be completely operationally founded in the following way. For each property $a \in \mathcal{L}$ we suppose that there is a yes/no experiment α that tests this property which means the following: A state $p \in \Sigma$ is contained in $\kappa(a)$ iff the outcome for the yes/no experiment α is yes with certainty. If $p \notin \kappa(a)$ then the outcome for α is uncertain (can be yes or no).

There exists two natural pre-order relations[4], one on $\mathcal{L}$ and one on Σ, defined as follows:

[2]There do exist a lot of finite dimensional generalized Hilbert spaces that are different from the three standard examples. But since a physical entity has to have at least a position observable, it follows that the generalized Hilbert space must be infinite dimensional. At the time of Piron's representation theorem, the only infinite dimensional cases that were known are the three standard Hilbert spaces, over the real, complex or quaternionic numbers.

[3]The idea of characterizing properties by the set of states that make them actual can be found in the work of Eli Cartan, and that it why we have called this function the Cartan map [8].

[4]A pre-order relation $<$ is a relation that is reflexive ($x < x$), and transitive ($x < y$ and $y < z$ implies $x < z$).

Definition 2 (pre-order relations). *Suppose that* $(\Sigma, \mathcal{L}, \kappa)$ *is the state property space describing the physical entity* S*. For* $a, b \in \mathcal{L}$ *and* $p, q \in \Sigma$ *we define:*

$$a < b \quad \Leftrightarrow \quad \kappa(a) \subset \kappa(b) \tag{2}$$

$$p < q \quad \Leftrightarrow \quad \forall a \in \mathcal{L} : q \in \kappa(a) \Rightarrow p \in \kappa(a) \tag{3}$$

and say 'a implies b' if $a < b$*, and 'p implies q' if* $p < q$*.*

The physical meaning of these two pre-order relation is obvious, for example $a < b$ means that whenever a is actual then also b is actual.

Let us see how the mathematical structure of a state property space is present in classical mechanics as well as in quantum mechanics. For a classical entity described by classical mechanics the set of states is the state space Ω, and for a quantum entity described by standard quantum mechanics the set of states is the set of unit vectors of the complex Hilbert space $\mathcal{H}$, which we denote by $\Sigma(\mathcal{H})$. We mentioned already that a property of a quantum entity described by standard quantum mechanics is represented by the closed subspace which is the range of the projection operator that describes the yes/no experiment testing this property. Hence $\mathcal{L}$ equals $\mathcal{P}(\mathcal{H})$, the set of all closed subspaces of $\mathcal{H}$. For a classical entity described by classical mechanics each subset of the state space represents a property, which shows that $\mathcal{L}$ equals $\mathcal{P}(\Omega)$, the set of all subsets of Ω. The Cartan map κ in the classical case is the identity, and in the quantum case it is the function that maps a closed subspace onto the set of unit vectors contained in this closed subspace. To conclude, classical mechanics has a state property space $(\Omega, \mathcal{P}(\Omega), \kappa)$, where $\kappa(A) = A$, and quantum mechanics has a state property space $(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}), \kappa)$ where $\kappa(A) = \{u \mid u \in \Sigma(\mathcal{H}), u \in A\}$.

By means of these two examples we see already that the shift of attention introduced by Birkhoff and von Neumann, Mackey, Piron and others to the set of operational propositions (called properties now), and hence mathematically the shift of attention from $\mathcal{H}$ to $\mathcal{P}(\mathcal{H})$ for a quantum entity, and the shift of attention from Ω to $\mathcal{P}(\Omega)$ for a classical entity, makes it possible to consider a classical entity and a quantum entity within the same formalism. The state space of a classical physical entity is a very different mathematical structure as compared to the complex Hilbert space of a quantum mechanical physical entity, but the sets of properties of both entities are just variations on a similar mathematical structure. This means that we have gained a great potential to eventually understand the difference between classical and quantum, and also a real potential to work out a theory that integrates both descriptions, and where both classical mechanics and quantum mechanics appear as special cases.

4 The Axioms

The state property space $(\Sigma, \mathcal{L}, \kappa)$ is a purely operational structure, which means that no axioms are necessary to get it. Any physical entity has it. To come closer to the two cases, classical mechanics and quantum mechanics, we have to introduce axioms. Some of them will have a well defined operational interpretation and other will be of a purely technical mathematical nature.

4.1 Axiom 1: State Property Determination

The first axiom comes to demanding that (1) the set of states that make a certain property actual determine this property and (2) the set of properties that are actual in a certain state determine this state. That is why we call it the axiom of state and property determination. It is satisfied as well in standard quantum mechanics as in classical mechanics. Let us formulate it:

Axiom 1 (state property determination). *Suppose that we have a physical entity* S *described by a state property space* $(\Sigma, \mathcal{L}, \kappa)$*. The first axiom of state and property determination is satisfied iff for* $p, q \in \Sigma$ *and* $a, b \in \mathcal{L}$ *we have:*

$$\kappa(a) = \kappa(b) \quad \Rightarrow \quad a = b \tag{4}$$

$$\{a \mid a \in \mathcal{L}, p \in \kappa(a)\} = \{a \mid a \in \mathcal{L}, q \in \kappa(a)\} \quad \Rightarrow \quad p = q \tag{5}$$

It is easy to verify that if **Axiom 1** is satisfied, the two pre-order relations of definition 2 become partial order relations[5]. An obvious matter to wonder about from a mathematical point of view for a partially

[5]A partial order relation $<$ is a pre-order relation that is apart from being reflexive and transitive also symmetric ($x < y$ and $y < x$ implies $x = y$).

ordered set is whether there exists infima and suprema for this partial order relation. From a physical operational point of view, for two properties $a, b \in \mathcal{L}$, the infimum, which we denote $a \wedge b$ if it exists, would normally play the role of the conjunction of the two properties a and b, hence the property a 'and' b. The supremum, which we denote $a \vee b$ if it exists, would normally play the role of the disjunction of the two properties a and b, hence the property a 'or' b. For the states the meaning of the infimum and supremum is less straightforward, but we will see later that we do not have to bother about this, because another axiom, more specifically **Axiom 2**, is satisfied that makes the question irrelevant. So let us concentrate on the structure of $\mathcal{L}$, the set of properties of the physical entity S, which is now, since we suppose **Axiom 1** to be satisfied, a partially ordered set, with partial order relation $<$ as introduced in definition 2.

We arrive here at the first aspect of quantum mechanics, related to the existence of the superposition principle, that can be understood and explained by means of our approach.

4.2 Conjunctions and Disjunctions

It turns out that, if we take into account that the general physical operational situation for two measurements is the situation where they cannot necessarily be carried out at once (or together), which in quantum jargon means that they are incompatible, we can show that in this case, the conjunction for these properties, of which one is measured by one of the measurements and the other by the other measurement, still exists as an operational property, but the disjunction does not necessarily exists as an operational property. This is a subtle matter and one that is not easy to explain in few words, but we will make an attempt. Suppose that we have two yes/no experiments α and β testing properties a and b. If we think of the standard way to define conjunction and disjunction in logic, by means of the truth tables, it is obvious that both conjunction and disjunction can only be defined operationally if both yes/no experiments can be performed together, because that is the only way to form operationally the truth tables. For the conjunction the outcome (yes, yes) for the joint measurement of α and β is substituted by 'yes', and the outcome (yes, no), (no, yes) and (no, no) are substituted by 'no', while for the disjunction the outcome (yes, yes), (yes, no) and (no, yes) are substituted by 'yes', while the outcome (no, no) is substituted by 'no'. This procedure can however not be applied when the two yes/no measurements α and β cannot be performed together. The subtility of the matter is that for the conjunction there is another procedure available which can always be applied, while for the disjunction this is not the case. To make this clear, consider for the two yes/no experiments α and β the yes/no experiment $\alpha \cdot \beta$, which we call the product experiment, that consists of choosing (at random or not) one of the yes/no experiments, α or β, and performing the chosen experiment and giving the outcome, yes or no, that occurs in this way, to the product yes/no experiment $\alpha \cdot \beta$. When will $\alpha \cdot \beta$ give with certainty the outcome yes? Obviously if and only if both α and β give with certainty the outcome yes. This means that $\alpha \cdot \beta$ tests the property a 'and' b. Since $\alpha \cdot \beta$ exists always, also if α cannot be performed together with β, because we only have to choose α or β to perform it, it proves that for two operational properties a and b, the property a 'and' b always exists as an operational property.

In [8] we gave an example where this is particularly evident. Consider a piece of wood and two properties a and b of the piece of wood, where a is the property "the piece of wood burns well" and b is the property "the piece of wood floats on water". The yes/no experiment α, testing a, consists of putting the piece of wood on fire following a well defined procedure, and seeing whether it burns. If so the outcome yes occurs. The yes/no experiment β, testing b, consists of putting the piece of wood on water and seeing whether it floats. If so, the outcome 'yes' occurs. Obviously it is rather difficult to perform both yes/no experiments together, and if one would try anyhow, no reliable outcome would occur. But we all agree that there exist a lot of pieces of wood for which both properties a and b are actual at once. The reason that we all are convinced of this fact is that we unconsciously use the yes/no experiment $\alpha \cdot \beta$ to test the conjunction property a 'and' b. Indeed, we decide that for a specific piece of wood in a specific state both properties are actual, because if we would choose to perform one of the two yes/no experiments α or β, the outcome 'yes' would occur with certainty. This is exactly the same as performing $\alpha \cdot \beta$, the product yes/no experiment.

Hence, the reason that there is an asymmetry for the existence of an operational conjunction and disjunction, the conjunction existing always, while the disjunction only existing when the corresponding experiments can be performed together, is because to perform the product experiment $\alpha \cdot \beta$ of two experiments α and β we only need to be able to perform α or β, which is indeed always possible. We remark that the product yes/no experiment exist for any number of yes/no experiments $\{\alpha_i\}_i$, and defines operationally the conjunction of all the corresponding properties $\{a_i\}_i$. It can be proven that the conjunction is an infimum for the partial order relation $<$ existing on $\mathcal{L}$, and that is why we will denote it $\wedge_i a_i$. It is a mathematical theorem that a partial ordered set $\mathcal{L}, <$, such that for any family of elements $\{a_i\}_i$ there exists an infimum, is a complete lattice[6] if there exists a maximal element of $\mathcal{L}$. And this maximal element exist, for example the property "the physical entity under consideration exists" which we denote by I, is such a maximal element of $\mathcal{L}, <$. The supremum for a family of elements $\{a_i\}_i$, denoted $\vee_i a_i$, is then defined mathematically by the formula:

$$\bigvee_i a_i = \bigwedge_{x \in \mathcal{L}, a_i < x \ \forall i} x \tag{6}$$

As a conclusion we can say that if **Axiom 1** is satisfied, the set of properties $\mathcal{L}$ of the state property space $(\Sigma, \mathcal{L}, \kappa)$ describing the physical entity S, is a complete lattice. Let us remark that, as a consequence, $\mathcal{L}$ also contains a minimal element, that we denote by 0.

We promised that we would be able to explain something related to the superposition principle in quantum mechanics by what we have been analyzing in the foregoing. To do this, let us return to the state property spaces $(\Omega, \mathcal{P}(\Omega), \kappa)$ of classical mechanics and $(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}), \kappa)$ of quantum mechanics. In $\mathcal{P}(\Omega)$ the infimum and supremum of two subsets $A, B \in \mathcal{P}(\Omega)$ are given respectively by the intersection $A \cap B \in \mathcal{P}(\Omega)$ and the union $A \cup B \in \mathcal{P}(\Omega)$ of subsets. For the case of $\mathcal{P}(\mathcal{H})$ the infimum of two closed subspaces $A, B \in \mathcal{P}(\mathcal{H})$ is given by the intersection $A \cap B \in \mathcal{P}(\mathcal{H})$, because indeed the intersection of two closed subspaces is again a closed subspace. On the other hand, the union of two closed subspaces is in general not a closed subspace. This means that the union does not give us the supremum in this case. For two closed subspaces $A, B \in \mathcal{P}(\mathcal{H})$, the smallest closed subspace that contains both, is $\overline{A + B}$, the topological closure of the sum of the two subspaces. Hence this is the supremum of A and B in $\mathcal{P}(\mathcal{H})$. The vectors that are contained in the topological closure $\overline{A + B}$ of the sum of A and B are exactly the vectors that are superpositions of vectors in A and vectors in B. Hence for a quantum entity, described by $(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}, \kappa)$, there are additional vectors in the supremum of A and B, contained neither in A nor in B, while for the classical entity, described by $(\Omega, \mathcal{P}(\Omega), \kappa)$, there are no such additional elements, because the supremum of A and B is the union $A \cup B$. This is due to the fact that for quantum mechanics experiments that cannot be performed together are an essential ingredient of the theory, while for classical mechanics, although such experiments exist, think of the example of the piece of wood, they can always be substituted by other experiments that can be performed together. For the piece of wood we can for example break the piece in two pieces, and let one flow on water while put fire on the other.

The role of the superposition principle is even still somewhat more subtle than we explained here, because also for compatible properties, for example two closed subspaces A and B with respective projection operators that commute, there exists states that are neither contained in A and nor in B, superposition states of states in A and B, that are contained in the closure of the sum $\overline{A + B}$. This is due to the possibility of existence of EPR-like correlations in quantum mechanics. We have analyzed this effect in detail in [18], but can explain the crux of it in a few lines. If, for example, in performing the yes/no experiments α and β, testing properties a and b, there is a EPR-type of correlation, such that always (yes, no) or (no, yes) comes out for the experiment that performs both α and β together, then a 'or' b is actual as a property, following the rules of the truth tables, but neither a nor b is actual, because both (yes, no) or (no, yes) are possible outcomes. This shows that in the presence of EPR-like correlations the truth table defined 'or' is not the classical logic 'or'.

4.3 Axiom 2: Atomisticity

An element of a (complete) lattice is called an atom, if it is a smallest element different from the minimal element 0. Let us define precisely what we mean.

Definition 3 (atom of a complete lattice). *We say that $s \in \mathcal{L}$ is an atom of $\mathcal{L}$ if for $x \in \mathcal{L}$ we have:*

$$0 < x < s \Rightarrow x = 0 \text{ or } x = s \tag{7}$$

The atoms of $(\mathcal{P}(\Omega), \subset, \cap, \cup)$ are the singletons of the phase space Ω, and the atoms of $(\mathcal{P}(\mathcal{H}), \subset, \cap)$ are the one dimensional subspaces (rays) of the Hilbert space $\mathcal{H}$.

[6] A complete lattice is a partially ordered set such that for any family of elements there exists an infimum and a supremum for this partial order.

The second axiom consists in demanding that the states can be considered as atoms of the property lattice.

Axiom 2 (atomisticity). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. For $p \in \Sigma$ we have that*

$$\bigwedge_{p \in \kappa(a)} a \tag{8}$$

is an atom of $\mathcal{L}$.

Obviously this axiom is satisfied in classical mechanics as well as in quantum mechanics.

4.4 Axiom 3: Orthocomplementation

The third axiom introduces the structure of an orthocomplementation for the lattice of properties. At first sight the orthocomplementation could be seen as a structure that plays a similar role for properties as the negation in logic plays for propositions. But that is not a very careful way of looking at things. We cannot go into the details of the attempts that have been made to interpret the orthocomplementation in a physical way, and refer to [7, 19, 8, 9, 10] for those that are interested in this problem. Also in [20, 21, 22] the problem is considered in depth.

Axiom 3 (orthocomplementation). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The lattice $\mathcal{L}$ of properties of the physical entity is orthocomplemented. This means that there exists a function $' : \mathcal{L} \rightarrow \mathcal{L}$ such that for $a, b \in \mathcal{L}$ we have:*

$$(a')' = a \tag{9}$$

$$a < b \Rightarrow b' < a' \tag{10}$$

$$a \wedge a' = 0 \quad \text{and} \quad a \vee a' = I \tag{11}$$

For $\mathcal{P}(\Omega)$ the orthocomplement of a subset is given by the complement of this subset, and for $\mathcal{P}(\mathcal{H})$ the orthocomplement of a closed subspace is given by the subspace orthogonal to this closed subspace.

4.5 Axiom 4 and 5: Covering Law and Weak Modularity

The next two axioms are called the covering law and weak modularity. There is no obvious physical interpretation for them. They have been put forward mainly because they are satisfied in the lattice of closed subspaces of a complex Hilbert space.

Axiom 4 (covering Law). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The lattice $\mathcal{L}$ of properties of the physical entity satisfies the covering law. This means that for $a, x \in \mathcal{L}$ and $p \in \Sigma$ we have:*

$$a < x < a \vee p \Rightarrow x = a \text{ or } x = a \vee p \tag{12}$$

Axiom 5 (weak modularity). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The orthocomplemented lattice $\mathcal{L}$ of properties of the physical entity is weakly modular. This means that for $a, b \in \mathcal{L}$ we have:*

$$a < b \Rightarrow (b \wedge a') \vee a = b \tag{13}$$

It can be shown that both axioms, the covering law and weak modularity, are satisfied for the two examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ [6, 7].

4.6 Axiom 6: Plane Transitivity

The first five axioms are modelled following Piron's original representation theorem [6]. The sixth axiom that brings us directly to the structure of one of the three standard Hilbert spaces is much more recent [17].

Axiom 6 (plane transitivity). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The orthocomplemented lattice $\mathcal{L}$ of properties of the physical entity is plane transitive. This means that for all atoms $s, t \in \mathcal{L}$ there exist two distinct atoms s_1, s_2 and a symmetry f such that $f|_{[0, s_1 \vee s_2]}$ is the identity and $f(s) = t$.*

Both classical entities and quantum entities can be described by a state property space where the set of properties is a complete atomistic orthocomplemented lattice that satisfies the covering law, is weakly modular and plane transitive. Now we have to consider the converse, namely how this structure leads us to classical physics and to quantum physics.

5 The Representation Theorem

This section is more technical than what proceeded, because we want to show somewhat in detail how the standard quantum mechanical structure emerges from the simple operational structure of a state property space that satisfies the 6 axioms. Most of all we want to make clear in which way the classical and pure quantum parts of the general structure appear. So, first we show how the classical and nonclassical parts can be extracted from the general structure, and second we show how the nonclassical parts can be represented by generalized Hilbert spaces, if they are finite dimensional, and by one of the three standard Hilbert spaces if they are infinite dimensional. Since both examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ satisfy the six axioms, it is clear that a theory where the six axioms are satisfied can give rise to a classical theory, as well as to a quantum theory, but in general gives rise to a mixture of both, in the sense of a quantum theory with superselection rules.

5.1 The Classical Part

It is possible to filter out the classical part by introducing the notions of classical property and classical state. We introduce a classical property $a \in \mathcal{L}$ as a property for which for each state $p \in \Sigma$ of the physical entity this property a is actual or its orthocomplement property a' is actual. The idea is that a property $a \in \mathcal{L}$ is classical of no indeterminism exist for any test α testing this property, meaning that for each state $p \in \Sigma$ we have that α gives with certainty the outcome 'yes' or α gives with certainty the outcome 'no' if tested.

Definition 4 (classical property). *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property space representing a physical entity S, satisfying* **Axioms 1, 2** *and* **3**. *We say that a property $a \in \mathcal{L}$ is a classical property if for all $p \in \Sigma$ we have*

$$p \in \kappa(a) \text{ or } p \in \kappa(a') \tag{14}$$

The set of all classical properties we denote by $\mathcal{C}$.

Again considering our two examples, it is easy to see that for the quantum case, hence for $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have no nontrivial classical properties. Indeed, for any closed subspace $A \in \mathcal{H}$, different from 0 and $\mathcal{H}$, we have rays of $\mathcal{H}$ that are neither contained in A nor contained in A'. These are exactly the rays that correspond to a superposition of states in A and states in A'. It is the superposition principle in standard quantum mechanics that makes that the only classical properties of a quantum entity are the trivial ones, represented by 0 and $\mathcal{H}$. It can also easily be seen that for the case of a classical entity, described by $\mathcal{P}(\Omega)$, all the properties are classical properties. Indeed, consider an arbitrary property $A \in \mathcal{P}(\Omega)$, then for any singleton $\{p\} \in \Sigma$ representing a state of the classical entity, we have $\{p\} \subset A$ or $\{p\} \subset A'$, since A' is the set theoretical complement of A.

Next we introduce the idea of 'classical state' in the following way. For each state $p \in \Sigma$ of the entity we consider the set of all classical properties that are actual when the entity is in this state. The infimum of this set of classical properties is a property that is also actual, and that is the greatest property that makes all these classical properties actual when actual itself. Hence it plays perfectly the role of 'classical state' corresponding to p, namely the state in which only the classical properties (classical part) of the entity is considered. In an obvious way we introduce the 'classical Cartan map' as the map that makes correspond with each classical property the set of classical states that make this property actual.

Definition 5 (classical state). *Suppose that* $(\Sigma, \mathcal{L}, \kappa)$ *is the state property space of a physical entity* S *satisfying* **Axiom 1, 2** *and* **3**. *For* $p \in \Sigma$ *we introduce*

$$\omega(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{C}} a \tag{15}$$

$$\kappa_c(a) = \{\omega(p) \mid p \in \kappa(a)\} \tag{16}$$

and call $\omega(p)$ *the classical state of the physical entity whenever it is in a state* $p \in \Sigma$, *and* κ_c *the classical Cartan map. The set of all classical states will be denoted by* Ω.

We have now introduced all that is needed to define the 'classical state property space' of the entity under consideration.

Definition 6 (classical state property space). *Suppose that* $(\Sigma, \mathcal{L}, \kappa)$ *is the state property space of a physical entity satisfying* **Axiom 1, 2** *and* **3**. *The classical state property space corresponding with* $(\Sigma, \mathcal{L}, \kappa)$ *is* $(\Omega, \mathcal{C}, \kappa_c)$.

Let us look at our two examples. For the quantum case, with $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only two classical properties, namely 0 and $\mathcal{H}$. This means that there is only one classical state, namely $\mathcal{H}$. It is the classical state that corresponds to 'considering the quantum entity under study' and the state does not specify anything more than that. For the classical case, every state is a classical state.

It can be proven that $\kappa_c : \mathcal{C} \to \mathcal{P}(\Omega)$ is an isomorphism[8, 10]. This means that if we filter out the classical part and limit the description of our general physical entity to its classical properties and classical states, the description becomes a standard classical physical description.

5.2 The nonClassical Parts

Now that we have identified the classical parts, let us filter out the nonclassical part. The idea is that we consider the physical entity to be now in a specific classical state ω, and penetrate then further into the left over nonclassical aspects of this entity.

Definition 7 (nonclassical part). *Suppose that* $(\Sigma, \mathcal{L}, \kappa)$ *is the state property space of a physical entity satisfying* **Axiom 1, 2** *and* **3**. *For* $\omega \in \Omega$ *we introduce*

$$\mathcal{L}_\omega = \{a \mid a < \omega, a \in \mathcal{L}\} \tag{17}$$

$$\Sigma_\omega = \{p \mid p \in \kappa(\omega), p \in \Sigma\} \tag{18}$$

$$\kappa_\omega(a) = \kappa(a) \text{ for } a \in \mathcal{L}_\omega \tag{19}$$

and we call $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ *the nonclassical components of* $(\Sigma, \mathcal{L}, \kappa)$.

For the quantum case $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only one classical state $\mathcal{H}$, and obviously $\mathcal{L}_\mathcal{H} = \mathcal{L}$. Similarly we have $\Sigma_\mathcal{H} = \Sigma$. This means that the only nonclassical component is $(\Sigma, \mathcal{L}, \kappa)$ itself. For the classical case, since all properties are classical properties and all states are classical states, we have $\mathcal{L}_\omega = \{0, \omega\}$, which is the trivial lattice, containing only its minimal and maximal element, and $\Sigma_\omega = \{\omega\}$. This means that the nonclassical components are all trivial.

For the general situation of a physical entity described by $(\Sigma, \mathcal{L}, \kappa)$ it can be shown that $\mathcal{L}_\omega$ contains no classical properties with respect to Σ_ω except 0 and ω, the minimal and maximal element of $\mathcal{L}_\omega$, and that if $(\Sigma, \mathcal{L}, \kappa)$ satisfies **Axiom 1, 2, 3, 4, 5** and **6** then also $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ $\forall \omega \in \Omega$ satisfy **Axiom 1, 2, 3, 4, 5** and **6** (see [8, 10, 17]).

We remark that, if **Axiom 1, 2** and **3** are satisfied we can identify a state $p \in \Sigma$ with the element of the lattice of properties $\mathcal{L}$ given by:

$$s(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{L}} a \tag{20}$$

which is an atom of $\mathcal{L}$. More precisely, it is not difficult to verify that, under the assumption of **Axiom 1** and **2**, $s : \Sigma \to \Sigma_\mathcal{L}$ is a well-defined mapping that is one-to-one and onto, $\Sigma_\mathcal{L}$ being the collection of all atoms in $\mathcal{L}$. Moreover, $p \in \kappa(a)$ iff $s(p) < a$. We can call $s(p)$ the property state corresponding to p and define

$$\Sigma' = \{s(p) \mid p \in \Sigma\} \tag{21}$$

the set of state properties. It is easy to verify that if we introduce

$$\kappa' : \mathcal{L} \to \mathcal{P}(\Sigma') \tag{22}$$

where

$$\kappa'(a) = \{s(p) \mid p \in \kappa(a)\} \tag{23}$$

that

$$(\Sigma', \mathcal{L}, \kappa') \cong (\Sigma, \mathcal{L}, \kappa) \tag{24}$$

when **Axiom 1, 2** and **3** are satisfied.

To see in more detail in which way the classical and nonclassical parts are structured within the lattice $\mathcal{L}$, we make use of this isomorphism and introduce the direct union of a set of complete, atomistic orthocomplemented lattices, making use of this identification.

Definition 8 (direct union). *Consider a set $\{\mathcal{L}_\omega \mid \omega \in \Omega\}$ of complete, atomistic orthocomplemented lattices. The direct union $\textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega$ of these lattices consists of the sequences $a = (a_\omega)_\omega$, such that*

$$(a_\omega)_\omega < (b_\omega)_\omega \quad \Leftrightarrow \quad a_\omega < b_\omega \ \forall \omega \in \Omega \tag{25}$$

$$(a_\omega)_\omega \wedge (b_\omega)_\omega \quad = \quad (a_\omega \wedge b_\omega)_\omega \tag{26}$$

$$(a_\omega)_\omega \vee (b_\omega)_\omega \quad = \quad (a_\omega \vee b_\omega)_\omega \tag{27}$$

$$(a_\omega)'_\omega \quad = \quad (a'_\omega)_\omega \tag{28}$$

The atoms of $\textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega$ are of the form $(a_\omega)_\omega$ where $a_{\omega_1} = p$ for some ω_1 and $p \in \Sigma_{\omega_1}$, and $a_\omega = 0$ for $\omega \neq \omega_1$.

It can be proven that if $\mathcal{L}_\omega$ are complete, atomistic, orthocomplemented lattices, then also $\textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega$ is a complete, atomistic, orthocomplemented lattice (see [8, 10]). The structure of direct union of complete, atomistic, orthocomplemented lattices makes it possible to define the direct union of state property spaces in the case **Axiom 1, 2** and **3** are satisfied.

Definition 9 (direct union of state property spaces). *Let $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ be a set of state property spaces, where $\mathcal{L}_\omega$ are complete, atomistic, orthocomplemented lattices and for each ω we have that Σ_ω is the set of atoms of $\mathcal{L}_\omega$. The direct union $\textcircled{\vee}_\omega(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ of these state property spaces is the state property space $(\cup_\omega\Sigma_\omega, \textcircled{\vee}_\omega\mathcal{L}_\omega, \textcircled{\vee}_\omega\kappa_\omega)$, where $\cup_\omega\Sigma_\omega$ is the disjoint union of the sets Σ_ω, $\textcircled{\vee}_\omega\mathcal{L}_\omega$ is the direct union of the lattices $\mathcal{L}_\omega$, and*

$$\textcircled{\vee}_\omega\kappa_\omega((a_\omega)_\omega) = \cup_\omega\kappa_\omega(a_\omega) \tag{29}$$

The first part of a fundamental representation theorem can now be stated. For this part it is sufficient that **Axiom 1, 2** and **3** are satisfied.

Theorem 1 (representation theorem: part 1). *We consider a physical entity S described by its state property space $(\Sigma, \mathcal{L}, \kappa)$. Suppose that* **Axiom 1, 2** *and* **3** *are satisfied. Then*

$$(\Sigma, \mathcal{L}, \kappa) \cong \textcircled{\vee}_{\omega\in\Omega}(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \tag{30}$$

where Ω is the set of classical states of $(\Sigma, \mathcal{L}, \kappa)$ (see definition 5), Σ'_ω is the set of state properties, κ'_ω the corresponding Cartan map, (see (21) and (23)), and $\mathcal{L}_\omega$ the lattice of properties (see definition 7) of the nonclassical component $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$. If **Axiom 4, 5** *and* **6** *are satisfied for $(\Sigma, \mathcal{L}, \kappa)$, then they are also satisfied for $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ for all $\omega \in \Omega$.*

Proof: see [8, 10, 17]

From the previous section follows that if **Axiom 1, 2, 3, 4, 5** and **6** are satisfied we can write the state property space $(\Sigma, \mathcal{L}, \kappa)$ of the physical entity under study as the direct union $\textcircled{\vee}_{\omega\in\Omega}(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ over its classical state space Ω of its nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, and that each of these nonclassical components also satisfies **Axiom 1, 2, 3, 4, 5** and **6**. Additionally for each one of the nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ no classical properties except 0 and ω exist. It is for the nonclassical components that a further representation theorem can be proven such that a vector space structure emerges for each one of the nonclassical components. To do this we rely on the original representation theorem that Piron proved in [6] and on the more recent results proved in [17].

Theorem 2 (representation theorem: part 2). *Consider the same situation as in theorem 1, with additionally* **Axiom 4, 5** *and* **6** *satisfied. For each nonclassical component* $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$*, of which the lattice* $\mathcal{L}_\omega$ *has at least four orthogonal states*[7]*, there exists a vector space* V_ω*, over a division ring* K_ω*, with an involution of* K_ω*, which means a function*

$$^* : K_\omega \to K_\omega \tag{31}$$

such that for $k, l \in K_\omega$ *we have:*

$$(k^*)^* = k \tag{32}$$
$$(k \cdot l)^* = l^* \cdot k^* \tag{33}$$

and an Hermitian product on V_ω*, which means a function*

$$\langle \ , \ \rangle : V_\omega \times V_\omega \to K_\omega \tag{34}$$

such that for $x, y, z \in V_\omega$ *and* $k \in K_\omega$ *we have:*

$$\langle x + ky, z\rangle = \langle x, z\rangle + k\langle x, y\rangle \tag{35}$$
$$\langle x, y\rangle^* = \langle y, x\rangle \tag{36}$$
$$\langle x, x\rangle = 0 \Leftrightarrow x = 0 \tag{37}$$

and such that for $M \subset V_\omega$ *we have:*

$$M^\perp + (M^\perp)^\perp = V_\omega \tag{38}$$

where $M^\perp = \{y \mid y \in V_\omega, \langle y, x\rangle = 0, \forall x \in M\}$*. Such a vector space is called a generalized Hilbert space or an orthomodular vector space. And we have that:*

$$(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \cong (\mathcal{R}(V), \mathcal{L}(V), \nu) \tag{39}$$

where $\mathcal{R}(V)$ *is the set of rays of* V*,* $\mathcal{L}(V)$ *is the set of biorthogonally closed subspaces (subspaces that are equal to their biorthogonal) of* V*, and* ν *makes correspond with each such biorthogonal subspace the set of rays that are contained in it. If moreover a classical component is infinite dimensional, which means that it contains an infinite sequence of orthogonal atoms, the generalized Hilbert space is isomorphic to a real, complex or quaternionic Hilbert space.*

Proof: See [6, 7, 17].

6 Failing Axioms: Separated Entities

In the foregoing section we have explained in detail the representation theorem that makes it possible to go, by means of 6 axioms, from the completely operational structure of a state property space, to standard quantum mechanics with superselection rules. Since we only have presented the results here and not the proofs we want to mention that three hard and really long mathematical proofs are behind this. First there is the representation theorem of Piron, that brings us by means of 5 axioms to a generalized Hilbert space (with superselection rules) [6, 7]. This representation theorem makes use of the fundamental theorem of projective geometry [24], one of the hard and long standard mathematical proofs. Second there is Solèr's theorem that from axiom 6 brings us to one of the standard Hilbert spaces for the infinite dimensional nonclassical components [16]. Also this is a very hard and long mathematical theorem, not standard yet, because it is very recent, but certainly the important steps of it will become standard. Last, but not least, we did not mention quantum probability till now. But, since we have arrived now at the standard Hilbert spaces, we can use Gleason's theorem to derive in a unique way the standard quantum mechanical transition probability. This means that to get from our state property space with the six axioms to standard quantum mechanics with superselection rules hard and long mathematical proofs are necessary. It is a quite powerful situation to have been able to concentrate all this wealth of mathematical structure inside the transparent operational structure of a state property space and 6 axioms.

[7]Two states $p, q \in \Sigma_\omega$ are orthogonal if there exists $a \in \mathcal{L}_\omega$ such that $p < a$ and $q < a'$.

One of the aspects of this power is that we can investigate the status of these axioms. More specifically we can look at situations that have caused deep problems to be described by standard quantum mechanics, and investigate which of the axioms are at the origin of these problems. This is exactly what we have done in the past decades and we have been able to prove that **Axiom 4** and **5** are at the origin of two essential shortcomings of standard quantum mechanics: its incapacity to describe separated quantum entities and its incapacity to describe a continuous transition from quantum to classical. Within the approach of state property spaces where **Axiom 4** and **5** are relieved, the description of separated quantum entities is possible and it is also possible to describe a continuous transition from quantum to classical. We will explain now somewhat more in detail what really happens for the description of separated physical entities. How **Axiom 4** and **5** make it impossible to describe a continuous transition from quantum to classical we analyze in the next section, because we need the explanation for the quantum probabilities within our approach that we put forward in next section for this analysis.

6.1 The Impossibility to Describe Separated Entities

Let us first explain what is meant by separated physical entities. We consider the situation of a physical entity S that consists of two physical entities S_1 and S_2. The definition of 'separated' that has been used in [8, 9] is the following. Suppose that we consider two experiments e_1 and e_2 that can be performed respectively on the entity S_1 and on the entity S_2, such that the joint experiments $e_1 \times e_2$ can be performed on the joint entity S consisting of S_1 and S_2. We say that experiments e_1 and e_2 are separated experiments whenever for an arbitrary state p of S we have that (x_1, x_2) is a possible outcome for experiment $e_1 \times e_2$ if and only if x_1 is a possible outcome for e_1 and x_2 is a possible outcome for e_2. We say that S_1 and S_2 are separated entities if and only if all the experiments e_1 on S_1 are separated from the experiments e_2 on S_2.

Let us remark that S_1 and S_2 being separated does not mean that there is no interaction between S_1 and S_2. Most entities in the macroscopic world are separated entities. Let us consider some examples to make this clear. The earth and the moon, for example, are separated entities. Indeed, consider any experiment e_1 that can be performed on the physical entity earth (for example measuring its position), and any experiment e_2 that can be performed on the physical entity moon (for example measuring its velocity). The joint experiment $e_1 \times e_2$ consists of performing e_1 and e_2 together on the joint entity of earth and moon (measuring the position of the earth and the velocity of the moon at once). Obviously the requirement of separation is satisfied. The pair (x_1, x_2) (position of the earth and velocity of the moon) is a possible outcome for $e_1 \times e_2$ if and only if x_1 (position of the earth) is a possible outcome of e_1 and x_2 (velocity of the moon) is a possible outcome of e_2. This is what we mean when we say that the earth has position x_1 and the moon velocity x_2 at once. Clearly this is independent of whether there is an interaction, the gravitational interaction in this case, between the earth and the moon.

It is not easy to find an example of two physical entities that are not separated in the macroscopic world, because usually nonseparated entities are described as one entity and not as two. In earlier work we have given examples of nonseparated macroscopic entities [25, 26, 27]. The example of communicating vessels of water is a good example to give an intuitive idea of what nonseparation means. Consider two vessels V_1 and V_2 each containing 10 liters of water. The vessels are communicating by a tube, which means that they form a communicating set of vessels. Also the tube contains some water, but this does not play any role for what we want to show. Experiment e_1 consists of taking out water of vessel V_1 by a siphon, and measuring the amount of water that comes out. We give the outcome x_1 if the amount of water coming out is greater than 10 liters. Experiment e_2 consists of doing exactly the same on vessel V_2. We give outcome x_2 to e_2 if the amount of water coming out is greater than 10 liters. The joint experiment $e_1 \times e_2$ consists of performing e_1 and e_2 together on the joint entity of the two communicating vessels of water. Because of the connection, and the physical principles that govern communicating vessels, for e_1 and for e_2 performed alone we find 20 liters of water coming out. This means that x_1 is a possible (even certain) outcome for e_1 and x_2 is a possible (also certain) outcome for e_2. If we perform the joint experiment $e_1 \times e_2$ the following happens. If there is more than 10 liters coming out of vessel V_1 there is less than 10 liters coming out of vessel V_2 and if there is more than 10 liters coming out of vessel V_2 there is less than 10 liters coming out of vessel V_1. This means that (x_1, x_2) is not a possible outcome for the joint experiment $e_1 \times e_2$. Hence e_1 and e_2 are nonseparated experiments and as a consequence V_1 and V_2 are nonseparated entities.

The nonseparated entities that we find in the macroscopic world are entities that are very similar to the communicating vessels of water. There must be an ontological connection between the two entities, and that is also the reason that usually the joint entity will be treated as one single entity. A connection through dynamic interaction, as it is the case between the earth and the moon, interacting by gravitation, leaves the entities separated.

For quantum entities it can be shown that only when the joint entity of two quantum entities contains entangled states the entities are nonseparated quantum entities. It can be proven [25, 26, 27] that experiments are separated if and only if they do not violate Bell's inequalities. All this has been explored and investigated in many ways, and several papers have been published on the matter [25, 26, 27, 28, 29]. Interesting consequences for the Einstein Podolsky Rosen paradox and the violation of Bell's inequalities have been investigated [30, 31].

6.2 The Separated Quantum Entities Theorem

We are ready now to state the theorem about the impossibility for standard quantum mechanics to describe separated quantum entities [8, 9]. The demand of separation explained in the foregoing section can easily be transferred to the state property spaces by just demanding the yes/no experiments that test properties of one of the entities are separated from yes/no experiments testing properties of the other entity.

Theorem 3 (separated quantum entities theorem). *Suppose that S is a physical entity consisting of two separated physical entities S_1 and S_2. Let us suppose that* **Axiom 1, 2** *and* **3** *are satisfied and call $(\Sigma, \mathcal{L}, \kappa)$ the state property space describing S, and $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ and $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ the state property spaces describing S_1 and S_2.*
If **Axiom 4** *is satisfied, namely the covering law, then one of the two entities S_1 or S_2 is a classical entity, in the sense that one of the two state property spaces $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ or $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ contains only classical states and classical properties.*
If **Axiom 5** *is satisfied, namely weak modularity, then one of the two entities S_1 or S_2 is a classical entity, in the sense that one of the two state property spaces $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ or $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ contains only classical states and classical properties.*

Proof: See [8, 9]

The theorem proves that two separated quantum entities cannot be described by standard quantum mechanics. A classical entity that is separated from a quantum entity and two separated classical entities do not cause any problem, but two separated quantum entities need a structure where neither the covering law nor weak modularity are satisfied.

One of the possible ways out is that there would not exist separated quantum entities in nature. This would mean that all quantum entities are entangled in some way or another. If this is true, perhaps the standard formalism could be saved. Let us remark however that even standard quantum mechanics presupposes the existence of separated quantum entities. Indeed, if we describe one quantum entity by means of the standard formalism, we take one Hilbert space to represent the states of this entity. In this sense we suppose the rest of the universe to be separated from this one quantum entity. If not, we would have to modify the description and consider two Hilbert spaces, one for the entity and one for the rest of the universe, and the states would be entangled states of the states of the entity and the states of the rest of the universe. But, this would mean that the one quantum entity that we considered is never in a well-defined state. It would mean that the only possibility that remains is to describe the whole universe at once by using one huge Hilbert space. It goes without saying that such an approach will lead to many other problems. For example, if this one Hilbert space has to describe the whole universe, will it also contain itself, as a description, because as a description, a human activity, it is part of the whole universe. Another, more down to earth problem is, that in this one Hilbert space of the whole universe also all classical macroscopical entities have to be described. But classical entities are not described by a Hilbert space, as we have seen in section 5. If the hypothesis that we can only describe the whole universe at once is correct, it would anyhow be more plausible that the theory that does deliver such a description would be the direct union structure of different Hilbert spaces. But if this is the case, we anyhow are already using a more general theory than standard quantum mechanics. So we can as well use the still slightly more general theory, where axioms 4 and 5 are not satisfied, and make the description of separated quantum entities possible.

All this convinces us that the shortcoming of standard quantum mechanics to be able to describe separated quantum entities is really a shortcoming of the mathematical formalism used by standard quantum mechanics, and more notably of the vector space structure of the Hilbert space used in standard quantum mechanics.

7 An Explanation for Quantum Probability

The axiomatics that we have outlined in the foregoing sections still lacks a description of a very fundamental notion: probability. We may choose to derive the probabilistic features of quantum mechanics by means of Gleason's theorem, but this is really a great detour. It is our view that probability should be introduced on a much more profound level, because it is an operationally well-defined aspect of repeated measurements. The reason that the axiomatics was built in this way, neglecting probability, is mostly historical. As such, we felt an investigation into the probabilistic aspects of our approach was called for. What we have been able to show for quantum probability can best be illustrated by a very simple macroscopic example that, as we will see, also constitutes a model for the spin of a spin 1/2 quantum entity.

7.1 The Sphere Model

The example that we want to introduce consists of a physical entity constituted by a point particle P that can move on the surface of a sphere, denoted $surf$, with center O and radius 1. The unit-vector v giving the location of the particle on $surf$ represents the state p_v of the particle (see Fig. 1,a). Hence the collection of all possible states of the sphere model, that is how we will call our model, is given by $\Sigma = \{p_v \mid v \in surf\}$. We introduce the following yes/no experiments. For each point $u \in surf$, we introduce the experiment α_u. We consider the diametrically opposite point $-u$, and install an elastic band of length 2, such that it is fixed with one of its end-points in u and the other end-point in $-u$. Once the elastic is installed, the particle P falls from its original place v orthogonally onto the elastic, and sticks to it (Fig. 1,b). The elastic then breaks and the particle P, attached to one of the two pieces of the elastic (Fig. 1,c), moves to one of the two end-points u or $-u$ (Fig. 1,d). Depending on whether the particle P arrives in u (as in Fig. 1) or in $-u$, we give the outcome 'yes' or 'no' to α_u. The state p_v is changed by the experiment α_u into one of the two states p_u or p_{-u}.

We make the hypothesis that the elastic band breaks uniformly, which means that the probability that a particle in state p_v arrives in u, is given by the length of L_1 (which is $1 + \cos\theta$) divided by the total length of the elastic (which is 2). The probability that a particle in state p_v arrives in $-u$, is given by the length of L_2 (which is $1 - \cos\theta$) divided by the total length of the elastic. If we denote these probabilities respectively by $P(\alpha_u, p_v)$ and $P(\alpha_{-u}, p_v)$ we have:

$$P(\alpha_u, p_v) = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2} \tag{40}$$

$$P(\alpha_{-u}, p_v) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{41}$$

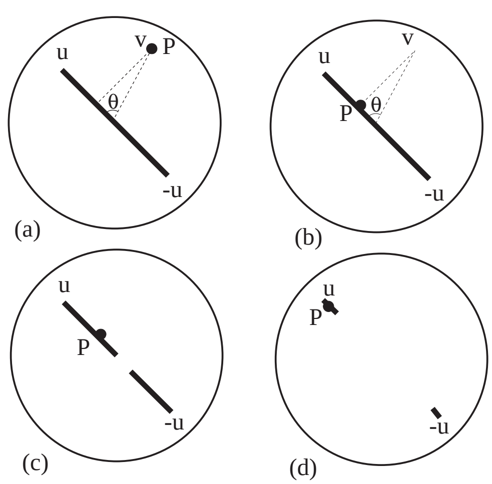

Fig. 1 : A representation the sphere model. In (a) the physical entity P is in state p_v in the point v, and the elastic corresponding to the experiment α_u is installed between the two diametrically opposed points u and $-u$. In (b) the particle P falls orthogonally onto the elastic and sticks to it. In (c) the elastic breaks and the particle P is pulled towards the point u, such that (d) it arrives at the point u, and the experiment α_u gets the outcome 'yes'.

In Figure 2 we represent the experimental process connected to α_u in the plane where it takes place, and we can easily calculate the probabilities corresponding to the two possible outcomes. In order to do so we remark that the particle P arrives in u when the elastic breaks in a point of the interval L_1, and arrives in $-u$ when it breaks in a point of the interval L_2 (see Fig. 2).

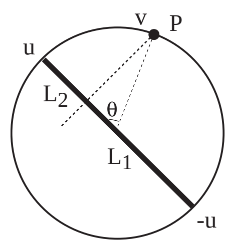

Fig. 2 : A representation of the experimental process in the plane where it takes place. The elastic of length 2, corresponding to the experiment α_u, is installed between u and $-u$. The probability, $P(\alpha_u,p_v)$, that the particle P ends up in point u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability, $P(\alpha_{-u},p_v)$, that the particle P ends up in point $-u$ is given by the length of the piece of elastic L_2 divided by the total length of the elastic.

We can easily show that the sphere model is an entity of which the description is isomorphic to the quantum description of the spin of a spin 1/2 particle, and as such delivers a model for this. This means that we can describe this macroscopic entity using the ordinary quantum formalism with a two-dimensional complex Hilbert space as the carrier for the set of states of the entity.

We remark that the sphere model is an elaboration of the well known Bloch or Poincaré model for the spin of a spin 1/2 particle, including also a modeling of the spin measurements. The sphere model as a model for an arbitrary quantum system described by a two dimensional Hilbert space was presented in [32, 33, 34]. It is possible to prove that for any arbitrary quantum entity one can construct a model like that of the sphere model [35, 36, 37, 38]. The explanation of the quantum structure that is given in the sphere model can thus also be used for general quantum entities. We have called this explanation the 'hidden measurement approach', hidden measurements referring to the fact that for a real measurement there is a 'lack of knowledge' about the measurement process in this approach. For the sphere model, for example, this lack of knowledge is the lack of knowledge about where the elastic will break during a measurement process.

7.2 What are Quantum Structures and Why Do They Appear in Nature?

The explanation for the quantum probabilities that we have put forward within the hidden measurement approach makes it possible to identify the reason why quantum structures appear in a natural way in nature.

The original development of probability theory aimed at a formalization of the description of the probabilities that appear as the consequence of a lack of knowledge. The probability structure appearing in situations of lack of knowledge was axiomatized by Kolmogorov and such a probability model is now called Kolmogorovian. Since the quantum probability model is not Kolmogorovian, it has now generally been accepted that the quantum probabilities are not associated with a lack of knowledge. Sometimes this conclusion is formulated by stating that the quantum probabilities are ontological probabilities, as if they were present in reality itself. In the hidden measurement approach we show that the quantum probabilities can also be explained as being due to a lack of knowledge, and we prove that what distinguishes quantum probabilities from classical Kolmogorovian probabilities is the nature of this lack of knowledge. Let us go back to the sphere model to illustrate what we mean.

If we consider again our sphere model (Fig. 1 and Fig. 2), and look for the origin of the probabilities as they appear in this example, we can remark that the probability is entirely due to a lack of knowledge about the measurement process. Namely the lack of knowledge of where exactly the elastic breaks during a measurement. More specifically, we can identify two main aspects of the experiment α_u as it appears in the sphere model.

(1) The experiment α_u effects a real change on the state p_v of the point P. Indeed, the state p_v changes into one of the states p_u or p_{-u} by the experiment α_u.

(2) The probabilities appearing are due to a lack of knowledge about a deeper reality of the individual measurement process itself, namely where the elastic breaks.

These two effects give rise to quantum-like structures, and the lack of knowledge about the deeper reality of the individual measurement process comes from 'hidden measurements' that operate deterministically in this deeper reality [32, 33, 34, 40, 42, 43].

One might think that our 'hidden-measurement' approach is in fact a 'hidden-variable' theory. In a certain sense this is true. If our explanation for the quantum structures is the correct one, quantum mechanics is compatible with a deterministic universe at the deepest level. There is no need to introduce the idea of an ontological probability. Why then the generally held conviction that hidden variable theories cannot be used for quantum mechanics? The reason is that those physicists who are interested in trying out hidden variable theories, are not at all interested in the kind of theory that we propose here. They want the hidden variables to be hidden variables of the state of the entity under study, so that the probability is associated to a lack of knowledge about the deeper reality of this entity; as we have mentioned already this gives rise to a Kolmogorovian probability theory. This kind of hidden variables relating to states is indeed impossible for quantum mechanics for structural reasons, with exception of course in the de Broglie-Bohm theory: there, in addition to the hidden state variables, a new spooky entity of 'quantum potential' is introduced in order to express the action of the measurement as a change in the hidden state variables.

If one wants to interpret our hidden measurements as hidden variables, then they are hidden variables of the measuring apparatus and not of the entity under study. In this sense they are highly contextual, since each experiment introduces a different set of hidden variables. They differ from the variables of a classical hidden variable theory, because they do not provide an 'additional deeper' description of the reality of the physical entity. Their presence, as variables of the experimental apparatus, has a well defined philosophical meaning, and expresses the fact that we, human beings, want to construct a model of reality independent of our experience of this reality. The reason is that we look for 'properties' or 'relations between properties', and these are defined by our ability to make predictions independent of our experience. We want to model the structure of the world, independently of our observing and experimenting with this world. Since we do not control these variables in the experimental apparatus, we do not allow them in our model of reality, and the probability introduced by them cannot be eliminated from a predictive theoretical model. In the macroscopic world, because of the availability of many experiments with negligible fluctuations, we find an 'almost' deterministic model. For a detailed study of other aspects of the hidden measurement approach we refer to [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47].

7.3 A Transition from Quantum to Classical

It is a great mystery how the macroscopic classical world around us has emerged from the microscopic quantum world, to our knowledge no satisfactory explanation or model has been given for this. What is certain is that somewhere and somehow a transition has to have happened. Taking into account the sphere model we can propose a simple model for such a transition: we introduce a parameter ϵ that parameterizes the amount of fluctuation that is present on the interaction between measuring apparatus and physical entity. The extended sphere model that appears in this way we have called the ϵ-model.

More specifically we introduce two real parameters $\epsilon \in [0,1]$, and $d \in [-1+\epsilon, 1-\epsilon]$, and consider the experiment $\alpha^\epsilon_{u,d}$ that consists of the particle P falling from its original place v orthogonally onto the line between u and $-u$, and arriving in a point, coordinated by the real number $v \cdot u$. The hypothesis is that the rubber band never breaks outside the interval $[d-\epsilon, d+\epsilon]$. In the interval $[d-\epsilon, d+\epsilon]$ we consider a uniformly distributed random variable λ, and the experiment proceeds as follows. If $\lambda \in [d-\epsilon\ ,\ v \cdot u[$, the particle P moves to the point u, and the experiment $\alpha^\epsilon_{u,d}$ gives outcome 'yes'. If $\lambda \in]v \cdot u\ ,\ d+\epsilon]$, it moves to the point $-u$, and the experiment $\alpha^\epsilon_{u,d}$ gives outcome 'no'. If $\lambda = v \cdot u$ it moves with probability $\frac{1}{2}$ to the point u, and the experiment $\alpha^\epsilon_{u,d}$ gives outcome 'yes', and it moves with probability $\frac{1}{2}$ to the point $-u$, and then the experiment $\alpha^\epsilon_{u,d}$ gives outcome 'no'. This completes the description of the experiment $\alpha^\epsilon_{u,d}$.

We shall consider now different situations labeled by the parameter ϵ.

1. $d+\epsilon \leq v \cdot u$. Then $P(\alpha^\epsilon_{u,d}, p_v) = 1$ and $P(\alpha^\epsilon_{-u,d}, p_v) = 0$.

2. $d-\epsilon < v \cdot u < d+\epsilon$

$$P(\alpha^\epsilon_{u,d}, p_v) = \frac{1}{2\epsilon}(v \cdot u - d + \epsilon) \quad (42)$$

$$P(\alpha^\epsilon_{-u,d}, p_v) = \frac{1}{2\epsilon}(d + \epsilon - v \cdot u) \quad (43)$$

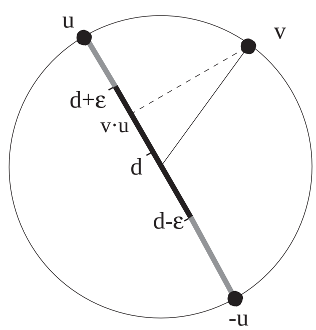

Fig. 3 : A representation of the experiment $\alpha^\epsilon_{u,d}$. We have chosen a case where d=0.2, and ϵ=$\frac{1}{2}$. If we consider the elastic-sphere-example realizing this situation, then the elastic breaks uniformly inside the interval $[d-\epsilon, d+\epsilon]$, and is unbreakable outside this interval, in the points of the set $[-1, d-\epsilon] \cup [d+\epsilon, 1]$.

3. $v \cdot u \leq d - \epsilon$. Then $P(\alpha^\epsilon_{u,d}, p_v) = 0$ and $P(\alpha^\epsilon_{-u,d}, p_v) = 1$.

We have the following situation: There are regions of eigenstates, one region centered around the point u, which we denote $eig(\alpha^\epsilon_{u,d})$, and another one centered around the point $-u$, which we denote $eig(\alpha^\epsilon_{-u,d})$. These regions of eigenstates are spherical sectors of $surf$. Let us denote a closed spherical sector centered around the point $u \in surf$ with angle θ by $sec(u, \theta)$. We remark that in the classical situation, for $\epsilon = 0$, $eig(\alpha^\epsilon_{u,d})$ and $eig(\alpha^\epsilon_{-u,d})$ are given by open spherical sectors centered around u and $-u$. We denote an open spherical sector centered around u and with angle θ by $sec^o(u, \theta)$. We call λ^ϵ_d the angle of the spherical sector corresponding to $eig(\alpha^\epsilon_{u,d})$ for all u, hence for $0 \neq \epsilon$ we have $eig(\alpha^\epsilon_{u,d}) = \{p_v \mid v \in sec(u, \lambda^\epsilon_d)\}$, and $eig(\alpha^0_{u,d}) = \{p_v \mid v \in sec^o(u, \lambda^0_d)\}$. We can verify easily that $eig(\alpha^\epsilon_{-u,d})$ is determined by a spherical sector centered around the point $-u$. We call μ^ϵ_d the angle of this spherical sector, hence for $0 < \epsilon$ we have $eig(\alpha^\epsilon_{-u,d}) = \{p_v \mid v \in sec(-u, \mu^\epsilon_d)\}$ and $eig(\alpha^0_{-u,d}) = \{p_v \mid v \in sec^o(-u, \mu^\epsilon_d)\}$. We denote by σ^ϵ_d the angle of superposition states. We have:

$$\cos \lambda^\epsilon_d = \epsilon + d \quad (44)$$

$$\cos \mu^\epsilon_d = \epsilon - d \quad (45)$$

$$\sigma^\epsilon_d = \pi - \lambda^\epsilon_d - \mu^\epsilon_d \quad (46)$$

$$\lambda^\epsilon_{-d} = \mu^\epsilon_d \text{ and } \sigma^\epsilon_{-d} = \sigma^\epsilon_d \quad (47)$$

The quantum situation ($\epsilon = 1$).

For $\epsilon = 1$ we always have $d = 0$, and the ϵ-model becomes the sphere model of foregoing section, hence a quantum model for the spin of a spin 1/2 quantum entity. For the eigenstate sets we find:

$$eig(\alpha^1_{u,0}) = \{p_v \mid +1 \leq v \cdot u\} = \{p_u\} \quad (48)$$

$$eig(\alpha^1_{-u,0}) = \{p_v \mid v \cdot u \leq -1\} = \{p_{-u}\} \quad (49)$$

which shows that the eigenstates are the states p_u and p_{-u}, and all the other states are superposition states.

The classical situation ($\epsilon = 0$).

The classical situation is the situation without fluctuations. If $\epsilon = 0$, then d can take any value in the interval $[-1, +1]$, and we have:

$$eig(\alpha^0_{u,d}) = \{p_v \mid d < v \cdot u\} \tag{50}$$

$$eig(\alpha^0_{-u,d}) = \{p_v \mid v \cdot u < d\} \tag{51}$$

which shows that for the classical situation, the only superposition states are the states p_v such that $v \cdot u = d$, and all the other states are eigenstates.

A continuous transition from a pure quantum entity to a pure classical entity is possible if we consider the ϵ-model and the transition from $\epsilon = 1$ to $\epsilon = 0$. For values of ϵ such that $0 < \epsilon < 1$ we have an entity that is neither quantum nor classical, but 'intermediate'.

As we have mentioned in the foregoing, we can prove that the intermediate situations do not satisfy **Axiom 4** and **5**. Let us put forward the two theorems that we have proven in this sense and make reference to where the proofs can be found.

Theorem 4. *If the entity described by the ϵ model* **Axiom 4** *is satisfied then $\epsilon = 0$ or $\epsilon = 1$.*

Proof: See [49]

Theorem 5. *If the entity described by the ϵ model* **Axiom 5** *is satisfied then $\epsilon = 0$ or $\epsilon = 1$.*

Proof: See [49]

For a detailed study of other aspects of the ϵ-model we refer to [48, 49, 50, 51, 52, 53]

8 Reflections on the New Theory

In the foregoing section we have seen that the quantum probability structure can be explained as being the consequence of the presence of fluctuations on the interaction between the measurement apparatus and the physical entity under consideration. The amount and the nature of the distribution of the these fluctuations determines the amount of deviation of the probability structure from a classical Kolmogorovian structure. The quantum mechanical probability structure appears in the situation of maximal amount of fluctuations with a uniform distribution. There exist intermediate situations 'in between classical and quantum' where the amount of fluctuation is neither zero nor maximal, and we have shown that there intermediate situations cannot be described within an axiomatic approach where the 6 axioms are satisfied, and it are **Axiom 4** and **5** that are at the origin of this impossibility.

More specifically the intermediate situations cannot be described when the nonclassical parts of the state property space are represented by Hilbert spaces as in standard quantum mechanics. This indicates that a satisfactory modelling of the classical limit will only be possible within a more general theory where **Axiom 4** and **5** are relieved. Since **Axiom 4** and **5** are also the axioms that stand in the way for a description of the situations of a joint entity consisting of two separated quantum entities, we believe that these two axioms should really be relieved and replaced by other axioms if we want to proceed ahead. Since **Axiom 5** (the covering law) is equivalent with the existence of a vector space structure for the set of states of the physical entity under consideration, we believe that the superposition principle cannot be seen any longer as a general principle which is always satisfied.

8.1 Nonlocality is Nonspatiality

In this section we want to put forward another fundamental consequence of the hidden measurement hypothesis. Suppose we consider the hidden measurement explanation for the concrete situation of the state of a quantum entity described by a wave function $\psi(x, y, z)$, element of $L^2(\mathbb{R}^3)$, the Hilbert space of the square integrable complex functions of three real variables. Suppose that the wave function is well spread out and hence has the form of a Gaussian. In quantum jargon this means that the quantum entity under consideration is well delocalized. Suppose that we make a measurement of position, localizing the quantum entity within an region A of space. The new wave function, after the measurement, is given by

$$\phi(x, y, z) = \frac{\chi_A \circ \psi(x, y, z)}{\|\chi_A \circ \psi(x, y, z)\|} \tag{52}$$

where χ is the characteristic function of the region A of space, and hence an orthogonal projection operator of the Hilbert space $L^2(\mathbb{R}^3)$. The quantum entity after the measurement is much more localized, namely it is localized within the region A of space. From the hidden measurement hypothesis follows that the original wave function $\psi(x, y, z)$ describes a reality of the quantum entity under consideration, that is changed by the measurement into another reality of the quantum entity. Hence this is a reality that is not inside space. Even the wave function $\phi(x, y, z)$ describes a reality that is still not inside space. What happens is that, starting from a nonspatial state $\psi(x, y, z)$, the measurement transforms this state into another state $\phi(x, y, z)$, which is somewhat more spatial, because representing the quantum entity within the region A of space, but still nonspatial inside A. Only a delta function would describe a reality that is inside space, as a limiting case hence.

This means that the hidden measurement hypothesis has as a consequence that the 'locus' of a quantum entity is created by the position measurement itself and does not exists before the measurement has been performed. Nonlocality has to be interpreted as nonspatiality, and space cannot be seen as the theatre of all of reality. Reality is much bigger than those parts of it that are contained inside space. Space should be interpreted as a structure that has emerged together with the macroscopic material entities that have emerged from the microscopic quantum entities, and it has emerged as 'their' space, meaning the 'space' in which these macroscopic entities exist and interact, as an emergent structure.

We have developed this picture in great detail in our group in Brussels and have called the philosophical view that corresponds to it the 'creation discovery view' [54, 55, 56, 57, 58].

8.2 Why General Relativity is not a Good Starting Point

From the hidden measurement hypothesis follows also the the whole of the universe can still supposed to be deterministic. The hidden measurement hypothesis does not imply this, but leaves it open as a possibility. In other words, quantum probabilities, that cannot be avoided if one describes a physical entity where there are intrinsic fluctuations present on the interaction between the measurement apparatus and the entity, can disappear if one focuses on a description of the whole of reality. In this sense, quantum structure and quantum probability appear as a consequence of considering a piece of the universe, and such a piece that it can only be studied by means of measurements that contain intrinsic fluctuations on their interactions with this piece. Classical entities, in this view, are special pieces of the universe, pieces such that there are measurements available that do not have these intrinsic fluctuations.

Having said this, it would mean that a deterministic theory as general relativity could eventually be the starting point for the new theory to be developed. This is true, except that there is a very important and somewhat hidden assumption in relativity theory as it has been elaborated now, namely the assumption that the set of events has to be described by the points of a four dimensional manifold, the points being interpreted as space-time coordinates of the events. This assumption is the one that makes it impossible, in our opinion, to use the approach of general relativity theory as a starting point for the new theory to be developed. In [59, 60, 61, 62] we have worked out an approach that could remedy this state of affairs. The main idea is that the set of happenings, we have introduced the concept 'happening' in [59, 60, 61, 62] to substitute for the concept 'event', to make it clear that what we do is different from general relativity, a priori does not coincide with the set of space-time points. In the next section we explain the ideas and approach presented in [59, 60, 61, 62].

9 A Possible Framework

In this section we propose a framework that could be used to construct 'the new theory' that would have quantum mechanics and relativity theory as special cases. To elaborate this framework we make use of all the insights that an operational and realistic approach to quantum mechanics has given us. More specifically, we also use these insights for the case of relativity theory.

To do so we want to analyze the way in which we penetrate, clothe and decorate reality starting from our personal experiences. The main point we want to make is that there is complex and mostly forgotten process at the origin of how we penetrate, clothe and decorate reality, and it is by analyzing in detail this process that we will be able to see clear in many of the paradoxical aspects of reality. Reality is out there. But the way that we know reality is through our experience of it. We order these experiences in a certain way, and are finally left over with a world view, in which what is 'real' has its specific place and function. We will see that physical theories, classical mechanics, quantum mechanics and relativity theory, have a lot of difficulties to recover and restate carefully what reality is, as we have introduced it within our pre-scientific personal world views. We will also see that a lot of the paradoxical aspects of our physical theories are due to a bad and fuzzy, and even sometimes wrong, understanding of this process.

9.1 Personal Experiences, Creations and Happenings

All the data that we gather about reality have come to us through our experiences. We consider an experience to be an interaction between a participator and a piece of the world. When the participator lives his or her experience, we say that this experience is present, and we call it the present experience of the participator. We remark that we consciously use the word 'participator' instead of the word 'observer' to indicate that we consider the cognitive receiver to participate creatively in his or her cognitive act. For the situation of a measurement, we consider the experimentalist and his or her experimental apparatus together to constitute the participator, and the physical entity under study is the piece of the world that interacts with the participator. The experiment is part of the experience.

Let us consider again the example that we mentioned in section 4.2, but now in terms of experiences. Consider a piece of wood and two experiences that we can have with the piece of wood. One experience consists of testing whether the piece of wood 'burns well'. The test consists of putting a sufficiently amount of fire during a sufficiently amount of time to the piece of wood en seeing whether it burns. Let us suppose that indeed the piece of wood burns well, and let us call this experience E_1 (I put fire to the piece of wood and it burns). Another experience that we consider of testing whether the piece of wood floats on water. The test consists of putting the piece of wood on water and seeing whether it floats. Let us suppose that indeed the piece of wood floats on water, and let us call this experience E_2 (I put the piece of wood on water and it floats). We deliberately have chosen these two experiences, because it is clear that we cannot experience them at once. If we would try to make the piece of wood burn and float on water, this would not work out well. So parts of both experience are clearly incompatible, in the sense that they cannot be realized at the same time. Even though this is obvious for everybody that considers our example, there are parts of both experiences that we do consider to be present at the same time. Indeed, we do attribute two 'properties' to the piece of wood, one property expressed as follows: 'The piece of wood has the property of burning well', and another property expressed as follows: 'The piece of wood has the property of floating on water'. We believe that this one piece of wood, with which we do not have any of the two experiences E_1 and E_2, has at the same time the properties of 'burning well' and 'floating on water'.

Let us give a second example, that originally was introduced in [59]. Consider the following situation: I am inside my house in Brussels. It is night, the windows are shut. I sit in a chair, reading a novel. I have a basket filled with walnuts at my side, and from time to time I take one of them, crack it and eat it. New York exists and is busy. Let us enumerate the experiences that are relevant in this situation: E_3 (I read a novel), E_4 (I experience the inside of my house in Brussels), E_5 (I experience that it is night), E_6 (I take a walnut, crack it and eat it), E_7 (I experience that New York is busy).

As in the case with the piece of wood, where it is impossible to experience E_1 and E_2 at once, also here I do not experience all these experiences at once. On the contrary, in principle, I only experience one experience at once, namely my present experience. Let us suppose that my present experience is E_3 (I read a novel). Then a lot of other things happen while I am living this present experience. These things happen in my present reality. While 'I am reading the novel' some of the happenings that happen are the following: H_3 (the novel exists), H_4 (the inside of my house in Brussels exists), H_5 (it is night), H_6 (the basket and the walnuts exist, and are at my side), H_7 (New York exists and is busy). All the happenings, and much more, happen while I live the present experience E_3 (I read a novel).

Why is the structure of reality such that what I am just saying is evident for everybody? Certainly it is not because I experience also these other happenings. My only present experience is the experience of reading the novel. But, and this is the origin of the specific structure of reality as it appears in my world view, I could have chosen to live an experience including one of the other happenings in replacement of my present experience. Let me recapitulate the list of the experiences that I could have chosen to experience in replacement of my present experience: E_4 (I observe that I am inside my house in Brussels), E_5 (I see that it is night), E_6 (I take a walnut, crack it and eat it), E_7 (I take the plane to New York and see that it is busy). The same is true for the example of the piece of wood. While I live the experience E_1 (I put fire on the wood and it burns), I could have lived in replacement of this experience the other experience E_2 (I put the wood on water and it floats), but I would have had to take another decision in my past, before I decided to start putting fire on the piece of wood.

These examples indicate how reality is structured within my world view. First of all we have to identify two main aspects of an experience. The aspect that is controlled and created by me, and the aspect that just happens to me and can only be known by me. Let us introduce this important distinction in a formal way. To see what I mean, let us consider the experience E_6 (I take a walnut, crack it and eat it). In this experience, there is an aspect that is an action of me, the taking and the cracking, and the eating. There is also an aspect that is an observation of me, the walnut and the basket. By studying how our senses work, I can indeed say that it is the light reflected on the walnut, and on the basket, that gives me the experience of walnut and the experience of basket. This is an explanation that only now can be given; it is, however, not what was known in earlier days when the first world views of humanity were constructed. But without knowing the explanation delivered now by a detailed analysis, we could see very easily that an experience contains always two aspects, a creation aspect, and an observation aspect, simply because our will can only control part of the experience. This is the creation aspect. For example, in E_3 (I read a novel) the reading is created by me, but the novel is not created by me. In general we can indicate for an experience the aspect that is created by me and the aspect that is not created by me. The aspect not created by me lends itself to my creation. We can reformulate an experience in the following way: E_6 (I take a walnut, crack it and eat it) becomes E_6 (The walnut is taken by me, and lends itself to my cracking and eating) and E_3 (I read a novel) becomes E_3 (The novel lends itself to my reading). The taking, cracking, eating, and reading will be called creations or actions and will be denoted by C_6 (I take, crack and eat) and C_3 (I read). The walnut and the novel will be called happenings and will be denoted by H_6 (The walnut) and H_3 (The novel).

> *A creation is that aspect of an experience created, controlled, and acted upon by me, and a happening is that aspect of an experience lending itself to my creation, control and action.*

An experience is determined by a description of the creation and a description of the happening. Creations are often expressed by verbs: to take, to crack, to eat, and to read, are the verbs that describe my creations in the examples. The walnut and the novel are happenings that have the additional property of being objects, which means happening with a great stability. Often happenings are expressed by a substantive.

> *Every one of my experiences E consists of one of my creations C and one of my happenings H, so we can write $E = (C, H)$.*

A beautiful image that can be used as a metaphor for our model of the world is the image of the skier. A skier skis downhill. At every instant he or she has to be in complete harmony with the form of the mountain underneath. The mountain is the happening. The actions of the skier are the creation. The skier's creation, in harmony fused with the skier's happening, is his or her experience.

9.2 How We Penetrate and Clothe Reality

Let us again consider the collection of experiences: E_3 (I read a novel), E_4 (I observe that I am inside my house in Brussels), E_5 (I see that it is night), E_6 (I take a walnut, crack it and eat it) and E_7 (I take the plane to New York and see that it is busy). Let us now represent in which way we penetrate and clothe reality that is made out of this small collection of experiences. E_3 (I read a novel) is my present experience. In my past I could, however, at several moments have chosen to do something else and this choice would have led me to have another present experience than E_3 (I read a novel). For example: One minute ago I could have decided to stop reading and observe that I am inside the house. Then E_4 (I observe that I am inside my house in Brussels) would have been my present experience. Two minutes ago I could have decided to stop reading and open the windows and see that it is night. Then E_5 (I see that it is night) would have been my present experience. Three minutes ago I could have decided to stop reading, take a walnut from the basket, crack it, and eat it. Then E_6 (I take a walnut, crack it and eat it) would have been my present experience. Ten hours ago I could have decided to take a plane and fly to New York and see how busy it was. Then E_7 (I go to New York and see that it is busy) would have been my present experience.

> *Even when they are not the happening aspect of my present experience, happenings 'happen' at present if they are the happening aspect of an experience that I could have lived in replacement of my present experience, if I had so decided in my past.*

The fact that a certain experience E consisting of a creation C and an happening H is for me a possible present experience depends on two factors: (1) I have to be able to perform the creation, (2) the happening has to be available. For example, the experience E_4 (I observe that I am inside my house in Brussels) is a possible experience for me, if: (1) I can perform the creation that consists in observing the inside of my house in Brussels. In other words, if this creation is in my personal power. (2) The happening 'the inside of my house in Brussels' has to be available to me. In other words, this happening has to be contained in my personal reality.

> *The collection of all creations that I can perform at the present I will call my present personal power. The collection of all happenings that are available to me at the present I will call my present personal reality.*

I define as my present personal reality the collection of these happenings, the collection of happenings that are available to one of my creations if I had used my personal power in such a way that at the present I fuse one of these creations with one of these happenings.

> *My present personal reality consists of all happenings that are available to me at present. My past reality consists of all happenings that were available to me in the past. My future reality consists of all happenings that will be available to me in the future. My present personal power consists of all creations that I can perform at present. My past personal power consists of all the creations that I could perform in the past. My future personal power consists of all creations I shall be able to perform in the future.*

Happenings can happen 'together and at once', because to happen a happening does not have to be part of my present experience. It is sufficient that it is available, and things can be available simultaneously. Therefore, although my present experience is only one, my present personal reality consists of an enormous amount of happenings all happening simultaneously. This concept of reality is not clearly understood in present physical theories. Physical theories know how to treat past, present and future. But reality is a construction about the possible. It is a construction about the experiences I could have lived but probably will never live.

9.3 Material Time and Material Happenings

From ancient times humanity has been fascinated by happenings going on in the sky, the motion of the sun, the changes of the moon, the motions of the planets and the stars. These happenings in the sky are periodic. By means of these periodic happenings humans started to coordinate the other experiences. They introduced the counting of the years, the months and the days. Later on watches were invented to be able to coordinate experiences of the same day. And in this sense material time was introduced in the reality of the human species. Again we want to analyze the way in which this material time was introduced, to be able to use it operationally if later on we analyze the paradoxes of time and space. My present experience is seldom a material time experience. But in replacement of my present experience, I always could have consulted my watch, and in this way live a material time experience E_8 (I consult my watch and read the time). In this way, although my present experience is seldom a material time experience, my present reality always contains a material time happening, namely the happening H_8 (The time indicated by my watch), which is the happening to which the creation C_8 (I consult) is fused to form the experience E_8. It is in this way that time coordination is introduced into my personal reality.

> *The collection of all creations that I can perform at time t, I will call my personal power at time t. The collection of all happenings that are available to me at time t, I will call my personal reality at time t.*

Of course, as we mentioned already, at time t, only one of my creations will be fused with one of the happenings available, that will lead to one experience that I live at time t.

9.4 Penetration in Depth and Width, Entity and Space

The two examples that we have considered give rise to seemingly different aspects of reality. We consider two happenings of the piece of wood, H_1 (the piece of wood entails the property of burning well) and H_2

(the piece of wood entails the property of floating on water), and two creations connected to this piece of wood, C_1 (I put fire on the piece of wood) and C_2 (I put the piece of wood on water). Then the two experiences that we have considered are $E_1 = (C_1, H_1)$ and $E_2 = (C_2, H_2)$. The reason that we attribute the two properties 'the piece of wood burns well' and 'the piece of wood floats on water' to the piece of wood, is because we know that the two happenings H_1 and H_2 are available at once for one of the two creations C_1 or C_2 that I would choose to fuse to give rise to one of the experiences E_1 or E_2. The example of the piece of wood shows us how we penetrate reality in depth, attributing properties to entities. Of course, we have to be aware that a more profound way of seeing this process of penetration in depth is the following. Certain happenings, as for example H_1 and H_2 cluster together, and the entity 'piece of wood' is the collection of all these happenings. We call this way of clustering happenings together into an entity, our 'penetration in depth' of reality. All other properties of the piece of wood are linked to happenings that we have classified within the process of penetration in depth. For example the weight of the piece of wood, the fact that it is constituted of cells, of molecules etc... The fact that the piece of wood constitutes an entity, is due to the fact that all these happenings indeed cluster together. Of course, as we know, this clustering together is not absolute. We can break the piece of wood into two pieces of wood, and destroy some of the clustering.

Our other example is a typical example of what we will call 'penetration in width'. The happenings H_3, H_4, H_5, H_6 and H_7 are not clustered together, and we will indeed not consider them as part of an entity. At first sight we could say that these happenings are situated in 'space'. But again we have to correct ourselves. It is indeed the other way around that we should proceed. Exactly as we have ordered the happenings that we collect by penetration in depth into an entity, because they are clustered together, we have ordered the happenings that we find by penetration in width into space. That is the way that we finally arrive at an image of my present reality existing of space being filled up with different entities, where each entity is a cluster of happenings ordered by penetration in depth, and the different entities are spread over space, in this way attributing to the ordering of the happenings that we have collected by penetration in width.

It would be very fruitful to perform an analysis of reality where the division in penetration in depth and penetration in width would be explicitly seen as two specific processes of penetration. In future work we want to engage in such an analysis, because we believe that it will reveal us deep and new insights into the nature of reality. In this paper we want to analyze some other aspects of this penetration. In our penetration in width something remarkable occurs. We find entities, like our fellow human beings, occupying places in space other than the space we occupy ourselves. And we call these entities 'fellow human beings' because we believe that they also penetrate their personal reality in a similar way than we penetrate our own personal reality. The big adventure of communication and dialogue starts here. Remark that we do not experience something similar in our personal penetration in depth.

10 The Nature of the Present

Before being able to analyze in which way we fuse personal realities into an encompassing inter subjective reality, we have to analyze in a detailed way what is the nature of space within our approach. To do so we have to take into account the results of relativity theory.

10.1 Relativity Theory and My Personal Present

Let us suppose that I am here and now in my house in Brussels, and it is April 10, 2003, 3 PM exactly. I want to find out 'what is the material reality for me now?'. Let us use the definition of reality given in the foregoing section and consider a place in New York, for example at the entrance of the Empire State building, and let us denote, the center of this place by (x_1, x_2, x_3). I also choose now a certain time, for example April 10, 2003, 3 PM exactly, and let me denote this time by x_0. I denote the happening that corresponds with the spot (x_1, x_2, x_3) located at the entrance of the Empire State building, at time x_0 by H_9. I can now try to investigate whether this happening H_9 is part of my personal present.

The question I have to answer is, can I find a creation of localization l, in this case this creation is just the observation of the spot (x_1, x_2, x_3) at the entrance of the Empire State building, at time x_0, that can be fused with this happening m. The answer to this question can only be investigated if we take into account the fact that I, who want to try to fuse a creation of localization to this happening, am bound to my body, which is also a material entity. I must specify the question introducing the material time coordinate that I coordinate by my watch. So suppose that I coordinate my body by the four numbers (y_0, y_1, y_2, y_3), where y_0 is my material time, and (y_1, y_2, y_3) is the center of mass of my body. We apply now our operational definition of reality. At this moment, April 10, 2003 at 3 PM exactly, my body is in my house in Brussels, which means that (y_0, y_1, y_2, y_3) is a point such that y_0 equals April 10, 2003, 3 PM, and (y_1, y_2, y_3) is a point, the center of mass of my body, somewhere in my house in Brussels. This shows that (x_0, x_1, x_2, x_3) is different from (y_0, y_1, y_2, y_3), in the sense that (x_1, x_2, x_3) is different from (y_1, y_2, y_3) while $x_0 = y_0$. The question is now whether (x_0, x_1, x_2, x_3) is a point of my personal present, hence whether it makes sense to me to claim that now, April 10, 2003, 3 PM, the entrance of the Empire State building 'exists'. If our theoretical framework corresponds in some way to our pre scientific construction of reality, the answer to the foregoing question should be affirmative. Indeed, we all believe that 'now' the entrance of the Empire State building exists. Let us try to investigate in a rigorous way this question in our framework.

We have to verify whether it was possible for me to decide somewhere in my past, hence before April 10, 2003, 3 PM, to change some of my plans of action, such that I would decide to travel to New York, and arrive exactly at April 10, 2003, 3 PM at the entrance of the Empire State building, and observe the spot (x_1, x_2, x_3). There are many ways to realize this experiment, and we will not go into details here, because we shall come back later to the tricky parts of the realization of this experiment. I could thus have experienced the spot (x_1, x_2, x_3) at April 10, 2003, 3 PM, if I had decided to travel to New York at some time in my past. Hence (x_0, x_1, x_2, x_3) is part of my reality. It is sound to claim that the entrance of the Empire State building exists right now. And we note that this does not mean that I have to be able to experience this spot at the entrance of the Empire State building now, April 10, 2003, 3 PM, while I am inside my house of Brussels. I repeat again, reality is a construction about the possible happenings that I could have fused with my actual creation. And since I could have decided so in my past, I could have been at the entrance of the Empire State building, now, April 10, 2003, 3 PM. Until this moment one could think that our framework only confirms our intuitive notion of reality, but our next example shows that this is certainly not the case.

Let us consider the same problem as above, but for another point of time-space. We consider the point (z_0, z_1, z_2, z_3), where $(z_1, z_2, z_3) = (x_1, x_2, x_3)$, hence the spot we envisage is again the entrance of the Empire State building, and z_0 is April 11, 2003, 3 PM exactly, hence the time that we consider is, tomorrow 3 PM. If I ask now first, before checking rigorously by means of our operational definition of reality, whether this point (z_0, z_1, z_2, z_3) is part of my personal present, the intuitive answer here would be 'no'. Indeed, tomorrow at the same time, 3 PM, is in the future and not in the present, and hence it is not real, and hence no part of my personal present (this is the intuitive reasoning). If we go now to the formal reasoning in our framework, then we can see that the answer to this question depends on relativity theory. Indeed, let us first analyze the question in a Newtonian conception of the world to make things clear. Remark that in a Newtonian conception of the world (which has been proved experimentally wrong, so here we are just considering it for the sake of clarity), my personal present coincides with 'the present', namely all the points of space that have the same time coordinate April 10, 2003, 3 PM. This means that the entrance of the Empire State building tomorrow 'is not part of my personal present'. The answer is here clear and in this Newtonian conception, my present personal reality is just the collection of all (u_0, u_1, u_2, u_3) where $u_0 = y_0$ and (u_1, u_2, u_3) are arbitrary. The world is not Newtonian, this we now know experimentally; but if we put forward an ether theory interpretation of relativity theory (let us refer to such an interpretation as a Lorentz interpretation) the answer again remains the same. In a Lorentz interpretation, my present personal reality coincides with the present reality of the ether, namely all arbitrary points of the ether that are at time y_0, April 10, 2003, 3 PM, and again tomorrow the entrance of the Empire State building is not part of my personal present.

For an Einsteinian interpretation of relativity theory the answer is different. To investigate this I have to ask again the question of whether it would have been possible for me to have made a decision in my past such that I would have been able to make coincide (y_0, y_1, y_2, y_3) with (z_0, z_1, z_2, z_3). The answer here is that this is very easy to do, because of the well known, and experimentally verified, effect of 'time dilatation'. Indeed, it would for example be sufficient that I go back some weeks in my past, let us say March 15, 2003, 3 PM, and then decide to step inside a space ship that can move with almost the speed of light, so that the time when I am inside this space ship slows down in such a way, that when I return with the space ship to planet earth, still flying with a speed close to the velocity of light, I arrive in New York at the entrance of the Empire State building with my personal material watch indicating April 10, 2003, 3 PM, while the watch that remained at the entrance of the Empire State building indicates April

11, 2003, 3 PM. Hence in this way I make coincide (y_0, y_1, y_2, y_3) with (z_0, z_1, z_2, z_3), which proves that (z_0, z_1, z_2, z_3) is part of my personal present. First I could remark that in practice it is not yet possible to make such a flight with a space ship. But this point is not crucial for our reasoning. It is sufficient that we can do it in principle. We have not yet made this explicit remark, but obviously if we have introduced in our framework an operational definition for reality, then we do not have to interpret such an operational definition in the sense that only operations are allowed that actually, taking into account the present technical possibilities of humanity, can be performed. If we were to advocate such a narrow interpretation, then even in a Newtonian conception of the world, the star Sirius would not exist, because we cannot yet travel to it. What we mean with operational is much wider. It must be possible, taking into account the actual physical knowledge of the world, to conceive of a creation that can be fused with the happening in question, and then this happening pertains to our personal reality.

10.2 Einstein versus Lorentz

We can come now to one of the points that we want to make, clarifying the time paradox that distinguishes an ether interpretation of relativity (Lorentz) from an Einsteinian interpretation. To see clearly in this question, we must return to the essential aspect of the construction of reality in our framework, namely, the difference between a creation and a happening. In order to clarify this, we first give another example. Suppose that I am a painter and I consider again my personal present, at April 10, 2003, 3 PM, as indicated on my personal material watch. I am in my house in Brussels and let us further specify: the room where I am is my workshop, surrounded by paintings, of which some are finished, and others I am still working on. Clearly all these paintings exist in my presents reality, April 10, 2003, 3 PM. Some weeks ago, when I was still working on a painting that now is finished, I could certainly have decided to start to work on another painting, a completely different one, that now does not exist. Even if I could have decided this some weeks ago, everyone will agree that this other painting, that I never started to work on, does not exist now, April 10, 2003, 3 PM. The reason for this conclusion is that the making of a painting is a 'creation' and not a happening. It is not so that there is some 'hidden' space of possible paintings such that my choice of some weeks ago to realize this other painting would have made me to detect it. If this were to be the situation with paintings, then indeed also this painting would exist now, in this hidden space. But with paintings this is not the case. Paintings that are not realized by the painter are potential paintings, but they do not exist. With this example of the paintings we can explain very well the difference between Lorentz and Einstein. For an ether interpretation of relativity the fact that my watch is slowing down while I decide to fly with the space ship nearly at the speed of light and return to the entrance of the Empire State building when my watch is indicating April 10, 2003, 3 PM while the watch that remained at the Empire State building indicates April 11, 2003, 3 PM, is interpreted as a 'creation'. It is seen as if there is a real physical effect of creation on the material functioning of my watch while I travel with the space ship, and this effect of creation is generated by the movement of the space ship through the ether. Hence the fact that I can observe the entrance of the Empire State building tomorrow April 11, 2003, 3 PM, if had decided some weeks ago to start travelling with the space ship, only proves that the entrance of the Empire State building tomorrow is a potentiality. Just like the fact that this painting that I never started to paint could have been here in my workshop in Brussels is a potentiality. This means that as a consequence the spot at the entrance of the Empire State building tomorrow is not part of my present reality, just as the possible painting that I did not start to paint is not part of my present reality. If we however put forward an Einsteinian interpretation of relativity, then the effect on my watch during the space ship travel is interpreted in a completely different way. There is no physical effect on the material functioning of the watch - remember that most of the time dilatation takes place not during the accelerations that the space ship undergoes during the trip, but during the long periods of flight with constant velocity nearly at the speed of light - but the flight at a velocity close to the speed of light 'moves' my space ship in the time-space continuum in such a way that time coordinates and space coordinates get mixed. This means that the effect of the space-ship travel is an effect of a voyage through the time-space continuum, which brings me at my personal time of April 10, 2003, 3 PM at the entrance of the Empire State building, where the time is April 11, 2003, 3 PM. And hence the entrance of the Empire State building is a happening, an actuality and not just a potentiality, and it can be fused with my present creation. This means that the happening (z_0, z_1, z_2, z_3) of April 11, 2003, 3 PM, entrance of the Empire State building, is an happening that can be fused with my creation of observation of the spot around me at April 10, 2003, 3 PM. Hence it is part of my personal present.

The entrance of the Empire State building at April 11, 2003, 3 PM exists for me today, April 10, 2003, 3 PM. If we advocate an Einsteinian interpretation of relativity theory we have to conclude from the foregoing section that my personal reality is four dimensional. This conclusion will perhaps not amaze those who always have considered the time-space continuum of relativity as representing the new reality. Now that we have however defined very clearly what this means, we can start investigating the seemingly paradoxical conclusions that are often brought forward in relation with this insight.

10.3 The Process View and the Geometric View

The paradoxical situation that we can now try to resolve is the confrontation of the process view of reality with the geometric view. It is often claimed that an interpretation where reality is considered to be related to the four-dimensional time-space continuum contradicts another view of reality, namely the one where it is considered to be of a process-like nature. By means of our framework we can now understand exactly what these two views imply and see that there is no contradiction. Let us repeat now what in our framework is the meaning of the conclusion that my personal reality is four dimensional. It means that, at a certain specific moment, that I call my 'present', the collection of places that exist, and that I could have observed if I had decided to do so in my past, has a four-dimensional structure, well represented mathematically by the four dimensional time-space continuum. This is indeed my personal present. This does not imply however that this reality is not constantly changing. Indeed it is constantly changing. New entities are created in it and other entities disappear, while others are very stable and remain into existence. This in fact is the case in all of the four dimensions of this reality. Again I have to give an example to explain what I mean. We came to the conclusion that now, at April 10, 2003, 3 PM the entrance of the Empire State building exists for me while I am in my house in Brussels. But this is not a statement of deterministic certainty. Indeed, it is quite possible that by some chain of events, and without me knowing of these events, that the Empire State building had disappeared, for example because it had been rebuilt; thus my statement about the existence of the entrance of the Empire State building 'now', although almost certainly true, is not deterministically certain[8]. The reason is again the same, namely that reality is a construction of what I would have been able to experience, if I had decided differently in my past. The knowledge that I have about this reality is complex and depends on the changes that go on continuously in it. What I know from experience is that there do exist material objects, and the Empire State building is one of them, that are rather stable, which means that they remain in existence without changing too much. To these stable objects, material objects but also energetic fields, I can attach the places from where I can observe them. The set of these places has the structure of a four-dimensional continuum. At the same time all these objects are continuously changing and moving in this four-dimensional scenery. Most of the objects that I have used to shape my intuitive model of reality are the material objects that surround us here on the surface of the earth. They are all firmly fixed in the fourth dimension (the dimension indicated by the 0 index, and we should not call it the time dimension) while they move easily in the other three dimensions (those indicated by the 1, 2, and 3 index). Other objects, for example the electromagnetic fields, have a completely different manner of being and changing in this four-dimensional scenery. This means that in our framework there is no contradiction between the four-dimensionality of the set of places and the process-like nature of the world. When we come to the conclusion that the entrance of the Empire State building, tomorrow, April 11, 2003, 3 PM also exists for me now, then our intuition reacts more strongly to this statement, because intuitively we think that this implies that the future exists, and hence is determined and hence no change is possible. This is a wrong conclusion which comes from the fact that during a long period of time we have had the intuitive image of a Newtonian present, as being completely determined. We have to be aware of the fact that it is the present, even in the Newtonian sense, which is not determined at all. We can only say that the more stable entities in our present reality are more strongly determined to be there, while the places where they can be are always there, because these places are stable with certainty.

10.4 The Singularity of My Personal Present

We now come back to the construction of reality in our framework which we have confronted here with the Einsteinian interpretation of relativity theory. Instead of wondering about the existence of the entrance

[8] We remark that our example of the entrance of the Empire State building was given in [60, 61], long before the terrible chain of events that led to the destruction of the Twin Towers came into being.

of the Empire State building tomorrow, April 11, 2003, 3 PM, I can also question the existence of my own house at the same place of the time-space continuum. Clearly I can make an analogous reasoning and come then to the conclusion that my own house, and the chair where I am sitting while reading the novel, and the novel itself, and the basket of wall nuts beside me, *etc* . . . , all exist in my present reality at April 11, 2003, 3 PM, hence tomorrow. If we put it like that, we are even more sharply confronted with a counter-intuitive aspect of the Einsteinian interpretation of relativity theory. But in our framework, it is a correct statement . We have to add however that all these objects that are very close to me now April 10, 2003, 3 PM, indeed also exist in my present reality at April 11, 2003, 3 PM, but the place in reality where I can observe them is of course much further away for me. Indeed, to be able to get there, I have to fly away with a space ship at nearly the velocity of light. We now come to a very peculiar question that will confront us with the singularity of our reality construction. Where do I myself exist? Do I also exist tomorrow April 11, 2003, 3 PM? If the answer to this question is affirmative, we would be confronted with a very paradoxical situation. Because indeed I, and this counts for all of you also, cannot imagine myself to exist at different instants of time. But our framework clarifies this question very easily. It is impossible for me to make some action in my past such that I would be able to observe myself tomorrow April 11, 2003, 3 PM. But if I had chosen to fly away and come back with the space-ship, it would be quite possible for me to observe now, on April 10, 2003, at 3 PM on my personal watch, the inside of my house tomorrow April 11, 2003, 3 PM. As we remarked previously, this proves that the inside of my house tomorrow is part of my personal reality today. But I will not find myself in it. Because to be able to observe my house tomorrow April 11, 2003, 3 PM, I have had to leave it. Hence, in this situation I will enter my house, being myself still at April 10, 2002, 3 PM, but with my house and all the things in it, being at April 11, 2002, 3 PM. This shows that there is no contradiction. In fact, if it comes to a much more common happening, the existence of the Empire State building at April 10, 2003, 3 PM, nobody would even make the confusion, and think that since to experience this existence I would have to take the plane and fly to New York 10 hours before, that this would imply that I would be at two places at once, in my house and in New York.

We can conclude this analysis of the nature of our personal present with the following observation. The nature of the present does not correspond very well to our intuitive idea of what this nature is. Indeed, intuitively we think of the present - our intuition being guided by a Newtonian world view - as the collection of all entities and their interactions that are inside space, 'now', simultaneously with our personal 'now'. This is a wrong conception. First of all the present has a four dimensional structure and not a three dimensional one as would follow from a Newtonian world view. We note that this is mainly a consequence of the exactness of relativity theory. But secondly, and this is more important, and also not understood by scientists that are aware of relativity theory, the present is more like we intuitively think about the future. It is not determinate in the simple way that we imagine. It is the collection of all happenings that I could have fused with one of my present creations, if I would have decided to do something different in my personal past. This collection is determined is some way, but in a rather complicated way, which is equivalent with how we intuitively feel the future to be determined in some complicated way. Let us try to see more clear what is the nature of this determination.

10.5 The Structure of My Personal Present

The basic structure of my personal present is demonstrated in Fig. 4. I live the experience E_3 at time t_5, and this is my only present experience. But at respectively times t_4, t_3, t_2, and t_1, I could have chosen another action than the one that lead to experience E_3, and that would lead me then respectively to experience E_4, E_5, E_6, or E_7. That is the reason why happenings H_4, H_5, H_6, and H_7, exist at time t_5 in my personal present.

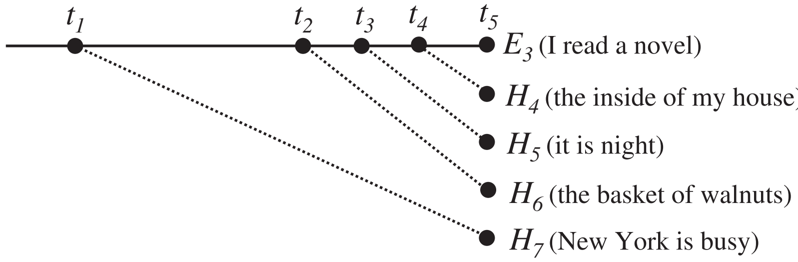

Fig. 4 : A representation of my personal present. I experience the experience E_3 at time t_5. At time t_5 also happenings H_4, H_5, H_6, and H_7 happen, because I could have decided, respectively at times t_4, t_3, t_2, and t_1, to take another action than the one that leads me to experience E_3 at time t_5.

Let us represent the situation that we have analyzed in section 10.1 and 10.2 in Fig. 5.

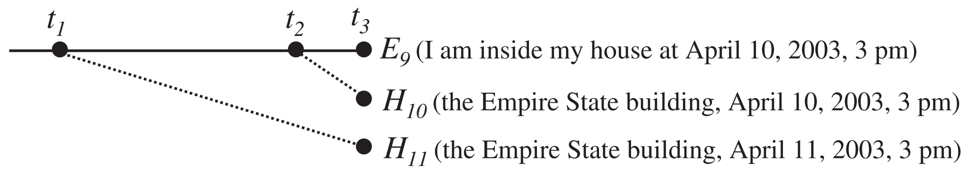

Fig. 5 : A representation of my personal present. I experience the experience E_9 at time t_3. At time t_3 also happenings H_{10} and H_{11} happen, because I could have decided, respectively at time t_2 and time t_1, to take another action than the one that leads me to experience E_9 at time t_3.

We have shown in section 10.4 that the situation related to the fact that also happenings that in my intuitive view on the present I would classify in the future are in my present does not lead to a paradox of 'being able to meet myself'. In the next chapter we will see that our intuitive view on the structure of the present has its roots in a further development of my personal present, namely the development connected with joining different personal presents into one inter subjective present.

10.6 Fusing of Different Personal Realities

In the foregoing we have analyzed in which way I penetrate and clothe reality. Now an extra hypothesis comes into play. We know that the world is populated with other people, that also penetrate and clothe reality in this way, by forming their personal reality. Let us analyze some of the fundamental problems that appear when we attempt to fuse two such personal realities together, into one encompassing reality.

A first remark that we have to make is the following. If we consider Fig. 5, we see that the Empire State Building at April 10, 2003, 3 PM, as well as the Empire State building at April 11, 2003, 3 PM, are both happenings that are part of my personal present at April 11, 2003, 3 PM. What is the problem with this. I, myself, I am only present 'now', living my present experience. And a moment later, this present experience has become one of my past experiences, and I am into a new present experience. My stream of experiences is hence a stream that moves from past to present, and what I have called my personal material time takes track of this stream of experiences. If we now believe that the Empire State building is also an entity, as I am, then we have to make the hypothesis that the Empire State building exists only at one moment within its personal reality, namely its present. This means that there is a difference between 'exist' for the Empire State Building, within my personal present, and 'exist' for the Empire State Building within its own personal present. We remark that we have already used the personal presents of the Empire State Building by indicating its times, April 11, 2003, 3 PM and April 10, 2003, 3 PM, which are, within the personal reality of the Empire State Building two happenings that are not in the same reality. The problem becomes more obvious when we consider another person involved. So suppose that I consider my student Bart D'Hooghe, with whom I have discussed some of the problems treated in this article. Then in my personal present Bart exists at all 'future' times that are indicated by his watch. In Bart's personal present, I exist at all future times that are indicated by my watch as is shown in Fig. 6.

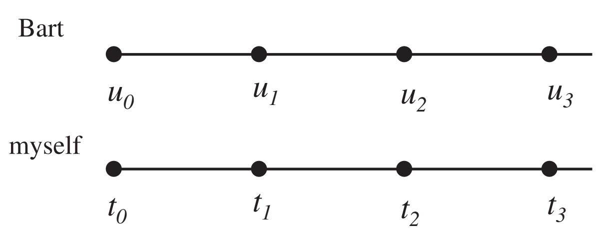

Fig. 6 : A presentation of the personal present of myself and of Bart.

We indicate the personal presents of myself, as they run through time, with a parameter t, that indicates the time of my personal watch, and in a similar way, we indicate the personal presents of Bart, as they run through time, with a parameter u, that indicates the time at Bart's personal watch. Then, Bart, at all times u_0, u_1, u_2, u_3, is part of my personal present at time t_0, while myself, at all times t_0, t_1, t_2, t_3, I am part of Bart's personal present at time u_0.

When time would be Newtonian the two types of existence can be fused together without problems. Indeed, in a Newtonian time frame, it will be the case that, for example, t_0 and u_0 can be said to be simultaneous, as well as t_1 and u_1, t_2 and u_2, and t_3 and u_3. And, if this is the case, we make a special slice within - for example - my personal present at time t_2, namely the slice that contains exactly Bart's personal present at time u_2 (and in a similar way t_0, t_1, and t_3, are identified with u_0, u_1, and u_3). Due to relativity theory, such a simple synchronization of the two watches, my watch and the one of Bart, cannot be made. Even though we have the deep intuition that it should be possible, suppose that I am at my personal present at time t_1, to elect 'one' and 'only one' personal present moment of Bart (hence one of the u's) to make it coincide with the t_1 of my personal present. We have to be aware however that this deep intuition is not correct taken into account the analysis about the way we penetrate reality. My personal present is the collection of all the happenings that I could have lived if I would have decided something different in my past, something that would lead me to experience the specific happening that I am considering. The only way in which a subcollection of this collection of happenings could stand out, and form a special subcollection, each of the happenings of the subcollection being in a certain sense more specifically related to my personal experience, is when it would be possible to classify the things that I could have done in my past, such that certain things stand out above the others. In the general scheme that we consider this is not possible. We can put forward the image that the personal presence of Bart, moment of personal time of Bart after following moment of personal time of Bart, runs through the set of happenings that are connected with Bart in my personal present. And in a similar way, I run through the collection of happenings that are connected with me in the personal present of Bart. It is till now an unresolved problem, linked to the problem of synchronization in relativity theory [63, 64], whether this view can be upheld. We believe that this problem should be analyzed taking into account the two subtle and different notions of reality that we consider in this paper. We want to investigate this problem in future work.

11 Conclusion

We believe that the new theory that will have quantum mechanics and general relativity as special cases and also explain why they are special cases will have to be formulated within a structural and mathematical context that is very different from the existing physical theories. The operational axiomatic approach that we have exposed in sections 2, 3, 4, 5 and 6 can deliver a framework for this new theory. This approach makes it possible to formalize the 'subject-object actions' that are possible in our world in the most general way. In this sense it is similar to general relativity in that it formalizes the 'geometrical beings' in our world in the most general way, but it is more general, because geometrical beings are special cases of subject-object interactions, where the subject-object action is reduced to a simple observation. Since we have identified two failing axioms within the set of six axioms that make the general approach equivalent to standard quantum mechanics with superselection rules, in relation with well defined physical situations, *i.e.* the situation of separated quantum entities (see section 6.1), and the situation of the continuous transition from quantum to classical (see section 7.3), it will not be possible to use standard quantum mechanics for this general enterprise, its mathematical structure being too limited. That is why the operational axiomatic approach may be an essential tool to build the necessary structure for the new theory. The explanation for quantum probability that we have developed, and that is presented in section 7, shows that the nonKolmogorovian nature of quantum probability can be explained as being the consequence of a lack of knowledge about the interaction between the measuring apparatus and the physical entity under consideration, hence due to the presence of fluctuations on this interaction. This means that there is no incompatibility with the hypothesis of a universe as a whole that is deterministic, and the presence of quantum probability, as an irreducible probability for the description of a piece of this universe that we call physical entity. The third important aspect of our approach is the interpretation of nonlocality. From our explanation for the quantum probability structure follows that nonlocality has to be interpreted as nonspatiality. Nonlocal states of a quantum entity are nonspatial states, meaning literally that a quantum entity in such a state is not inside space. As a consequence, space cannot be seen as an all embracing theatre for reality, but must be interpreted as a macroscopic structure that has emerged in the same process of emergence of macroscopic physical entities from the micro-world. Space is 'the space' of the macroscopic physical entities and not of the microscopic quantum entities. In our opinion this is the main reason why the global and fundamental approach of general relativity cannot incorporate the quantum world. One of the steps of general relativity is indeed to identify the set of all events with the set of all time-space points of the four dimensional time-space continuum (section 8). In section 9 and 10 we make the first steps for a theory that uses the insights and methodology of the operational axiomatic approach to quantum mechanics within a scheme the aims at a description in the style of relativity theory.

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