Macroscopic models for quantum systems and computers

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Abstract. We present examples of macroscopic systems entailing a quantum mechanical structure. One of our examples has a structure which is isomorphic to the spin structure for a spin $1/2$ and another system entails a structure isomorphic to the structure of two spin $1/2$ in the entangled singlet state. We elaborate this system by showing that an arbitrary tensor product state representing two entangled qubits can be described in a complete way by a specific internal constraint between the ray or density states of the two qubits, which describes the behavior of the state of one of the spins if measurements are executed on the other spin. Since any n-qubit unitary operation can be decomposed into 2-qubit gates and unary operations, we argue that our representation of 2-qubit entanglement contributes to a better understanding of the role of n-qubit entanglement in quantum computation. We illustrate our approach on two 2-qubit algorithms proposed by Deutsch, respectively Arvind et al. One of the advantages of the 2-qubit case besides its relative simplicity is that it allows for a nice geometrical representation of entanglement, which contributes to a more intuitive grasp of what is going on in a 2-qubit quantum computation.

1. Introduction

Richard Feynman has contributed to different fields of science in a fundamental way. Almost 30 years ago he wrote a paper on the possible advantages of using quantum systems for certain computational tasks and by this became one of the founding fathers of the field of quantum computation [1]. He described how simulating compound quantum systems on a classical computer quickly becomes intractable as the number of particles becomes large, and suggested how quantum systems themselves could be more suitable to implement such tasks [2]. Deutsch defined the Quantum Turing Machine and provided a 2-qubit quantum algorithm, which could solve a problem faster than possible on a classical digital computer [3]. The interest for quantum algorithms got a boost when Shor presented a factoring algorithm for a quantum computer, which has real life application since it would allow to break cryptographic codes faster than by any means classical possible [4]. Grover showed the existence of a fast quantum algorithm for database search [5]. After the work of Deutsch, Shor, Grover and others it became clear that the possible advantages of computers based on quantum principles could well go far beyond the

efficient simulation of quantum systems. Although still much work needs to be done on both the theoretical and the experimental level before a real-life quantum computer can be built, quantum computation has become over the last decade one of the most fascinating fields of science, both on the level of applications (cracking codes) as well as on the more fundamental philosophical level (Quantum Turing Machine). In our paper we would like to pay tribute to Feynman's groundbreaking ideas, by exploring what could be considered as the opposite of Feynman's original idea, namely how in principle certain macroscopic models could be used to implement quantum algorithms. As we shall demonstrate, these macroscopic systems simulate quantum behaviour in a specific way, inaccessible to classical, digital computers. Also, the physical realization of such macroscopic devices is not a trivial task at all, such that these 'macroscopic quantum-like computers' are still in their preliminary steps.

We adopt an operational approach to quantum mechanics, in which a physical entity is described by its set of states, its set of properties and a relation of 'actuality' between these two sets which expresses which properties are actual when the system is in a specific state [6]. Following this operational approach, it is a natural step to consider models which exhibit 'quantum behavior' on a structural level, even if they are not microscopic in size. This has lead to a 'quantum-like' model representing the spin properties of a quantum spin 1/2 particle [7]. Putting two such models together in a suitable way, one obtains a model for the singlet state of two entangled spin 1/2 [8], which is crucial if one wants to implement quantum algorithms in an efficient way (i.e. using the same number of steps as a quantum computer). By generalizing this 'putting together' procedure in a more formal way, one recovers the full set of states of the compound system of two entangled spin 1/2 [9]. This means that in principle one can represent any entangled state of a 2-qubit quantum computer with this model. We illustrate this by two quantum algorithms and show in which way the result is obtained by running the quantum algorithm. The first algorithm is the famous algorithm of Deutsch [3]. It is well-known that for the 2-qubit case this algorithm only uses product states and does not require entangled states. This is why we consider a second algorithm, proposed by Arvind et al., which does need entangled states to obtain the outcome [10, 11]. In this sense this algorithm is an ideal candidate to show how problems could be solved on our model of a macroscopic 2-qubit quantum computer. In general a quantum register contains n qubits. It can be shown that any n-qubit interaction can be decomposed into 2-qubit gates and unary operations [12], so in a sense, the 2-qubit quantum computer is already sufficient to demonstrate the most elementary functioning of a general quantum computer.

2. The SPS-formalism: an operational approach to quantum mechanics

The basic notions for the description of a physical entity S are as follows [6]. First, we consider that at any moment the entity S is in a (known or unknown to the observer) state $p \in \Sigma$. Also, S has a set of properties $\mathcal{L}$, defined by the set of available experiments which can be performed on S. A property a is either 'actual' or 'potential' for the entity S, meaning that if the property a is actual in the state p, then whenever one would perform the corresponding experiment, one finds the positive outcome with certainty. Between the set of states and (power)set of properties is a relation $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ of actuality that maps each state $p \in \Sigma$ onto the set $\xi(p)$ of those properties that are actual in this state. Dually, one can consider the map $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$, which maps a property $a \in \mathcal{L}$ onto the set of states $\kappa(a)$ that make this property actual. Depending on the nature of the entity S, one obtains a different structure on the set of states Σ, the set of properties $\mathcal{L}$ and the relation between these two sets. Hence, if we are only concerned with the structural behaviour of an entity, we can focus on the triple $(\Sigma, \mathcal{L}, \xi)$. More abstractly, even without an underlying physical entity S we can consider any two sets Σ and $\mathcal{L}$ and a function $\xi : \Sigma \to \mathcal{L} : p \to \xi(p)$ and study the emerging structure. We call $(\Sigma, \mathcal{L}, \xi)$ a State Property Space (**SPS**). If one considers the **SPS** of a quantum entity, one observes that certain 'quantum

axioms' hold, i.e. the mathematical structure of the **SPS** needs to obey certain specific rules. Conversely, one could start from an abstract **SPS**, and by imposing a suitable set of axioms one can derive a quantum structure on the set of properties, i.e. it becomes isomorphic with the Hilbert space representation of quantum mechanics. In Appendix A we briefly present these quantum axioms to give an idea of how the operational approach works and how quantum structure arises. In the operational **SPS** approach, a physical system is determined by the structure on its set of states and properties, and the actuality map ξ between these two. This means that it is not necessary (or even meaningful) to make a distinction between quantum and classical systems on basis of their size (classical macroscopic versus microscopic quantum) but rather on the different structure on their **SPS**.

3. 'Quantum' versus 'classical' systems

At this stage, it is useful to clarify what we mean by a 'quantum-like system' as compared to a 'classical' system. Traditionally, one often refers to the size of the system to identify it as a classical or quantum system. However, within the operational approach the concept of 'classical' versus 'quantum' have very different meanings depending on the context or framework in which these concepts are used. If one considers the set of states, then quantum states are represented by rays in a complex Hilbert space, while classical states are represented by points (reflecting both position and momentum) in phase space. Quantum properties are represented by closed subspaces in Hilbert space, and give rise to a non distributive lattice (quantum logic). Classical properties on the other hand can be represented by subsets in phase space and give rise to a Boolean algebra (i.e., distributive lattice). In classical theory, probabilities can only arise due to a lack of knowledge about the state of the system which yields a Kolmogorovian probability calculus [13]. On the other hand, quantum probabilities result in a non Kolmogorovian probability calculus. Finally, also the description of compound systems is fundamentally different for classical and quantum systems. The classical phase space of a compound classical system is constructed using the Cartesian product, which means that only product states occur. On the other hand, the Hilbert space of a compound quantum system is given by the tensor product of the Hilbert spaces of the subsystems, such that non-product states occur (so-called entangled states).

The possibility of (non)existence of quantum entangled states has been a long debate in physics. Already in their seminal paper of 1935 [14], Einstein, Podolsky and Roosen (EPR) argued that if such entanglement correlation is present in nature then quantum mechanics should be considered an incomplete theory. This was the beginning of a long debate in which defenders of local realism and hidden variable theories stood against defenders of the Copenhagen quantum interpretation. For many years, this debate could be considered a philosophical debate, since it was not clear at all how could be decided experimentally on the possible (non)existence of underlying hidden variable theories for quantum mechanics. However, in 1964 Bell put forward a set of inequalities which are violated in quantum mechanics, but which needs to hold if local hidden variables exist [15]. This shifted the debate from a purely philosophical level to the experimental level. Aspect and his team showed that indeed (generalized) Bell inequalities are violated [16], which for many scientists seemed to close the debate in favour of Bohr and his view on quantum mechanics, and against the beliefs of EPR. Also, on a more mathematical level local hidden variable theories were shown to be incompatible with the structures of quantum mechanics. Starting with the first no-go theorems (of von Neumann [17]) till the more elaborated theorems of Kochen-Specker and others [18, 19], which showed that if such hidden variables existed — giving quantum theory an underlying, hidden reality — that then these hidden variables should at least be contextual, i.e. dependent on the measurement context. Again, for many people this seemed to close the debate since it was hard to imagine what the meaning of such contextual variables could be, since they were no longer variables *only* pertaining to the

system.

However, these arguments didn't convince everybody, and non-standard interpretations of quantum mechanics were proposed. One of the more successful hidden variable theories is the Bohm(-de Broglie) theory in which contextuality is reflected by the presence of a 'strange' or 'spooky' potential which implements the non-local behavior of quantum systems [20]. Another interpretation is the Hidden Measurement Approach, in which (quantum) probabilities occur in experiments due to an irreducible lack of knowledge on the interaction between system and measurement apparatus [7]. More precisely, with each (quantum) experiment corresponds a set of deterministic sub-measurements such that a lack of knowledge about which sub-measurement occurs results in the quantum probability distribution. As a consequence, the Hidden Measurement approach is contextual by its very nature. Within this approach various macroscopic models have been constructed which have a quantum structure on their set of states and properties, and a quantum probability distribution on the set of outcomes.

4. The single qubit

4.1. The quantum sphere model

The sphere model is a generalization of the Bloch sphere representation[1] such that also the measurements are represented [7, 21]. The main differences with the standard use of the Bloch representation are as follows. First, in our approach all points of the Bloch sphere represent states of the spin, such that points on the surface correspond to pure states, while interior points correspond to density states. This is because an arbitrary point $u(r, \theta, \phi)$, $r \in [0, 1], \theta \in [0, \pi], \phi \in [0, 2\pi]$, of the Bloch sphere can be expressed as a convex linear combination

$$u(r, \theta, \phi) = ru(1, \theta, \phi) + (1 - r)u(0, \theta, \phi)$$

from which follows the corresponding density state

$$
\begin{aligned}
D(r, \theta, \phi) &= rD(1, \theta, \phi) + (1 - r)D(0, \theta, \phi) \\
&= \frac{1}{2} \begin{pmatrix} 1 + r\cos\theta & r\sin\theta e^{-i\phi} \\ r\sin\theta e^{i\phi} & 1 - r\cos\theta \end{pmatrix}
\end{aligned}
$$

In this expression $D(1, \theta, \phi) = |\psi\rangle\langle\psi|$ is the usual density state representation of a pure state, while $D(0, \theta, \phi)$ is the density matrix representing the center of the sphere (or, alternatively, the singlet state). The spin up state (the pure state $|0\rangle$) corresponds with the 'North pole' $D(1, 0, \phi) = |0\rangle\langle0|$, while the spin down state (hence pure state $|1\rangle$) is represented by the 'South pole' $D(1, \pi, \phi) = |1\rangle\langle1|$.

Next to this, the sphere model allows a representation of measurements (see Figure 1). The experiments e_u are defined as follows. We put an elastic of length 2 centered in the origin o of the sphere S^2 between the point u and its antipode $-u$. Let us denote the segment between u and $-u$ with the interval $[-u, u]$. Next, the particle falls from its initial position p orthogonally onto the interval $[-u, u]$ in the point p' where it stays attached to the elastic. Then the elastic breaks randomly in some point $\lambda \in [-u, u]$ such that two possibilities can occur. If the elastic breaks between p' and $-u$, the elastic will pull the point particle towards u where it stays attached and the experiment is said to yield the outcome 'spin up'. If on the other hand the elastic breaks between u and p', then the elastic will pull the particle towards $-u$, where it stays attached, and the measurement is said to yield outcome 'spin down'. To make the description of the experiment e_u complete, one could specify that in the event that the elastic breaks at exactly the point p' where the particle is attached, we assume that the measurement always

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[1] In this representation, a qubit $|\psi\rangle = \cos\frac{\theta}{2}e^{\frac{-i\phi}{2}}|0\rangle + \sin\frac{\theta}{2}e^{\frac{i\phi}{2}}|1\rangle$ is represented by the point $u(1, \theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the surface of a 3-dimensional sphere, called the Bloch sphere.
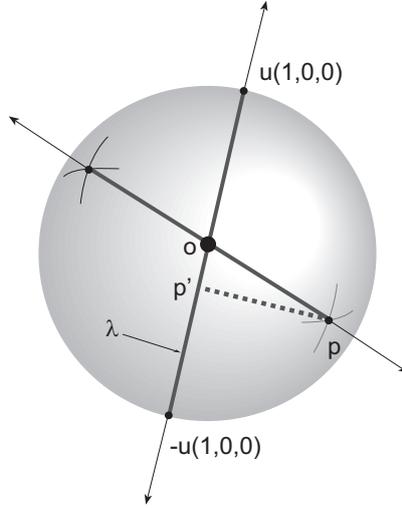
Figure 1. The macroscopic spin 1/2 model.

yields the outcome 'spin up'. Notice however that this event has measure zero to occur, and in such sense it is physical irrelevant with respect to the resulting probability distribution over the set of outcomes. Let θ denote the angle between the state p of the system and the direction u of the measurement device. The probability for outcome 'spin up' is given by the length of the elastic between the projection point p' and the point $-u$, normalized by dividing by the total length of the elastic. This yields following probability $P(u \mid p)$ for the 'spin up' outcome and corresponding state transition from initial state p towards final state u, eigenstate of the 'spin up' outcome:

$$P(u \mid p) = \frac{\cos\theta + 1}{2} = \cos^2 \frac{\theta}{2}$$

Similarly we can calculate the probability for the outcome 'spin down' as

$$P(-u \mid p) = \frac{1 - \cos\theta}{2} = \sin^2 \frac{\theta}{2}$$

These probabilities coincide with the quantum probabilities for a spin experiment on a spin-$\frac{1}{2}$ particle. Note that if one would have knowledge about where the elastic breaks, the measurement procedure happens deterministic. If we call the measurement e_u^λ the measurement that consists in performing e_u and such that the elastic breaks in the point λ for some $\lambda \in [-u, u]$, then, each time e_u is performed, it is actually one of the e_u^λ that takes place. We do not control this, in the sense that the e_u^λ are really 'hidden measurements' that we cannot choose to perform. The probability $\mu(e_u, p, o_1)$ that the experiment e_u gives the outcome o_1 if the entity is in state p is a randomization over the different situations where the hidden measurements e_u^λ gives the outcome o_1 with the entity in state p. This is exactly the way we have calculated the probability for the quantum sphere model. Remark that if we have a physical entity S and we have a lack of knowledge about the state p of the physical entity S, then a theory describing this situation is necessarily a classical statistical theory with a Kolmogorovian probability model. However, if we have a physical entity S and a lack of knowledge about the measurement e_u to be performed on the physical entity S, and to be changing the state of the entity S, then we cannot describe this situation by a classical statistical theory, because the probability model that arises is non Kolmogorovian. For the sphere model it can be shown that Bayes' axiom is violated, rendering the probability model non Kolmogorovian. Clearly, this is due to a lack of knowledge about

the measurement e_u, which can be regarded as a 'non pure' measurement, in the sense that there are hidden properties of the measurement, such that the performance of e_u introduces the performance of a 'hidden' measurement, denoted e_u^λ, with the same set of outcomes as the measurement e_u. The measurement e_u consists then in fact of choosing one way or another between one of the hidden measurements e_u^λ and then performing the chosen measurement e_u^λ. More generally, one could regard any quantum experiment as a class of 'hidden measurements' such that each hidden measurement by itself is deterministic, but a lack of knowledge about which hidden measurement is actually going on leads to a lack of knowledge on the level of the measurement outcomes, i.e. quantum probability can be explained as due to an uncontrollable (and irreducible) lack of knowledge on the interaction between the measurement device and the system. This interpretation of quantum mechanics is called the 'hidden measurement interpretation' of quantum mechanics. Since it locates the lack of knowledge in the measurement interaction, it goes beyond the reach of standard 'no go theorems' for hidden variable theories for quantum mechanics. Indeed, by its very nature the hidden measurement approach is highly contextual and hence non local from the very start.

4.2. The epsilon model

In principle, one could modify the sphere model by considering experiments in which the lack of knowledge about which sub-measurement is occurring, is controlled by a couple of parameters (ϵ, d), such that ϵ controls the fluctuations on the measurement interaction and d controls the 'symmetry' of the measurements [22]. The experiment $e_u^{\epsilon,d}$ is defined similarly as for the quantum sphere model, but now an elastic of length 2ϵ is centered around the point d on the interval $[-u, u]$ (see Figure 2).
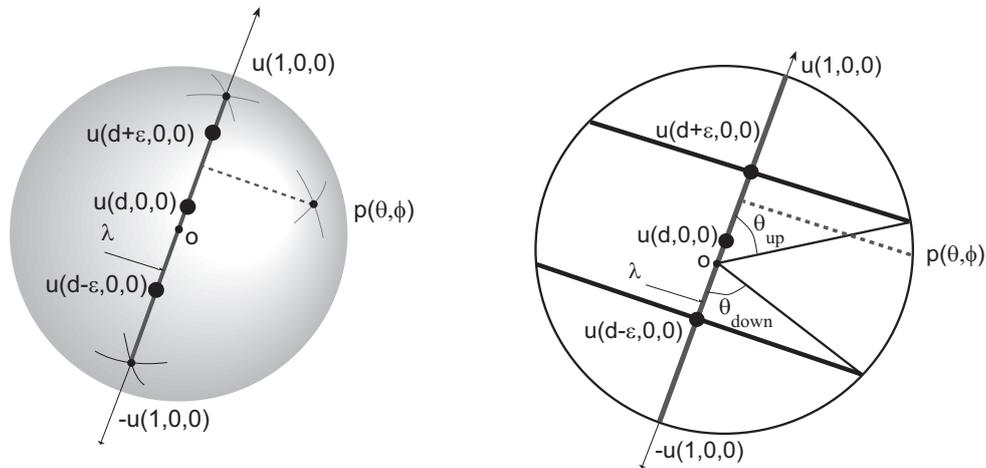

Figure 2. The epsilon model.

One can show that for this model, in contrast with the sphere model, quantum axioms can be violated such that it is impossible to represent it in a 2-dimensional complex Hilbert space. More precisely, the axioms of weak modularity and the covering law only hold in the quantum limit ($\epsilon = 1$) and the deterministic limit ($\epsilon = 0$). In the intermediate cases ($0 < \epsilon < 1$) of the epsilon model these axioms are violated. This shows how within the hidden measurement approach using the **SPS** formalism one can go beyond the reach of standard quantum mechanics. Such a two-level state system was called a Contextual Bit (ConBit) [23], but we will not go deeper into this subject, because it would lead us too far away from the main scope of this paper, namely simulating 2-qubit quantum computation on a macroscopic model. Also, it is still not clear to

the authors what kind of algorithms could be implemented using these ConBits, so we leave it for future research.

5. Two entangled qubits

5.1. Description of compound systems

We consider the situation of a physical entity S that consists of two physical entities S_1 and S_2. Let us consider two experiments e_1 and e_2 for the entity S_1, S_2 resp., from which we can define the joint experiment $e_1 \times e_2$ for the compound entity S which consists of performing the experiment e_1 on entity S_1 and experiment e_2 on entity S_2. Experiments e_1 and e_2 are called 'separated experiments' iff for an arbitrary state p of S holds that (x_1, x_2) is a possible outcome for the joint experiment $e_1 \times e_2$ iff x_1 is a possible outcome for e_1 and x_2 is a possible outcome for e_2. The entities S_1 and S_2 are 'separated entities' iff all experiments e_1 on S_1 are separated from all experiments e_2 on S_2. It was an unexpected and also non trivial result that standard quantum mechanics *can not* describe the situation of separated quantum systems. In fact, if the compound system obeys the rules of quantum mechanics, then one of the subsystems needs to be classical, i.e. all experiments are deterministic. This was proven within an operational approach which was the predecessor of the **SPS** formalism (Aerts' theorem [24]):

Theorem 1 *Suppose Axioms 1, 2 and 3 are satisfied and $(\Sigma, \mathcal{L}, \xi)$ is the **SPS** describing S and $(\Sigma_1, \mathcal{L}_1, \xi_1)$ and $(\Sigma_2, \mathcal{L}_2, \xi_2)$ the **SPS** describing S_1 and S_2 then:*

- *If Axiom 4 (covering law) is satisfied then one of the two entities S_1 or S_2 is a classical entity.*

- *If Axiom 5 (weak modularity) is satisfied then one of the two entities S_1 or S_2 is a classical entity.*

It is important to notice that the result of this theorem points at a failure of standard quantum mechanics, and is not just a characteristic of the axiomatic approach itself. Indeed, one can show that the structure on the set of states and set of properties of a compound system of two separated systems does not fit in Hilbert space, i.e., the quantum axioms are violated [25, 26]. This also sheds new light on the claims of EPR, namely that quantum mechanics is incomplete. This is true, but in a slightly different way: if separated quantum systems do exist, then the corresponding compound system cannot be described with quantum mechanics.

5.2. Macroscopic violation of Bell inequalities

Let us consider a very simple example of a compound system, namely the vessels of water model which is an example of a non-separated and non-micro model [27]. The entity S consists of two vessels V_1 and V_2 connected by a tube containing 20 liters of transparent water (see Figure 3). Experiment e_1 (resp. e_3) consists of putting a siphon K_1 (resp. K_2) in the vessel V_1 (resp. V_2) and pouring out water and collecting it in reference vessel R_1 (resp. R_2). If more than 10 liter is collected in the reference vessel, then the measurement yields outcome $o_1(up)$ (resp. $o_3(up)$). If less than 10 liter is collected, the measurement is said to yield the outcome $o_1(down)$ (resp. $o_3(down)$). Experiment e_2 (resp. e_4) consists of taking a spoonful of water from the vessel V_1 (resp. V_2) and determining whether it is transparent or not. If the collected water is transparent the measurement is said to yield outcome $o_2(up)$ (resp. $o_4(up)$). Otherwise the outcome is $o_2(down)$ (resp. $o_4(down)$). In order to derive Bell inequalities, we consider joint experiments e_{ij}, which consists of performing e_i and e_j together. Following Bell [15], we define expectation values $\mathbb{E}(i, j)$:

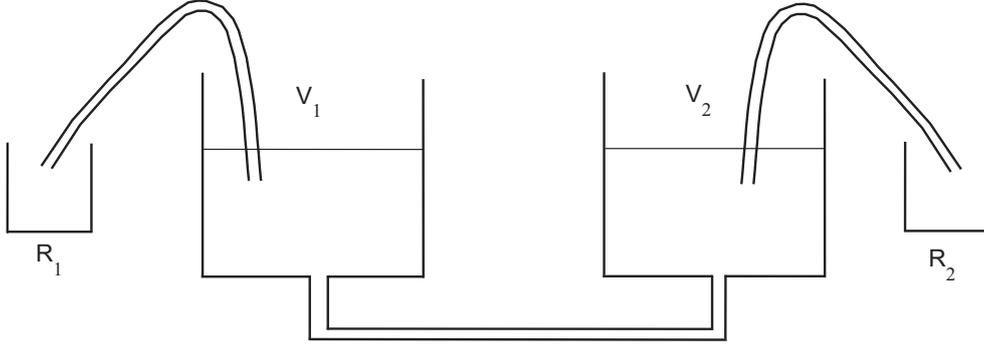$$\mathbb{E}(i, j) = +1.P(o_i(up), o_j(up)) + 1.P(o_i(down), o_j(down))$$

Figure 3. The vessels of water model.

$$-1.P(o_i(up), o_j(down)) - 1.P(o_i(down), o_j(up))$$

From the assumption that outcomes are either $+1$ or -1 and that $\mathbb{E}(i,j)$ can be written as an integral over some hidden variable of a product of the two local assignments, one derives the Bell inequalities:

$$|\mathbb{E}(1,3) - \mathbb{E}(1,4)| + |\mathbb{E}(2,3) + \mathbb{E}(2,4)| \leq 2$$

One can show that 2 entangled spin-1/2 particles in the singlet state yield a possible violation of Bell inequalities with maximum value $2\sqrt{2}$. However, if one calculates Bell inequalities for the vessel of water model, one finds:

$$|\mathbb{E}(1,3) - \mathbb{E}(1,4)| + |\mathbb{E}(2,3) + \mathbb{E}(2,4)| = 4$$

which is a stronger violation than quantum violation, namely value 4 instead of $2\sqrt{2}$. This not only shows that the quantum violation is not maximal, but also that Bell inequalities can be violated in the non-micro world too.

Some will remark that the violation of Bell-inequalities has a different nature than the quantum violation. Indeed, it is not obvious that the 'faster than light aspect' identified in the quantum experiments is also present in this vessels of water experiment. Although the correlations can be made to happen in two distinct regions of space, such that the correlated events are not part of each others causal cones, also for the vessels of water experiment, what is not clear for the vessels of water case is 'which are the internal interactions which cause these correlations'. A classical model, for example using fluid mechanics, of the vessels of water, will only describe physical interactions of causal nature, propagating with a speed smaller than the speed of light, and hence a priori will not be able to model the speed effects within the quantum experiment. It might well be however that such classical models for the water are not complete models and that other types of speed effects are also possible in water. However, apart from this intriguing question about the nature of completeness of classical theories for the description of macroscopic phenomena, we do not really have to focus on this aspect to make the point we want to make in the realm of the present article. Indeed, it is not the speed effects present in this entanglement correlation which is vital in a quantum computation, but rather its algebraic structure. In this sense, if one could build a model of 2-qubit entanglement (even with 'slower than light' implementation), it is this possibility to implement entanglement in the compound system which makes it non-classical, and therefore impossible to implement by classical means.

5.3. Macroscopic model of two entangled spin 1/2

Next, let us show how by putting together two sphere models one can construct a macroscopic model for a compound system of two entangled spin 1/2 in the singlet state [8]. The compound system consists of two point particles in the center of two sphere models for a single spin 1/2 which are connected by a rigid but extendable rod, centered in c, i.e. in the middle of the line connecting the two point particles (see Figure 4). The joint experiment $e(a,b)$ is performed as follows. First, the two spheres reach the measurement apparatuses, with measurement direction a respectively b chosen by the experimenter. The spin measurements e_a and e_b are defined similarly as for the single sphere model. When one side is measured, the measurement apparatus draws one of the entities to one of the two possible outcomes with probability 1/2 (because initially the point particle is in the center of the sphere) (see Figure 5). Because of the rod the other entity is drawn toward the opposite side of the sphere as compared with the first entity (see Figure 6, in which we consider for the sake of illustration the event that e_a has yielded outcome 'spin up'). The connecting rod is then taken away and the second spin measurement is performed (see Figure 7). The measurement $e(a,b)$ has 4 possible outcomes (a,b), $(a,-b)$, $(-a,b)$ and $(-a,-b)$. Giving value $+1$ for outcome (a,b) or $(-a,-b)$, and value -1 for outcome $(a,-b)$ or $(-a,b)$, one finds for the expectation value $\mathbb{E}(a,b) = -a.b$, i.e. this corresponds with the expectation value for the spin measurements along directions a, b on a compound system of two entangled spin 1/2 particles in the singlet state.
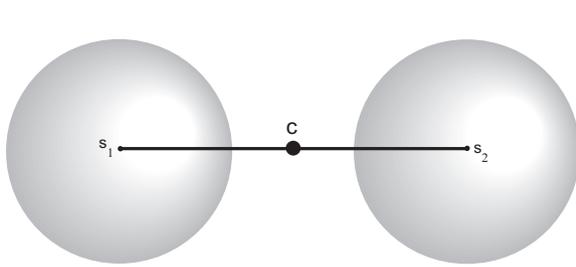

Figure 4. The initial configuration of the macroscopic model of two entangled spin 1/2 in the singlet state.
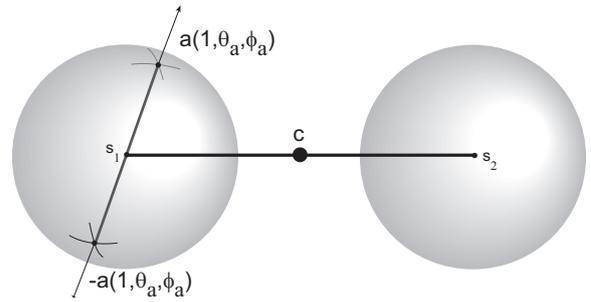

Figure 5. The measurement e_a is performed on the left point particle, drawing it towards the point a.
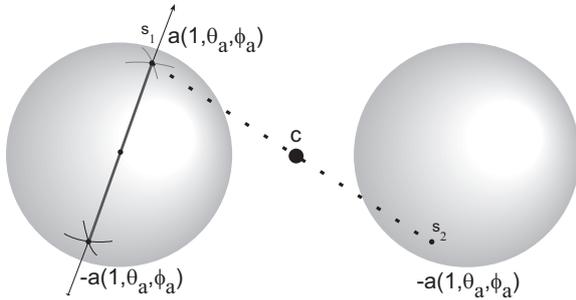

Figure 6. By means of the rigid rod, the right point particle is drawn towards the point $-a$ on its sphere. Next, the connecting rod is taken away.
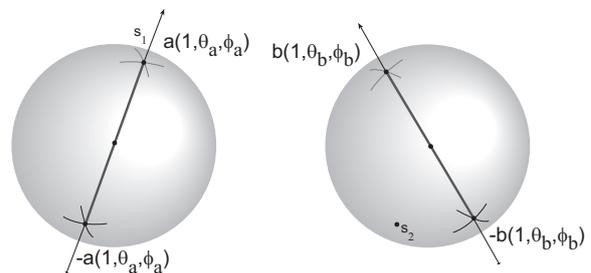

Figure 7. Finally, the measurement e_b is performed on the right point particle.

The maximum value of the Bell inequality is obtained for coplanar measurement directions a, a', b, b' with angles $a.b = a'.b = a'.b' = \frac{\pi}{4}$ and $a.b' = \frac{3\pi}{4}$. In such case, the expectation values

are given by $\mathbb{E}(a, b) = -\mathbb{E}(a, b') = \mathbb{E}(a', b) = \mathbb{E}(a', b') = \frac{\sqrt{2}}{2}$ such that Bell inequality has value:

$$|\mathbb{E}(a, b) - \mathbb{E}(a, b')| + |\mathbb{E}(a', b) + \mathbb{E}(a', b')| = 2\sqrt{2}$$

which is the maximum quantum violation of the Bell inequality.

There are two key-features of the singlet model. First, there is the rigid rod which expresses the non-separable wholeness of the singlet joint experiment (cfr. the connecting tube between the vessels of water). Secondly, there is an elastic that breaks during the measurements on the sphere models. This represents the probabilistic nature of the outcomes (cfr. the siphon in the vessels of water). To explore these two features this model was extended by introducing two parameters [28], parameter ϵ controlling the amount of indeterminism in the measurements and parameter ρ controlling the correlation between the two subsystems, i.e. the amount of change induced on the state of one particle when measurements are performed on the other. The Bell inequalities for this extended model have following value:

$$|\mathbb{E}(a, b) - \mathbb{E}(a, b')| + |\mathbb{E}(a', b) + \mathbb{E}(a', b')| = \begin{cases} \frac{2\sqrt{2}\rho}{\epsilon} & \text{if} \quad \frac{\epsilon}{\rho} > \frac{\sqrt{2}}{2} \\ 4 & \text{if} \quad \frac{\epsilon}{\rho} \leq \frac{\sqrt{2}}{2} \end{cases}$$

from which two important conclusions can be drawn for the compound system of two sphere models connected by a rigid rod. First of all, the correlation is the source of the violation. Let us remark that the non-local connectedness plays a crucial role for these correlations to appear, and not quantum indeterminism. In fact, if one increases the indeterminism in this model, one decreases the value the inequality takes. Hence for minimal indeterminism and maximal correlation, the modified model yields value 4, i.e., the same value as for the vessels of water example.

6. Geometrical representation of two entangled spin 1/2
6.1. Representing entanglement by constraint functions
In the previous section we discussed a compound system consisting of two connected sphere models which represented the singlet state of two entangled spin 1/2. In general, in a 2-qubit quantum algorithm the register will be in other entangled states as well. In [9] we show how this problem is solved for the coupled sphere models by introducing constraint functions. Let us briefly recall how these functions are introduced and some of their properties.

Let $\mathbb{C}_1^2$ and $\mathbb{C}_2^2$ be two copies of $\mathbb{C}^2$, which we label with indices 1 and 2 with the sole purpose of identifying them. A system of two entangled spin $\frac{1}{2}$ is described by means of an arbitrary unit vector $|\psi\rangle \in \mathbb{C}_1^2 \otimes \mathbb{C}_2^2$, which can always be written as the following linear combination

$$|\psi\rangle = \sum_{ij} \lambda_{ij} |e_1^i\rangle \otimes |e_2^j\rangle$$

where $\lambda_{ij} \in \mathbb{C}$, and $\{|e_1^i\rangle\}$ and $\{|e_2^j\rangle\}$ are bases of $\mathbb{C}_1^2$ and $\mathbb{C}_2^2$ respectively. Following von Neumann [17], when a measurement is performed on the first spin, it collapses into a spin state described by the unit vector $|x_1\rangle \in \mathbb{C}_1^2$ corresponding with the observed outcome, thus transforming the entangled state $|\psi\rangle$ into

$$(P_{|x_1\rangle} \otimes I)(|\psi\rangle)$$

where $P_{|x_1\rangle}$ is the orthogonal projector on $|x_1\rangle$ in $\mathbb{C}_1^2$, and I is the unit operator in $\mathbb{C}_2^2$. The result is that the entangled spins end up in the following product state

$$|x_1\rangle \otimes \sum_{ij} \lambda_{ij} \langle x_1, e_1^i\rangle |e_2^j\rangle$$

This means that as a consequence of the measurement on the first spin, collapsing its state to $|x_1\rangle$, the second spin collapses to the state

$$\sum_{ij} \lambda_{ij}\langle x_1, e_1^i\rangle |e_2^j\rangle$$

In an analogous way we can show that if a measurement is performed on the second spin, resulting in a collapse to the state $x_2 \in \mathbb{C}_2^2$, the state of the first spin becomes

$$\sum_{ij} \lambda_{ij}\langle x_2, e_2^j\rangle |e_1^i\rangle$$

Because of this, we arrive at the following definition.

Definition 1 (Constraint Functions) *The constraint functions $F_{12}(\psi)$ and $F_{21}(\psi)$ related to ψ are defined in the following way:*

$$F_{12}(\psi) \quad : \quad \mathbb{C}_1^2 \to \mathbb{C}_2^2 : |x_1\rangle \mapsto \sum_{ij} \lambda_{ij}\langle x_1, e_1^i\rangle |e_2^j\rangle$$

$$F_{21}(\psi) \quad : \quad \mathbb{C}_2^2 \to \mathbb{C}_1^2 : |x_2\rangle \mapsto \sum_{ij} \lambda_{ij}\langle x_2, e_2^j\rangle |e_1^i\rangle$$

The constraint functions map the state where one of the spins collapses to by a measurement to the state that the other spin collapses to under influence of the entanglement correlation. A detailed study of the constraint functions gives us a complete picture of how the entanglement correlation works as an internal constraint. Let us list some properties of the constraint functions:

- The constraint functions are canonically defined, i.e. the definition of $F_{12}(\psi)$ and $F_{21}(\psi)$ do not depend on the chosen bases.
- $F_{21}(\psi) \circ F_{12}(\psi) = D_1(\psi)$ is the partial trace density matrix over $\mathbb{C}_2^2$ and $F_{12}(\psi) \circ F_{21}(\psi) = D_2(\psi)$ is the partial trace density matrix over $\mathbb{C}_1^2$:

$$D_1(\psi) \equiv tr_{\mathbb{C}_1^2}|\psi\rangle\langle\psi|, D_2(\psi) \equiv tr_{\mathbb{C}_2^2}|\psi\rangle\langle\psi|$$

6.2. Constraint functions and the Schmidt diagonal form
To derive a complete view of how entanglement works as an internal constraint for a 2-particle system, we recall the relation between the Schmidt diagonal form [29] and the constraint functions. We begin by choosing the base

$$|x_1^1\rangle = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{\frac{i\phi}{2}})$$

$$|x_1^2\rangle = (-i\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}, i\cos\frac{\theta}{2}e^{\frac{i\phi}{2}})$$

in $\mathbb{C}_1^2$. With respect to this basis, the expression for a general density matrix becomes

$$D_1(\psi) = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$$

Define now the following basis $\{|x_2^1\rangle, |x_2^2\rangle\}$ in $\mathbb{C}_2^2$

$$\{|x_2^1\rangle, |x_2^2\rangle\} = \left\{ \frac{\sqrt{2}}{\sqrt{1+r}}F_{12}(\psi)(|x_1^1\rangle), \frac{\sqrt{2}}{\sqrt{1-r}}F_{12}(\psi)(|x_1^2\rangle) \right\}$$

One finds that $\|x_2^1\| = \|x_2^2\| = 1$ and

$$D_2(\psi)(|x_2^1\rangle) = \frac{1+r}{2}|x_2^1\rangle, D_2(\psi)(|x_2^2\rangle) = \frac{1-r}{2}|x_2^2\rangle$$

Hence $|x_2^1\rangle$ and $|x_2^2\rangle$ are normalized eigenvectors of $D_2(\psi)$ with eigenvalues $\frac{1+r}{2}$ and $\frac{1-r}{2}$ respectively. Therefore, with respect to the basis $\{|x_2^1\rangle, |x_2^2\rangle\}$, $D_2(\psi)$ is expressed as

$$D_2(\psi) = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$$

Finally, let us find the expression for ψ with respect to the basis $\{|x_1^1\rangle \otimes |x_2^1\rangle, |x_1^1\rangle \otimes |x_2^2\rangle, |x_1^2\rangle \otimes |x_2^1\rangle, |x_1^2\rangle \otimes |x_2^2\rangle\}$ of $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2$. In general, this expression is of the form

$$\psi = a|x_1^1\rangle \otimes |x_2^1\rangle + b|x_1^1\rangle \otimes |x_2^2\rangle + c|x_1^2\rangle \otimes |x_2^1\rangle + d|x_1^2\rangle \otimes |x_2^2\rangle$$

However, since

$$F_{12}(\psi)(|x_1^1\rangle) = a|x_2^1\rangle + b|x_2^2\rangle = \frac{\sqrt{1+r}}{\sqrt{2}}|x_2^1\rangle$$

$$F_{12}(\psi)(|x_1^2\rangle) = c|x_2^1\rangle + d|x_2^2\rangle = \frac{\sqrt{1-r}}{\sqrt{2}}|x_2^2\rangle$$

one obtains the Schmidt diagonal form of $|\psi\rangle$ given by

$$|\psi\rangle = \frac{\sqrt{1+r}}{\sqrt{2}}|x_1^1\rangle \otimes |x_2^1\rangle + \frac{\sqrt{1-r}}{\sqrt{2}}|x_1^2\rangle \otimes |x_2^2\rangle$$

6.3. Geometrical representation of entanglement

Let us remark that $F_{12}(\psi)$ does not conserve orthogonality (except for the basis vectors $\{|x_1^1\rangle, |x_1^2\rangle\}$ which are mapped onto the orthogonal basis vectors $\{|x_2^1\rangle, |x_2^2\rangle\}$ and in the case that $r = 0$, i.e. ψ is a product state). Also, $F_{12}(\psi)$ does not conserve the norm, since $\|F_{12}(\psi)(|x\rangle)\|^2 = \frac{1}{2}(1 + r\cos\theta)$. This means that we have to look at the *normalized* image of a vector $|x\rangle = x(\theta_1, \phi_1)$. This vector is mapped to $|y\rangle = y(\theta_2, \phi_2) = \frac{1}{\|F_{12}(\psi)(|x\rangle)\|}F_{12}(\psi)(|x\rangle)$. One derives following expression:

$$y(\theta_2, \phi_2) \cdot x_2^1(\theta, \phi) = \frac{r + \cos\theta_1}{1 + r\cos\theta_1}$$

from which one can deduce that straight lines containing the center o_1 on the first sphere are mapped to straight lines through the point $u(r, 0, 0)$ in the second sphere. This gives a geometrical representation of 'stretching' on the second sphere (see Figure 8). Hence the entanglement correlation is represented by a rotation (reflecting the basis vectors in the Schmidt diagonal form) and a stretching depending on the parameter r, i.e. 'amount of entanglement'.

Vice versa, given a general entangled state $|\psi\rangle = \lambda_{00}|00\rangle + \lambda_{01}|01\rangle + \lambda_{10}|10\rangle + \lambda_{11}|11\rangle$ expressed in the canonical basis, we can derive the Schmidt diagonal form using the constraint functions. From the reduced density matrix $D_1(\psi) = F_{21}(\psi) \circ F_{12}(\psi)$ one can calculate the basis vectors $\{|x_1^1\rangle, |x_1^2\rangle\}$, and consequently also the second pair of basis vectors by $\{|x_2^1\rangle, |x_2^2\rangle\} =$
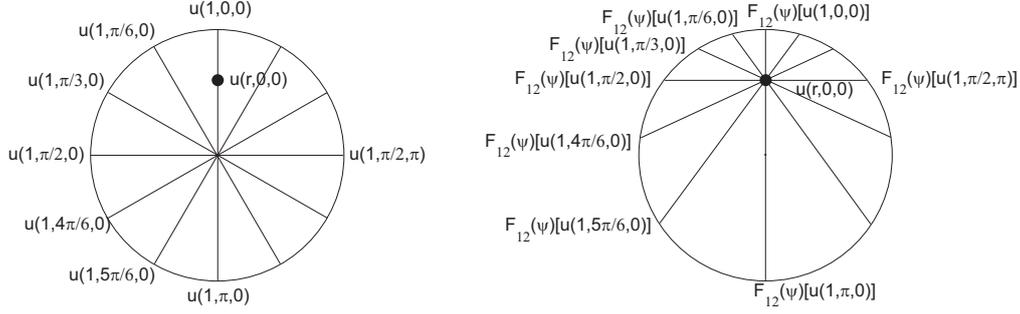
Figure 8. Geometrical representation of two entangled spin $1/2$.

$\left\{ \frac{\sqrt{2}}{\sqrt{1+r}} F_{12}(\psi)(|x_1^1\rangle), \frac{\sqrt{2}}{\sqrt{1-r}} F_{12}(\psi)(|x_1^2\rangle) \right\}$. In the canonical basis $\{|0\rangle, |1\rangle\}$ the reduced density matrix has following form:

$$D_1(\psi) = \begin{pmatrix} a & \bar{b} \\ b & 1-a \end{pmatrix}, a = |\lambda_{00}|^2 + |\lambda_{01}|^2, b = \overline{\lambda_{00}}\lambda_{10} + \overline{\lambda_{01}}\lambda_{11}$$

By expressing the reduced density matrix in the eigenvectors basis $\{|x_1^1\rangle, |x_1^2\rangle\}$ we obtain the value for the parameter r, i.e. the amount of entanglement:

$$D_1(\psi) = \begin{pmatrix} \frac{1+r}{2} & 0 \\ 0 & \frac{1-r}{2} \end{pmatrix}$$

7. Geometrical representation of 2-qubit algorithms

7.1. Deutsch problem

Deutsch problem is as follows [3]: given a $\{0,1\}$-valued function f defined on a two-point domain $\{0,1\}$, determine with a single call of the 'oracle' whether f is constant or balanced. On a classical digital computer, this problem is impossible to solve. However, on a quantum computer this is possible. Consider preparing a quantum register of two qubits in an input state:

$$\psi_0 = (H \otimes H)|01\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |10\rangle - |01\rangle - |11\rangle}{2} \tag{1}$$

Calling the oracle corresponds with applying a suitable unitary transformation U_f which maps $|xy\rangle$ onto $U_f|xy\rangle = |x(y \oplus f(x))\rangle$. Denoting $\overline{f(i)} = f(i) \oplus 1$ and noticing that $|f(i)\rangle - \left|\overline{f(i)}\right\rangle = (-1)^{f(i)}(|0\rangle - |1\rangle)$ one obtains that $\psi_1 = U_f(\psi_0) = \frac{|0f(0)\rangle + |1f(1)\rangle - |0\overline{f(0)}\rangle - |1\overline{f(1)}\rangle}{2}$ which can be written as a product state:

$$\psi_1 = \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \tag{2}$$

After applying a Hadamard transformation $H \otimes H$ on both qubits we obtain the following state:

$$\psi_2 = (H \otimes H) U_f(\psi_0) \tag{3}$$

- If f is constant, $f(0) = f(1)$ and the state ψ_2 can be rewritten as:

$$\psi_{2,c} = (-1)^{f(0)} |0\rangle \otimes |1\rangle \tag{4}$$

i.e. if f is constant, then the value of the first qubit is $|0\rangle$.
- If f is balanced, $f(1) = f(0) \oplus 1$ then the state ψ_2 can be rewritten as:

$$\psi_{2,b} = (-1)^{f(0)} |1\rangle \otimes |1\rangle \tag{5}$$

such that if f is balanced, then the first qubit is $|1\rangle$.

This shows that a quantum computer solves Deutsch problem with 1 call of the oracle. However, it does not achieve this task by calculating both values $f(0)$ and $f(1)$ 'parallel' and compare them simultaneously, but rather by acting on the state of the 2-qubit system as a whole. Also, it can be remarked that the final state of the second qubit is the same as its input state, such that this qubit could be regarded as an auxiliary qubit. Finally, it is important to notice that Deutsch algorithm uses product states only, which in principle makes it possible to simulate it on a 'classical' computer based on the polarisation states of classical light beams [30].

7.2. Deutsch problem on the geometric model
The intermediate states in Deutsch algorithm are given by the product states (1–5) such that one can represent these states in a trivial way on the geometric model (no entanglement is present, so the state of the compound system can be represented by the states of the individual qubits by their Cartesian product). Therefore, in the next section we consider a 2-qubit algorithm exploiting non-product states, such that the geometrical representation by two correlated sphere models becomes less trivial.

7.3. Arvind problem
The problem put forward by Arvind and his colleagues is the following [10, 11]: given a $\{0,1\}$-valued function f defined from a 2-qubit domain space to a one-bit range space (i.e. $f(x) : \{0,1\}^2 \to \{0,1\}$), determine with 2 calls of the 'oracle' whether f is even or odd. Since there are 4 possible input values $(00), (01), (10), (11)$ each of which has 2 possible output values $(0,1)$ there are 16 possible functions f. These can be divided into sub-classes depending on the number of values 0 respectively 1, i.e. $[0,4], [1,3], [2,2], [3,1], [4,0]$. One has 8 'even' functions $([0,4], [2,2], [4,0])$ and 8 'odd' functions $([1,3], [3,1])$. Clearly, on a classical computer Arvind's problem is impossible to solve, since it would require calling the oracle four times. However, Arvind et al. have shown how to solve this problem on a quantum computer. Again, calling the oracle corresponds with applying a unitary transformation $U_f : |x\rangle_{2-qubit} \to (-1)^{f(x)} |x\rangle_{2-qubit}$

$$U_f = \begin{pmatrix} (-1)^{f(00)} & 0 & 0 & 0 \\ 0 & (-1)^{f(01)} & 0 & 0 \\ 0 & 0 & (-1)^{f(10)} & 0 \\ 0 & 0 & 0 & (-1)^{f(11)} \end{pmatrix}$$

There are 16 U_f matrices, of which 8 are separable and 8 are non-separable, i.e. they are entangling, e.g. $U_f [1,1,1,-1]$. One could notice that the subclass of even functions $[0,4], [4,0], [2,2]$ reflects the problem of constant or balanced functions in Deutsch problem. Hence it should come as no surprise that for even f the matrix U_f is separable and no entanglement is used in Arvind's algorithm, just as in Deutsch algorithm. However, for odd

f the matrix U_f is non-separable which shows that in that case entanglement is necessary to solve Arvind's problem.

Let the quantum register of two qubits be prepared in input state $\psi_0 = |00\rangle$, then the consecutive unitary transformations in Arvind's algorithm are as follows:

$$\psi_0 = |00\rangle \rightarrow (H \otimes H)\, U_f\, (1 \otimes H)\, U_f\, (H \otimes H)\, |00\rangle$$

such that the consecutive 'intermediate' states in Arvind's algorithm are given by

$$\psi_1 = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$\psi_2 = \frac{(-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle}{2}$$

$$\psi_3 = \frac{1}{2\sqrt{2}} \begin{bmatrix} (-1)^{f(00)} + (-1)^{f(01)} \\ (-1)^{f(00)} - (-1)^{f(01)} \\ (-1)^{f(10)} + (-1)^{f(11)} \\ (-1)^{f(10)} - (-1)^{f(11)} \end{bmatrix}$$

$$\psi_4 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 + (-1)^{f(00)+(01)} \\ (-1)^{f(00)+(01)} - 1 \\ 1 + (-1)^{f(10)+(11)} \\ (-1)^{f(10)+(11)} - 1 \end{bmatrix}$$

$$\psi_5 = \frac{1}{2\sqrt{2}} \begin{bmatrix} \left((-1)^{f(00)\oplus f(01)} + (-1)^{f(10)+f(11)}\right) |00\rangle \\ +2\,|01\rangle \\ + \left((-1)^{f(00)\oplus f(01)} - (-1)^{f(10)+f(11)}\right) |10\rangle \end{bmatrix}$$

- If f is even, $(-1)^{f(00)\oplus f(01)} = (-1)^{f(10)+f(11)}$ such that the final state becomes

$$\psi_{5,e} = \frac{1}{\sqrt{2}} \left(\pm |00\rangle + |01\rangle\right)$$

- If f is odd, $(-1)^{f(00)\oplus f(01)} = -\,(-1)^{f(10)+f(11)}$ and the final state becomes

$$\psi_{5,o} = \frac{1}{\sqrt{2}} \left(\pm |10\rangle + |01\rangle\right)$$

It is important to notice that although $\psi_{5,e} \neq \psi_{5,o}$, these states are *not* orthogonal, which makes it not possible to distinguish them in the standard approach of quantum computation. However, on a NMR based quantum computer these two states are distinguishable [11].

7.4. Arvind problem on the geometric model

The density matrices for the intermediate states ψ_i in Arvind's algorithm are given by:

$$D_1(\psi_i) = \begin{pmatrix} a & \bar{b} \\ b & 1-a \end{pmatrix}$$

in which the respective values for a_i, b_i are given by following expressions.

$$
\begin{aligned}
a_i &= \frac{1}{2}, i = 1, \ldots, 4; a_5 = \frac{1}{4}\left[3 + (-1)^{\sum f(ij)}\right] \\
b_1 &= \frac{1}{2}, b_2 = b_3 = \frac{1}{4}\left[(-1)^{f(00)+f(10)} + (-1)^{f(01)+f(11)}\right], \\
b_4 &= \frac{1}{4}\left[1 + (-1)^{\sum f(ij)}\right], b_5 = 0
\end{aligned}
$$

- If f is even: $(-1)^{f(00)+f(10)} = (-1)^{f(01)+f(11)} \implies b_2 = b_3 = \frac{1}{2}(-1)^{f(00)+f(10)}, \ b_4 = \frac{1}{2}$.
  Hence the reduced density matrices $D_1(\psi_i)$ for even f are given by:

$$
\begin{aligned}
D_1(\psi_1) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, D_1(\psi_2) = D_1(\psi_3) = \begin{pmatrix} \frac{1}{2} & (-)^{f(00)+f(10)}\frac{1}{2} \\ (-)^{f(00)+f(10)}\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
D_1(\psi_4) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, D_1(\psi_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

such that the set of eigenstates for each matrix $D_1(\psi_i)$ is given by $\left\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}, i = 1, \ldots, 4$; the set of eigenstates for matrix $D_1(\psi_5)$ is $\{|0\rangle, |1\rangle\}$. For instance, for $D_1(\psi_2)$ we can distinguish two cases: $(-)^{f(00)+f(10)} = +1$ or -1. In the first case, we obtain $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ eigenvector with eigenvalue 1 and its image through the constraint function yields $F_{12}(\psi)(|x_1^1\rangle) = (-)^{f(00)}\frac{|0\rangle+(-)^{f(00)+f(01)}|1\rangle}{\sqrt{2}}$. In the second case, i.e. $(-)^{f(00)+f(10)} = -1$ we obtain $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ as eigenvector with eigenvalue 1 and its image through the constraint function given by $F_{12}(\psi)(|x_1^1\rangle) = (-)^{f(00)}\frac{|0\rangle+(-)^{f(00)+f(01)}|1\rangle}{\sqrt{2}}$. Hence the consecutive states in Arvind's theorem are given by

$$
\psi_1 = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}
$$

$$
\psi_2 = \frac{|0\rangle+(-)^{f(00)+f(10)}|1\rangle}{\sqrt{2}} \otimes (-)^{f(00)}\frac{|0\rangle+(-)^{f(00)+f(01)}|1\rangle}{\sqrt{2}}
$$

$$
\psi_3 = \frac{|0\rangle+(-)^{f(00)+f(10)}|1\rangle}{\sqrt{2}} \otimes (-)^{f(00)}\left[|0\rangle\left(\frac{1+(-)^{f(00)+f(01)}}{2}\right) + \left(\frac{1-(-)^{f(00)+f(01)}}{2}\right)|1\rangle\right]
$$

$$
\psi_4 = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \left[\left(\frac{(-)^{f(00)+f(01)}+1}{2}\right)|0\rangle + \left(\frac{(-)^{f(00)+f(01)}-1}{2}\right)|1\rangle\right]
$$

$$
\psi_5 = |0\rangle \otimes \frac{(-)^{f(00)+f(01)}|0\rangle+|1\rangle}{\sqrt{2}}
$$

which in fact are all product states, making the representation by two coupled sphere models trivial since 'there is no entanglement correlation'.

- If f is odd: $(-1)^{f(00)+f(10)} = -(-1)^{f(01)+f(11)}$ such that $b_i = 0, i = 2, \ldots, 5$. The reduced density matrices $D_1(\psi_i)$ for odd f are hence given by:

$$
D_1(\psi_1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, D_1(\psi_i) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, i = 2, \ldots, 5
$$

Again, ψ_1 is a product state since applying the Hadamard gate on both qubits in the ground state $|00\rangle$ results in a product state. However, for $i = 2, \ldots, 5$ the reduced density matrices $D_1(\psi_i)$ correspond with an entangled state, namely the singlet state. There is no unique Schmidt diagonal form for the singlet state and hence no unique representation of the intermediate states in Arvind's algorithm on the geometric model for odd f. Nevertheless, the fact that these are singlet states also has an unexpected advantage, because these singlet states can be represented by the rigid rod model in a mechanistic way, making it possible to represent Arvind's algorithm 'directly' on the macroscopic singlet model.

8. Conclusions

In this paper, we adopt the operational **SPS** approach to quantum mechanics, making it natural to consider macroscopic 'quantum-like' systems and their compositions. Although the entanglement correlation present in these models is of a different nature (i.e., slower than light) than quantum entanglement, they reproduce the same probability distributions over the set of outcomes as the standard quantum mechanical systems. Hence in principle these models can be used to represent entangled states in quantum algorithms. This results in a geometric model for entanglement which allows to illustrate the key intermediate states in a quantum algorithm. More specifically, we have shown how a compound system of two entangled qubits in a non-product state can be described in a complete way by extracting entanglement into an internal constraint between the two qubits. So-called constraint functions describe the behavior of the state of one of the spins if measurements are executed on the other spin. Therefore the non-product state can be substituted by means of the states of the individual qubits and this internal constraint function, which models the entanglement which is present in a direct way. By making use of the sphere model representation for the spin 1/2, this allows for an easy to grasp visual support for the developed formalism.

We have discussed two elementary 2-qubit quantum algorithms and illustrated them on the geometric model. As our geometric model illustrates nicely, Deutsch algorithm does not require any entanglement between the two qubits. However, Arvind's algorithm requires entangled states, and we have illustrated the intermediate steps in this algorithm by means of the corresponding entangled states in our geometric model. This shows how in principle modelling entanglement through constraint functions allows for a rigorous method to follow its form and effect in quantum computations. This is very important since at this stage it is not at all clear how entanglement can be exploited in a systematic way. In this sense, our geometric model is a step towards the better understanding of the exploitation of 2-qubit entanglement in a quantum algorithm. To our surprise the considered intermediate entangled states in Arvind's theorem are all singlet states. On the other hand, this allows us to represent these states with the macroscopic model in which a rigid rod explicitly realizes the entanglement correlation.

In future work we aim to study how the 2-qubit models can be generalized to represent the n-qubit quantum computer. Since any quantum system with a measurable set of outcomes has a Hidden Measurement representation [7, 31], this approach should be in principle feasible. Also, it is still an open question whether one can find a physical (rod-like?) implementation of the constraint functions for the non-singlet states. In our opinion it is an interesting question at what point, if any, this approach will encounter problems, i.e., whether there are some fundamental reasons why a 'quantum computer' should not be realizable by macroscopic means. Do we really need microscopic reality to construct non-classical computers?

Appendix A. Quantum axioms and Piron's representation theorem

In this appendix we briefly present the quantum axioms and show how quantum structure arises. Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. For properties $a, b \in \mathcal{L}$ we introduce an implication relation: $a < b \iff \kappa(a) \subset \kappa(b)$. Similarly, for states $p, q \in \Sigma$ we introduce $p < q \iff \xi(q) \subset \xi(p)$. It can be shown that for a **SPS** $(\Sigma, \mathcal{L}, \xi)$: $(\Sigma, \leq)$ and $(\mathcal{L}, \leq)$ are pre-ordered sets.

Axiom 1 (Property determination) *Consider a* **SPS** *$(\Sigma, \mathcal{L}, \xi)$. The axiom of property determination is satisfied iff for $a, b \in \mathcal{L}$:*

$$\kappa(a) = \kappa(b) \Rightarrow a = b$$

If the axiom of property determination is satisfied, then $\mathcal{L}, \leq$ is a partial order relation on $\mathcal{L}$.

Axiom 2 (Property completeness) *Consider a* **SPS** *$(\Sigma, \mathcal{L}, \xi)$. The axiom of property completeness is satisfied iff $\exists$ generating subset $\mathcal{T} \subseteq \mathcal{L}$ such that $\forall (a_i)_i \subseteq \mathcal{T}, \exists a \in \mathcal{L}$:*

$$\kappa(a) = \bigcap_i \kappa(a_i) \tag{A.1}$$

and $\forall a \in \mathcal{L} : \exists (a_i)_i \subseteq \mathcal{T}$ such that (A.1) is satisfied.

The property a is called the meet of $(a_i)_i$, denoted as $a = \bigwedge_i a_i$.

Axiom 3 (State completeness) *Consider a* **SPS** *$(\Sigma, \mathcal{L}, \xi)$. The axiom of state completeness is satisfied iff $\exists$ generating subset $\Gamma \subseteq \Sigma$, such that $\forall (p_j)_j \subseteq \Gamma : \exists p \in \Sigma$:*

$$\xi(p) = \bigcap_j \xi(p_j) \tag{A.2}$$

and $\forall p \in \Sigma : \exists$ subset $(p_j)_j \subseteq \Gamma$ such that (A.2) is satisfied.

$\Gamma \subseteq \Sigma$ is called the generating set of states of **SPS** $(\Sigma, \mathcal{L}, \xi)$, the state p is called the join of the set of states $(p_j)_j$, denote as $p = \bigvee_j p_j$. Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and property completeness are satisfied. Then $(\mathcal{L}, <, \wedge, \vee)$ is a complete lattice.

Axiom 4 (Orthocomplementation) *Consider a* **SPS** *$(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and property completeness are satisfied. The axiom of orthocomplementation is satisfied iff for a generating set of properties $\mathcal{T} \subseteq \mathcal{L}$ there exists a function*

$$^\perp : \mathcal{T} \to \mathcal{T} \quad a \mapsto a^\perp$$

such that for $a \in \mathcal{T}$ we have

$$(a^\perp)^\perp = a$$
$$\kappa(a)^\perp \subseteq \kappa(a^\perp) \subseteq \Sigma \setminus \kappa(a)$$

Definition 2 (Property state) *Consider a* **SPS** *$(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and completeness are satisfied. $\forall p \in \Sigma$: the 'property state' is the property $s(p) = \bigwedge_{a \in \xi(p)} a$.*

Theorem 2 (Orthocomplemented lattice) *Consider a* **SPS** *$(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination, property completeness and orthocomplementation are satisfied. Then $(\mathcal{L}, \leq, \wedge, \vee)$ is a complete orthocomplemented lattice.*

Axiom 5 (Covering Law) *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The covering law is satisfied iff for $a, x \in \mathcal{L}$ and $p \in \Sigma$:*

$$a < x < a \vee p \Rightarrow x = a \;\; or \;\; x = a \vee p$$

Axiom 6 (Weak Modularity) *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The orthocomplemented lattice $\mathcal{L}$ is weakly modular, i.e. $\forall a, b \in \mathcal{L} : a < b \Rightarrow (b \wedge a^{\perp}) \vee a = b$.*

Axiom 7 (Atoms) *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The element $a \in \mathcal{L}$ is called an atom iff $\forall x \in \mathcal{L} : 0 < x < a \Rightarrow x = 0$ or $x = a$, i.e. a covers 0.*

Axiom 8 (Plane Transitivity) *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The orthocomplemented lattice $\mathcal{L}$ is plane transitive. This means that for atoms $s, t \in \mathcal{L}$, $\exists$ two distinct atoms $s_1 \neq s_2$ and a symmetry f such that $f\big|_{[0, s_1 \vee s_2]}$ is the identity and $f(s) = t$.*

The **SPS** $(\Sigma, \mathcal{L}, \xi)$ of quantum and classical mechanical entities is such that $\mathcal{L}$ is a complete orthocomplemented lattice that satisfies the covering law, and is weakly modular and plane transitive. This can be checked straightforward from the Hilbert space representation of $\mathcal{L}$ by the lattice of closed subspaces in Hilbert space for quantum systems, or the phase space representation for classical mechanical entities. The reverse statement is less easy to prove. In fact, even the formulation of the correct set of axioms which force the quantum structure on the set of properties has been a scientific endeavour which lasted over a few decades. However, in honour of Constantin Piron who proved the first 'almost finished' representation theorem of quantum (and classical) systems, we name it here after him [32, 33]:

Theorem 3 ((generalized) Representation theorem of Piron) *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ such that $\mathcal{L}$ contains at least 4 elements and is a complete orthocomplemented lattice that satisfies the covering law, is weakly modular and plane transitive. Then $\mathcal{L}$ is isomorphic to a family of lattices (with superselection rules) of closed subspaces in a Hilbert space over the field of reals, complex numbers or the division ring of quaternions.*

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