When can a data set be described by quantum theory?

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Abstract

There have been a recent claims in various fields of research that aim to show the presence of quantum structure outside the generally accepted domain of quantum theory. We will take a pragmatic and probabilistic perspective to answer the question when it makes sense to describe a given data set by means of quantum theory.

Introduction

Recently we have seen a surge in new research results claiming quantum or quantum-like effects in fields outside the scope of physics. These fields of research include, among others, economics (Schaden 2002; Baaquie 2004; Haven 2005; Bagarello 2006), operations research and management sciences (Bordley 1998; Bordley and Kadane 1999; Mogiliansky 2006), psychology and cognition (Aerts and Aerts 1994; Grossberg 2000; Gabora and Aerts 2002a,b; Aerts and Gabora 2005a,b; Busemeyer, Wang and Townsend 2006; Aerts 2007a,b), game theory (Eisert, Wilkens and Lewenstein 1999; Piotrowski and Sladkowski 2003), and language and artificial intelligence (Widdows 2003, 2006; Widdows and Peters 2003; Aerts and Czachor 2004; Van Reisbergen 2004; Aerts, Czachor and D'Hooghe 2005, Bruza and Cole 2005). Generally speaking, such claims have been met with caution, if not plain resistance, from many practising physicists. It is not difficult to understand why this is the case. In its infancy stage, quantum theory itself was controversial and was only gradually accepted after many discussions and experimental tests. But once the community of physicists accepted this strange new theory, their attitude reversed: they cherished the new quantum theory as the only proper way to deal with the mysterious phenomena that pertain to the quantum world and for which absolutely no classical interpretation exists. The firm rejection of a classical interpretation should not be taken as due to a conservative or impaired view on the subject. Rather it reflects the failed mission of many bright physicists to come up with such an interpretation. Indeed, one of the main problems in the foundations of quantum theory, a consequence of the infamous measurement problem, is not to give a classical interpretation for quantum theory, but rather to give a quantum mechanical account of the classical world! In spite of many interesting results, both attempts have so far failed due to deep structural differences between the classical and quantum theory of physical systems. But in the course of finding experimentally verifiable criteria to discern the two situations, physicists devised the proper tools to help us provide a quantitative answer to the question: when is a set of data describing a given phenomenon best described by quantum theory?

Applications of quantum theory outside quantum physics?

Say that we have a theory about a set of phenomena pertaining to a field other than micro-physics and that some claim to have a certain quantum likeness . The first obvious proposal that comes to mind to see how much this theory resembles quantum theory, is to investigate and compare the set of axioms that reproduce the two theories. Such an approach is successful only if there are simple means to relate the two systems of axioms. There is however no uniquely accepted set of axioms for quantum theory. Orthodox, non-relativistic quantum theory can be reproduced in a great variety of ways, each coming with its own sets of axioms. Indeed, it is safe to say no theory has gone through so many reformulations as quantum theory. To illustrate this point, let us restrict our attention to a non-relativistic setting and name a few of the most important reformulations. We start with Heisenberg's matrix formalism (Heisenberg 1925) and Schrödinger's wave mechanics (Schrödinger 1926). From this two main formulations arose the Dirac transformation theory (Dirac 1930) which is still the most widely used formulation for physicists, followed by the more mathematically oriented axiomatic formulation of quantum theory as an abstract theory in Hilbert space by Von Neumann (von Neumann 1932). Wigner (Wigner 1932) devised a phase-space formalism for quantum theory based on a joint representation of momentum and position. Much effort was also devoted to the so-called algebraic formulation of quantum theory that starts form the algebra of operators that describe observable quantities (Mackey 1963). We have the de Broglie-Bohm (Bohm 1952) formulation that recasts quantum theory in a form similar to the Hamilton-Jacobi theory of classical physics. There is the Feynman path-integral approach. More mathematically inspired approaches can be found when we start from the logic of quantum systems in the sense of von Neumann and Birkhoff (Birkhoff and von Neumann 1936) and Mielnik's convex approach that starts from the convexity of states (Mielnik 1974). It is clear that all these formulations start with widely different sets of axioms and are from this perspective difficult to compare. This abundance of reformulations begs the question what precisely we mean with quantum theory, and which theory we are talking about when we say we have a quantum-like description? Or is it better to talk about *a* quantum theory? Naturally there is a common ground to all these formulations, for all deliver (at least) a description for the commonly treated archetypical quantum system, i.e. non-relativistic quantum particles with spin interacting with quantized fields and amongst themselves. In other words, a formulation of quantum mechanics is considered valid only to the extent that it manages to reproduce the observable aspects of these archetypical quantum phenomena. It seems therefore natural to direct our attention to those parts of the theory that lay out the relation of the theory to the experiment. However natural this may seem from a modern perspective, we note that there is no such interface between theory and experiment (or rather that it is trivial) in classical physics, where it is assumed that the predictions of the theory identically follow the experimental results, safe for experimental errors. It is of considerable importance that all quantum phenomena exhibit a characteristic randomness in the outcomes. Hence reproducing the results means reproducing the probabilities of outcomes. As measurements in quantum mechanics change the very nature of the system, it is often impossible to perform repeated measurements on the same system and expect the same probabilities to arise. To solve this problem quantum experimentalists prepare a large number of systems in an identical way and repeat the measurements on this ensemble of systems. Thus arises the concept of the *state* of the system S, which can be regarded as a shorthand to denote the ensemble preparation. Some people prefer to interpret the state as the "ontological mode of being" of a system, still others prefer to say the state represents our own knowledge of the system. However important the concept of *state* is, its precise interpretation is of little consequence for our purposes here. Say then we have a theory $\mathcal{T}$ that describes a system S. We think implicitly of a theory as a formal abstract system encoded in a set of axioms, along with the most important consequences of those axioms. As explained above, a given set of phenomena can be described by possibly very different sets of axioms. However, we require of any theory that it contains at least the following three parts which serve as a link between the theory and the experiment:

1. The set Σ_S of possible states for the system S

2. The quantities that one can infer from S by means of experiments, i.e. the set $\mathcal{O}_S$ of observable quantities (observables) $\mathcal{O}_S = \{A, B, C, ...\}$.To each observable A, there corresponds a (discrete or continuous) set of outcomes X_A.

3. A map $P : \Sigma_S \times \mathcal{O}_S \times X_A \to [0, 1]$

$$\begin{aligned} (\psi, A, x_i) &\mapsto P(\psi, A, x_i) \\ \sum_{x_i \in X_A} P(\psi, A, x_i) &= 1 \end{aligned}$$

   (and a similar normalization equation for the continuous case), such that repeatedly measuring observable $A \in \mathcal{O}_S$ on a system in the state $\psi \in \Sigma_S$, yields a relative frequency for the occurrence of outcome $x_i \in X_A$ that converges to $P(\psi, A, x_i)$.

It is not very difficult to determine these three parts for the formulations given above. For a classical deterministic theory, we have that P is two valued only: $P : \Sigma_S \times \mathcal{O}_S \times X_A \to \{0, 1\}$, but how are we going to distinguish between a classical statistical situation and a quantum mechanical one?

Classical probability

Classical probability sprang forth in the old times from an investigation into gambling. Only in the 20^{th} century several attempts to formalize it into a consistent mathematical theory were undertaken. The theory that was most successful in terms of mathematical rigor, was the measure-theoretic formalization by Andrei Kolmogorov in 1933 (Kolmogorov 1933). In fact, the theory is considered so successful, that to most present-day mathematicians, the term "probability" denotes not so much the limit of a relative frequency, or a degree of belief, but simply a normalized measure on an event space. Hence from a probabilistic perspective it is natural to call a theory *classical*, if there exists a Kolmogorovian model for it. The Kolmogorovian framework is like a very advanced reformulation of the original rule of probability as ratio of the number of cases favorable to a given event, to the total of possible events. In other words: Kolmogorovian probability always allows for a lack of knowledge interpretation. Likewise, if a probability theory is not Kolmogorovian, a lack of knowledge interpretation is often problematic or even impossible. Suppose we have a data set, describing the probability of certain events, as well as the probability of the conjunction or disjunction of these events. How will we know whether this data set allows for a Kolmogorovian model? Fortunately, it is not necessary to construct such a model explicitly. Indeed, as was shown already by Boole in 1854 (Boole 1958) and later generalized by Bonferroni, there exist sets of linear inequalities in the probabilities which, when satisfied, guarantee the existence of such a model. Let us give an example taken from Pitowsky (Pitowsky 1989). Suppose we have an urn with N balls in various colors and made from different materials. Call p_{red} the probability that a drawn ball is colored red and call p_{wood} the probability that it is a wooden ball. Then clearly we have: $0 \leq p_{red} \leq 1$ and $0 \leq p_{wood} \leq 1$. If we call $p_{red,wood}$ the joint probability that a drawn ball is both red and made of wood, then we obviously have $0 \leq p_{red,wood} \leq p_{red} \leq 1$ and $0 \leq p_{red,wood} \leq p_{wood} \leq 1$. Consider now the probability that a drawn ball is *either* red *or* made of wood, but not both at the same time. This probability is given by $0 \leq p_{red} + p_{wood} - p_{red,wood} \leq 1$. Hence, if we measure the quantities p_{red}, p_{wood} and $p_{red,wood}$ then there only exists an urn model (a Kolmogorovian model) if the measured frequencies do not violate any of the inequalities above. Nothing tells us that the measured frequencies actually result from an urn model. We only know they *could have been* obtained from such a model. The data never uniquely fix the theoretical description, but they may invalidate certain theoretical models and this seems the best we can do. One may wonder at this point whether not all sets of actually measured probabilities obey these inequalities? Boole seems to have thought so, for he called these inequalities "*conditions of possible experience*" (Boole 1958). It turns out the probabilities obtained from quantum mechanics do not necessarily obey the inequalities that result from the Kolmogorovian model. The question is: in what sense are they different?

EPR, incompleteness and the quest for hidden variables

Radically contrary to our intuition and classical theory, quantum theory tells us that physical systems do not simultaneously possess definite values for all their observable quantities(e.g. position, momentum, spin,...). According to orthodox quantum dogma, such observables have potential values only, meaning that it is not the case that they are simply unknown or incompletely known to us, but rather that properties become actual only when they are observed as such. Albert Einstein was one of the strongest opponents of this observer-dependent notion of reality, as is evident by a famous quote, recalled by physicist Abraham Pais as "*We often discussed his notions on objective reality. I recall that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it.*" In 1935 Einstein, Podolsky and Rosen (EPR 1935) suggested a simple experiment aimed to show that perhaps quantum mechanics gives such a peculiar picture of reality because it is an incomplete theory, not describing the fine details of the inner workings of the universe. The nature of the incompleteness was not specified in the famous EPR paper, but it seems likely that the authors believed the theory could be supplemented by additional variables such that a complete account was still possible, even though such variables could not be measured directly. If true, the state of affairs would be somewhat remniscent of (classical) statistical physics, in which the precise values of observable quantities are unknown until one measures them. Von Neumann was the first to prove a theorem showing that the mere restoration of all values, even if unknown to us, leads to an inconsistency in the formalism of quantum theory (von Neumann 1932). The von Neumann theorem was criticized mainly because it assumes the same linear structure for the hidden variables as for the usual quantum variables. However, it still stands as the first proof that shows the probability model of quantum mechanics is difficult to reconcile with the idea that the probability is due to a lack of knowledge. In other words, the probability model is not a Kolmogorovian one. Although the experiment in the original EPR paper is conceptually simple, it is difficult to obtain the correlations that quantum theory predicts experimentally. David Bohm simplified the EPR experiment to what nowadays is known as the EPRB experiment, so that the correlations pertained to discrete spin values rather than the original continuous variables. That was a great step forward towards an experimental realization, but the real breakthrough came with only the work of John Bell.

Bell

John Bell (Bell, 1964) started with the experimental set up of the EPRB experiment and showed in one of the most celebrated papers in the foundations of quantum mechanics how one could experimentally measure correlations that defy a classical interpretation. As Bell himself points out in this publication: *A motivation is in the peculiar character of some quantum-mechanical predictions, which seem almost to cry out for a hidden variable interpretation. This is the famous argument of Einstein, Podolsky and Rosen. (...) We will find, in fact, that no local deterministic hidden-variable theory can reproduce all the experimental predictions of quantum mechanics. This opens the possibility of bringing the question into the experimental domain, by trying to approximate as well as possible the idealized situations in which local hidden variables and quantum mechanics cannot agree.* The experimental setup is easy. Alice and Bob prepare a so-called singlet state and measure the spin at two different locations with polarizers settings denoted by the vector $\mathbf{a}$ in Alice's location and vector $\mathbf{b}$ in Bob's location. The outcome is denoted by A for Alice (respectively B for Bob) and is given the value $+1$ if they measure spin up and -1 if they measure a spin down. On of the strongest aspects of the Bell inequality, is that so few assumptions are necessary to prove the theorem. We briefly state the assumptions here. First, the outcome may depend on some hidden variable λ that takes values in Λ and on the local polarizer setting, but not on the setting at the other location:

$$\begin{aligned} A(\mathbf{a},\mathbf{b},\lambda) &= A(\mathbf{a},\lambda) \\ B(\mathbf{a},\mathbf{b},\lambda) &= B(\mathbf{b},\lambda) \end{aligned}$$

We then clearly have for all values of $\mathbf{a}, \mathbf{b}, \lambda$

$$|A(\mathbf{a},\lambda)| \leq 1, |B(\mathbf{b},\lambda)| \leq 1$$

Probably the most crucial assumption of the Bell paper, is that the measured correlation has to take the form

$$E(\mathbf{a},\mathbf{b}) = \int_\Lambda A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)d\lambda \qquad (1)$$

It then easily follows (Bell, 1964) that for all possible angles a and b of the polarizers the following inequality holds[1]:

$$|E(\mathbf{a},\mathbf{b}) + E(\mathbf{a},\mathbf{b}')| + |E(\mathbf{a}',\mathbf{b}) - E(\mathbf{a}',\mathbf{b}')| \leq 2$$

However quantum theory predicts that $E(\mathbf{a},\mathbf{b}) = -\mathbf{a}.\mathbf{b}$ so that, for certain values this threshold will be surpassed by as much as a factor $\sqrt{2}$ and this was indeed measured experimentally (Aspect *et al.* 1982) The reason is a matter

[1] In fact, this is not the Bell-inquality, but rather the closely related Clauser, Horne, Shimony and Holt (CHSH) inequality.

of much debate, but it is commonly accepted that it constitutes a violation of local realism. What is not disputable, is that the correlation (1) is seemingly not obeyed by quantum mechanics. It came as quite a shock to many that such a simple and elegant idea, based on so few assumptions, could be used experimentally to decide between two candidate underlying theoretical models. In this way the inequalities lose their attraction as "conditions of possible experience " a la Boole, but we have gained a new and powerful tool for investigating which model can faithfully reproduce the data. To the best of our knowledge, it was Mielnik (Mielnik 1968) who first understood that, due to the specific vector space model of quantum theory, the probabilities derived from a quantum model need also obey certain inequalities. We will not present the work of Mielnik, but the mathematically more complete form of Accardi and Fedullo and later that of Pitowsky.

Accardi and Fedullo

Let $A, B, C, \ldots$ denote a set of observable quantities that can take real values (a_α), (b_β), (c_γ), .. with $\alpha, \beta, \gamma, .. = 1, 2, ..., n < +\infty$. and consider the transition probabilities

$$
\begin{aligned}
P(A &= a_\alpha | B = b_\beta), && (2)\\
P(B &= b_\beta | C = c_\gamma), && (3)\\
P(C &= c_\gamma | A = a_\alpha) && (4)
\end{aligned}
$$

where $P(A = a_\alpha | B = b_\beta)$ denotes the probability that measurement of observable A yields the outcome a, if it is known that observable B assumes the value b. Since Accardi and Fedullo consider transition probabilities, they require the probabilities to be symmetric in their arguments:

$$P(A = a_\alpha | B = b_\beta) = P(B = b_\beta | A = a_\alpha) \qquad (5)$$

and strictly positive: $P(A = a_\alpha | B = b_\beta) > 0$. We then say the transition probabilities (2) allow for a Kolmogorovian probability model iff there exist

1. a probability space (Ω, θ, μ)
2. for each observable $A, B, C, ..$ a measurable partition $(A_\alpha), (B_\beta), (C_\gamma), \ldots$ of Ω

   such that for each $\alpha, \beta, \gamma, \ldots$ we can write

$$P(A = a_\alpha | B = b_\beta) = \frac{\mu(A_\alpha \cap B_\beta)}{\mu(B_\beta)}$$

Moreover, we will say that the transition probabilities (2) allow for a complex (real) Hilbert space model iff there exist

1. a complex (real) Hilbert space $\mathcal{H}$ of dimension n
2. for each observable $A, B, C, ..$ an orthonormal basis (φ_α), (φ_β), (φ_γ), ...of $\mathcal{H}$ such that, for each $\alpha, \beta, \gamma, \ldots$ we have $P(A = a_\alpha | B = b_\beta) = |\left\langle \varphi_\alpha, \varphi_\beta \right\rangle|^2$

Because there is always a Kolmogorovian model if we only consider two observables, the first case of interest is when we consider three observables. Accardi and Fedullo then prove the following assertion.

Theorem 1 (Kolmogorovian model) *Let A, B and C be three n -valued observables. The transition matrices $P(A = a_\alpha | B = b_\beta)$, $P(B = b_\beta | C = c_\gamma)$, $P(C = c_\gamma | A = a_\alpha)$ admit a Kolmogorovian model iff there are n^3 real numbers $\Gamma_{\alpha,\beta,\gamma}$ such that all $\Gamma_{\alpha,\beta,\gamma}$ are positive and*

$$
\begin{aligned}
\sum_\gamma \Gamma_{\alpha,\beta,\gamma} &= P(A = a_\alpha | B = b_\beta)\\
\sum_\alpha \Gamma_{\alpha,\beta,\gamma} &= P(B = b_\beta | C = c_\gamma)\\
\sum_\beta \Gamma_{\alpha,\beta,\gamma} &= P(C = c_\gamma | A = a_\alpha)
\end{aligned}
$$

The quantum case is hard to track in general. Hence Accardi and Fedullo restrict their attention to the important case of two-valued observables. In this case a full characterization of the possible probabilistic frameworks is simple to obtain.

Theorem 2 *Three two-valued observables with respective probabilities $p = P(A = a_\alpha | B = b_\beta)$, $q = P(B = b_\beta | C = c_\gamma)$ and $r = P(C = c_\gamma | A = a_\alpha)$ admit*

(i) a Kolmogorovian model iff

$$|p + q - 1| \leq r \leq 1 - |p - q|$$

(ii) a quantum or, equivalently, complex Hilbert space model iff

$$-1 < \frac{p + q + r - 1}{2\sqrt{pqr}} < +1$$

(iii) a real Hilbert space model iff

$$
\begin{aligned}
\sqrt{r} &= \sqrt{pq} + \sqrt{(1-p)(1-q)}\\
&\textit{or}\\
\sqrt{r} &= |\sqrt{pq} - \sqrt{(1-p)(1-q)}|
\end{aligned}
$$

We have studied extensively the probabilistic behavior of the so-called epsilon-model by means of the Accardi-Fedullo inequalities (Aerts *et al.* 1999) and given examples of how these inequalities could be violated in psychological decision procedures (Aerts and Aerts 1994). One can construct the conditional probability for this model and show it yields a continuous transition from a Kolmogorovian framework to a quantum one, intermediately passing a region that cannot be described by either one of them (Aerts 1996).

Pitowsky

Pitowsky (Pitowsky 1989) considered polytopes of joint probabilities and developed quite general criteria for the existence of a Kolmogorovian or Hilbert space model for families of joint probabilities. Suppose we have an experiment performed on an entity A_1 which tests the occurrence or nonoccurrence of a certain event. And suppose we call the probability of occurrence $P(A_1)$. We have another experiment performed on an entity A_2 similarly testing the occurrence or non-occurrence of another event, and we call the probability of occurrence $P(A_2)$. Let us suppose that the situation is such that we can also perform a joint experiment on the entity A_1 and A_2, which tests the occurrence or non-occurrence of both events simultaneously. We denote the probability of occurrence of this joint event by $P(A_1 \wedge A_2)$. The question we want to consider is the following: "Under what conditions can these three probabilities be represented as a Kolmogorovian probability theory? Consider n entities $A_1, A_2, \ldots, A_n$, and n experiments (each one performed on one of the entities) testing for the occurrence and non occurrence of an event, and let us denote $P(A_i)$ the probability of occurrence of the experiment on entity A_i. Consider then mutually joint experiments on pairs of the entities and denote the probability of occurrence of the joint event by $P(A_i \wedge A_j)$ in case of the experiment on the entity A_i and A_j. It is not necessary that joint events are considered for each one of the possible pairs of entities. Hence we consider a set S of pairs of indices:

$$S \subseteq \{(i,j) \mid i < j; i,j = 1,2,\ldots,n\} \tag{6}$$

corresponding to those pairs of entities for which a joint experiment is possible to be performed. As a consequence, the following set of probabilities is given:

$$p_i = P(A_i) \quad i = 1,2,\ldots,n$$
$$p_{ij} = P(A_i \wedge A_j) \quad (i,j) \in S \tag{7}$$

We say that the set of probabilities in (7) has a Kolmogorovian representation if there exists a Kolmogorovian probability model (Ω, F, μ) with $X_1, X_2, \ldots, X_n \in F$ elements of the event algebra, such that

$$p_i = \mu(X_i) \quad i = 1,2,\ldots,n \tag{8}$$
$$p_{ij} = \mu(X_i \wedge X_j) \quad (i,j) \in S \tag{9}$$

Pitowsky introduced an expressive geometric language to give an answer to the quantum of existence of a Kolmogorovian representation for the set of probabilities in (7). First Pitowsky introduces a $n + |S|$-dimensional correlation vector

$$\overrightarrow{p} = (p_1, p_2, \ldots, p_n, \ldots, p_{ij}, \ldots) \tag{10}$$

where $|S|$ is the cardinality of S. Denote $R(n,S) = \mathbb{R}^{n+|S|}$ the $n + |S|$ dimensional vector space over the real numbers. Let $\epsilon \in \{0,1\}^n$ be an arbitrary n-dimensional vector consisting of $0's$ and $1's$. For each ϵ we construct the following vector $\overrightarrow{u}^\epsilon \in R(n,S)$:

$$u_i^\epsilon = \epsilon_i \quad i = 1,2,\ldots,n \tag{11}$$
$$u_{ij}^\epsilon = \epsilon_i \epsilon_j \quad (i,j) \in S \tag{12}$$

The set of convex linear combinations of the $u's$ is called the classical correlation polytope:

$$\begin{aligned} c(n,S) &= \{\overrightarrow{f} \in R(n,S) \mid \\ \overrightarrow{f} &= \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon \overrightarrow{u}^\epsilon;\ \lambda_\epsilon \geq 0; \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon = 1\} \end{aligned}$$

Pitowsky proved in (Pitowsky 1989) the following theorem:

Theorem 3 *The set of probabilities*

$$\begin{aligned} p_i &= P(A_i) \quad i = 1,2,\ldots,n \\ p_{ij} &= P(A_i \wedge A_j) \quad (i,j) \in S \end{aligned}$$

admits a Kolmogorovian probability model if and only if its correlation vector $\overrightarrow{p}$ *belongs to the correlation polytope* $c(n,S)$.

A very nice feature of the Pitowsky correlation polytopes is that they form a direct generalization of the Bell-Wigner polytope and the Clauser-Horne polytope. Indeed, setting $n = 3$ and $S = \{(1,2),(1,3),(2,3)\}$ yields the Bell inequalities and setting $n = 4$ and $S = \{(1,3),(1,4),(2,3),(2,4)\}$ we obtain the Clauser-Horne Polytope. To introduce the quantum case, recall (6) and define $R(n,S)$ as the real space of all functions f such that $f : \{1,2,.,,,n\} \cup S \to \mathbb{R}$.

Definition: Let $p = (p_1, p_2, ...p_n, ..., p_{ij}, ...) \in R(n,S)$. We shall say p admits a Hilbert space model iff there exists a Hilbert space $\mathcal{H}$ and a state ρ on $\mathcal{H}$ and projections $E_1, ..., E_n \in L(\mathcal{H})$ such that

$$\begin{aligned} p_i &= Tr(\rho E_i), i = 1,...,n \\ p_{ij} &= Tr(\rho[E_i \wedge E_j]), (i,j) \in S \end{aligned}$$

This is simply a reformulation of the well-known rules of quantum probability. Here $E_i \wedge E_j$ is the projection onto the subspace $E_i(\mathcal{H}) \cap E_j(\mathcal{H})$ such that ψ is an eigenstate of $E_i \wedge E_j$ iff it is an eigenstate of both E_i and E_j. We need one more definition.

Definition: Let $l(n,S)$ be the set of all vectors $p = (p_1, ...p_n,, ...p_{ij}, ...) \in R(n,S)$ such that

$$\begin{aligned} 0 &\leq p_i \leq 1, i = 1,...,n \\ 0 &\leq p_{ij} \leq \min(p_i, p_j) \text{ with } (i,j) \in S \end{aligned}$$

Obviously $l(n,S)$ is convex and closed. Then Pitowsky shows the following theorem:

Theorem 4 *Call* $q(n,S)$ *the set of all probabilities* $p \in R(n,S)$ *that admit a Hilbert space model. Then*

$$c(n,S) \subseteq q(n,S) \subseteq l(n,S)$$

Moreover $q(n,S)$ *is convex but not closed and contains the interior of* $l(n,S)$.

We see that the use of the joint probabilities allows for a characterization of the polytopes involved. Suppose we have measured set of p_i and their joint probabilities p_{ij} for which the closure is the polytope $e(n,S)$, then the ratio of the volume of this set to the volume of $q(n,S)$ and $c(n,S)$ could be used as a crude measure of non-classicality. Indeed, define

$$\begin{aligned} \xi &= \frac{\mu(e(n,S))}{\mu(c(n,S))} \\ \kappa &= \frac{\mu(e(n,S))}{\mu(q(n,S))} \end{aligned}$$

then the closer ξ is to unity, the more classical the data set is, the closer κ is to unity, the more quantum-like the data set is. It is entirely possible that both parameters are larger than one. Data obtained from quantum mechanical experiments surely will make ξ larger than one ($c(n,S) \subseteq q(n,S)$ and the monotonicity of measures) and some models violate the inequalities more than quantum mechanics does (Aerts 1982, Aerts 1996). We note that it is not clear how to interpret $E_i \wedge E_j$ as a real experiment if the projectors do not commute. However, Pitowsky was aiming at a generalization of the Bell inequality in which case there is no problem with interpreting $E_i \wedge E_j$ as performing E_i on one wing of the experiment and E_j on the second wing. This begs the question of whether it is possible to give a criterion for non-classicality if we consider a single system.

The quantumness of single systems

Alicki and van Ryn (Alicki and van Ryn 2007) consider the expectation values of two observables, but measured on a single system. They note that, for any two given functions satisfying

$$0 \leq f(x) \leq g(x)$$

and a given (classical) probability distribution $\rho(x)$, we have that the following inequality holds:

$$\langle f^2 \rangle_\rho = \int f^2(x)\rho(x)dx \leq \int g^2(x)\rho(x)dx \leq \langle g^2 \rangle_\rho$$

In quantum theory however, it is entirely possible that we have that that for all states ψ

$$0 \leq \langle\psi|A|\psi\rangle \leq \langle\psi|B|\psi\rangle$$

but for which there exists a state ϕ such that

$$\langle\psi|A^2|\psi\rangle > \langle\psi|B^2|\psi\rangle$$

If this is case, the observables A and B indicate the presence of a non-Kolmogorovian probability model for a single system. In order to experimentally establish estimate for the expectation values of a given set of observables, we need acces to an ensemble of identically prepared systems, hence the name "single system" in this context does not mean we have only one unique entity to perform our measurements on, but rather that, in the physical sense, the system is localized in a small spatial region. When applied to other fields of research, it denotes we will perform the measurements corresponding to observables A and B on the same system rather than on different subsystems of a larger system.

Conclusions

Unless a phenomenon is truly occurring at the micro-level of physics, it can be difficult to defend that it is still a quantum phenomenon. A strict separation is necessary between synthetic and natural phenomena. One can construct models that exhibit quantum or quantum-like behavior (synthetic) or one could argue that an experimentally established set of data can best be represented using a quantum theoretical model (natural). We have restricted our attention to the latter case here. For a given data set the interesting and pragmatic question we have focussed on, is not whether the phenomenon that gave rise to that data is quantum in origin or in its workings, but simply whether we can take advantage of the rich theory of quanta to model that phenomenon. Generally speaking, classical probability theory is much more simple than its quantum counterpart and we should have good reasons before we decide to use the quantum probabilistic model. We have drawn attention to the fact that well-known criteria exist in the literature aimed at classifying the underlying probabilistic model. In order to verify the presence of a non-Kolmogorovian probability structure, it is sufficient that certain inequalities that are linear in the probabilities are violated. Two well-known examples of schemes, one due to Accardi and Fedullo and one to Pitowsky, that are able to reproduce sets of such inequalities are given. If one can show at least one of these inequalities is violated, then there is reason to believe the underlying model is not Kolmogorovian and perhaps the probabilities are not due to a lack of knowledge of the system under study. An alternative hypothesis that allows for a special lack of knowledge interpretation can be found in (Aerts 1986). Whether a quantum model is possible at all can also be decided by sets of inequalities that are quadratic in the probabilities, but whether a quantum model does better at reproducing the data is a question that is best answered on a case by case basis. Of course, there may be structural similarities with quantum phenomena and this can also be conisdered a criterion to seek for a quantum description. However, the strong point of the inequalities is precisely that they do not bother with many of the interpretational problems that plague much of the discussion on quantum theory. It is outside the field of microphysics more often than not of little importance whether the Bell-inequality is violated because of issues with realism or locality, or that some intricate measurement loophole is at play. Singlet states may exist only in the quantum world, but the Bell-inequalities can be violated in many different systems (see, for example, Aerts 1991). As Riefel pointed out in the Quantum Interactions 2007 symposium, such systems do not satisfy the no-signaling condition and hence cannot be thought to reproduce the peculiar non-local effects of quantum physics. It would be ludicrous indeed to state that the violation of the Bell-inequalities in, for example, language (Aerts *et al.* 2000) is due to non-locality. Nor does that matter for our purposes. What counts at the end of the day is which framework best describes the measured data.

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