Quantum Axiomatics

Diederik Aerts

Center Leo Apostel (CLEA) and
Department of Mathematics (FUND),
Brussels Free University,
Krijgskundestraat 33,
1160 Brussels, Belgium.
diraerts@vub.ac.be

Abstract

We present an axiomatic and operational theory of quantum mechanics. The theory is founded on the axiomatic and operational approach started in Geneva mainly by Constantin Piron and his students and collaborators and developed further in Brussels by myself and different students and collaborators. A physical entity, which a priori can be a classical entity or a quantum entity or a combination of both, is described by means of its set of states, its set of properties and a physical notion of 'actuality of a property the entity being in a state'. This leads to the mathematical structure of a state property space $(\Sigma, \mathcal{L}, \xi)$, where Σ is the set of states of the entity, $\mathcal{L}$ the set of properties, and $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$ a function mapping each state on the set of properties that are actual if the entity is in that state. We introduce seven axioms such that if satisfied the state property space can be represented by the direct union over a classical state space of irreducible state property spaces, where each one of the irreducible state property spaces is a Hilbert space state property space of standard quantum mechanics, over the real, complex or quaternionic numbers. The axioms are introduced in an as much as possible operational way, such that we can analyze their physical meaning.

1 Introduction

Quantum axiomatics has its roots in the work of John von Neumann, in collaboration with Garett Birkhoff, that is almost as old as the standard formulation of quantum mechanics itself [1]. Indeed already during the beginning years of quantum mechanics, the formalism that is now referred to as standard quantum mechanics [2], was thought to be too specific by the founding fathers themselves. One of the questions that obviously was at the origin of this early dissatisfaction is: 'Why would a complex Hilbert space deliver the unique mathematical structure for a complete description of the microworld? Would that not be amazing? What is so special about a complex Hilbert space that its mathematical structure would play such a fundamental role?'

Let us turn for a moment to the other great theory of physics, namely general relativity, to raise more suspicion towards the fundamental role of the complex Hilbert space for quantum mechanics. General relativity is founded on the mathematical structure of Riemann geometry. In this case however it is much more plausible that indeed the right fundamental mathematical structure has been taken. Riemann developed his theory as a synthesis of the work of Gauss, Lobatsjevski and Bolyai on non-Euclidean geometry, and his aim was to work out a theory for the description of the geometrical structure of the world in all its generality. Hence Einstein took recourse to the work of Riemann to express his ideas and intuitions on space time and its geometry and this lead to general relativity. General relativity could be called in this respect 'the geometrization of a part of the world including gravitation'.

There is, of course, a definite reason why von Neumann used the mathematical structure of a complex Hilbert space for the formalization of quantum mechanics, but this reason is much less profound than it is for Riemann geometry and general relativity. The reason is that Heisenberg's matrix mechanics and Schrödinger's wave mechanics turned out to be equivalent, the first being a formalization of the new mechanics making use of l_2, the set of all square summable complex sequences, and the second making use of $L_2(\mathbb{R}^3)$, the set of all square integrable complex functions of three real variables. The two spaces l_2 and $L_2(\mathbb{R}^3)$ are canonical examples of a complex Hilbert space. This means that Heisenberg and Schrödinger were working already in a complex Hilbert space, when they formulated matrix mechanics and wave mechanics, without being aware of it. This made it a straightforward choice for von Neumann to propose a formulation of quantum mechanics in an abstract complex Hilbert space, reducing matrix mechanics and wave mechanics to two specific cases.

One problem with the Hilbert space representation was known from the start. A (pure) state of a quantum entity is represented by a unit vector or ray of the complex Hilbert space, and not by a vector. Indeed vectors contained in the same ray represent the same state or one has to renormalize the vector that represents the state after it has been changed in one way or another. It is well known that if rays of a vector space are called points and two dimensional subspaces of this vector space are called lines, the set of points and lines corresponding in this way to a vector space, form a projective geometry. What we just remarked about the unit vector or ray representing the state of the quantum entity means that in some way the projective geometry corresponding to the complex Hilbert space represents more intrinsically the physics of the quantum world as does the Hilbert space itself. This state of affairs is revealed explicitly in the dynamics of quantum entities, that is built by using group representations, and one has to consider projective representations, which are representations in the corresponding projective geometry, and not vector representations [3].

The title of the article by John von Neumann and Garett Birkhoff [1] that we mentioned as the founding article for quantum axiomatics is 'The logic of quantum mechanics'. Let us explain shortly what Birkhoff and von Neumann do in this article. First of all they remark that an operational proposition of a quantum entity is represented in the standard quantum formalism by an orthogonal projection operator or by the corresponding closed subspace of the Hilbert space $\mathcal{H}$. Let us denote the set of all closed subspaces of $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. Next Birkhoff and von Neumann show that the structure of $\mathcal{L}(\mathcal{H})$ is not that of a Boolean algebra, the archetypical structure of the set of propositions in classical logic. More specifically it is the distributive law between conjunction and disjunction

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \tag{1}$$

that is not necessarily valid for the case of quantum propositions $a, b, c \in \mathcal{L}(\mathcal{H})$. A whole line of research, called quantum logic, was born as a consequence of the Birkhoff and von Neumann article. The underlying philosophical idea is that, in the same manner as general relativity has introduced non-Euclidean geometry into the reality of the physical world, quantum mechanics introduces non-Boolean logic. The quantum paradoxes would be due to the fact that we reason with Boolean logic about situations with quantum entities, while these situations should be reasoned about with non-Boolean logic.

Although fascinating as an approach [4], it is not this idea that is at the origin of quantum axiomatics. Another aspect of what Birkhoff and von Neumann did in their article is that they shifted the attention on the mathematical structure of the set of operational propositions $\mathcal{L}(\mathcal{H})$ instead of the Hilbert space $\mathcal{H}$ itself. In this sense it is important to pay attention to the fact that $\mathcal{L}(\mathcal{H})$ is the set of all operational propositions, *i.e.* the set of yes/no experiments on a quantum entity. They opened a way to connect abstract mathematical concepts of the quantum formalism, namely the orthogonal projection operators or closed subspaces of the Hilbert space, directly with physical operations in the laboratory, namely the yes/no experiments.

George Mackey followed in on this idea when he wrote his book on the mathematical foundations of quantum mechanics [5]. He starts the other way around and considers as a basis the set $\mathcal{L}$ of all operational propositions, meaning propositions being testable by yes/no experiments on a physical entity. Then he introduces as an axiom that this set $\mathcal{L}$ has to have a structure isomorphic to the set of all closed subspaces $\mathcal{L}(\mathcal{H})$ of a complex Hilbert space in the case of a quantum entity. He states that it would be interesting to invent a set of axioms on $\mathcal{L}$ that gradually would make $\mathcal{L}$ more and more alike to $\mathcal{L}(\mathcal{H})$ to finally arrive at an isomorphism when all the axioms are satisfied. While Mackey wrote his book results as such were underway. A year later Constantin Piron proved a fundamental representation theorem. Starting from the set $\mathcal{L}$ of all operational propositions of a physical entity and introducing five axioms on $\mathcal{L}$ he proved that $\mathcal{L}$ is isomorphic to the set of closed subspaces $\mathcal{L}(V)$ of a generalized Hilbert space V whenever these five axioms are satisfied [6]. Let us elaborate on some of the aspects of this representation theorem to be able to explain further what quantum axiomatics is about.

We mentioned already that Birkhoff and von Neumann had noticed that the set of closed subspaces $\mathcal{L}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ is not a Boolean algebra, because distributivity between conjunction and disjunction, like expressed in (1), is not satisfied. The set of closed subspaces of a complex Hilbert space forms however a lattice, which is a more general mathematical structure than a Boolean algebra, moreover, a lattice where the distributivity rule (1) is satisfied is a Boolean algebra, which indicates that the lattice structure is the one to consider for the quantum mechanical situation. As we will see more in detail later, and to make again a reference to general relativity, the lattice structure is indeed to a Boolean algebra what general Riemann geometry is to Euclidean geometry. And moreover, meanwhile we have understood why the structure of operational propositions of the world is not a Boolean algebra but a lattice. This is strictly due to the fact that measurements can have an uncontrollable influence on the state of the physical entity under consideration. We explain this insight in detail in the following of this article, and mention it here to make clear that the intuition of Birkhoff and von Neumann, and later Mackey, Piron and others, although only mathematical intuition at that time, was correct.

When Piron proved his representation theorem in 1964, he concentrated on the lattice structure for the formulation of the five axioms. Meanwhile much more research has been done, both physically motivated in an attempt to make the approach more operational, as well as mathematically, trying to get axiomatically closer to the complex Hilbert space. In the presentation of quantum axiomatics we give in this article, we integrate the most recent results, and hence deviate for this reason from the original formulation, for example when we explain the representation theorem of Piron.

Axiomatic quantum mechanics is more than just an axiomatization of quantum mechanics. Because of the operational nature of the axiomatization, it holds the potential for 'more general theories than standard quantum mechanics' which however are 'quantum like theories'. In this sense, we believe that it is one of the candidates to generate the framework for the new theory to be developed generalizing quantum mechanics and relativity theory [7]. Let us explain why we believe that quantum axiomatics has the potential to deliver such a generalization of relativity theory and quantum mechanics. General relativity is a theory that brings part of the world that in earlier Newtonian mechanics was classified within dynamics to the geometrical realm of reality, and more specifically confronting us with the pre-scientific and naive realistic vision on space, time, matter and gravitation. It teaches us in a deep and new way, compared to Newtonian physics, 'what are the things that exists and how they exist and are related and how they influence each other'. But there is one deep lack in relativity theory: it does not take into account the influence of the observer, the effect that the measuring apparatus has on the thing observed. It does not confront the subject-object problem and its influence on how reality is. It cannot do this because its mathematical apparatus is based on the Riemann geometry of time-space, hence prejudicing that time-space is there, filled up with fields and matter, that are also there, independent of the observer. There is no fundamental role for the creation of 'new' within relativity theory, everything just 'is' and we are only there to 'detect' how this everything 'is'. That is also the reason why general relativity can easily be interpreted as delivering a model for the whole universe, whatever this would mean. We know that quantum mechanics takes into account in an essential way the effect of the observer through the measuring apparatus on the state of the physical entity under study. In a theory generalizing quantum mechanics and relativity, such that both appear as special cases, this effect should certainly also appear in a fundamental way. We believe that general relativity has explored to great depth the question 'how can things **be** in the world'.

Quantum axiomatics explores in great depth the question 'how can be **acted** in the world'. And it does explore this question of 'action in the world' in a very similar manner as general relativity theory does with its question of 'being of the world'. This means that operational quantum axiomatics can be seen as the development of a general theory of 'actions in the world' in the same manner that Riemann geometry can be seen as a general theory of 'geometrical forms existing in the world'. Of course Riemann is not equivalent to general relativity, a lot of detailed physics had to be known to apply Riemann resulting in general relativity. This is the same with operational quantum axiomatics, it has the potential to deliver the framework for the theory generalizing quantum mechanics and relativity theory.

We want to remark that in principle a theory that describes the possible actions in the world, and a theory that delivers a model for the whole universe, should not be incompatible. It should even be so that the theory that delivers a model of the whole universe should incorporate the theory of actions in the world, which would mean for the situation that exists now, general relativity should contain quantum mechanics, if it really delivers a model for the whole universe. That is why we believe that Einstein's attitude, trying to incorporate the other forces and interactions within general relativity, contrary to common believe, was the right one, globally speaking. What Einstein did not know at that time was 'the reality of non-locality in the micro-world'. Non-locality means non-spatiality, which means that the reality of the micro-world, and hence the reality of the universe as a whole, is not time-space like. Time-space is not the global theatre of reality, but rather a cristallization and structuration of the macro-world. Time-space has come into existence together with the macroscopic material entities, and hence it is 'their' time and space, but it is not the theatre of the microscopic quantum entities. This fact is the fundamental reason why general relativity, built on the mathematical geometrical Riemannian structure of time-space, cannot be the canvas for the new theory to be developed. A way to express this technically would be to say that the set of events cannot be identified with the set of time-space points as is done in relativity theory. Recourse will have to be taken to a theory that describes reality as a kind of pre-geometry, and where the geometrical structure arises as a consequence of interactions that collapse into the time-space context. We believe that operational quantum axiomatics, as presented in this article, can deliver the framework as well as the methodology to construct and elaborate such a theory. In the next section we introduce the basic notions of operational quantum axiomatics.

Mackey and Piron introduced the set of yes/no experiments but then immediately shifted to an attempt to axiomatize mathematically the lattice of (operational) propositions of a quantum entity, Mackey postulating right away an isomorphism with $\mathcal{L}(\mathcal{H})$ and Piron giving five axioms to come as close as possible to $\mathcal{L}(\mathcal{H})$. Also Piron's axioms are however mostly motivated by mimicking mathematically the structure of $\mathcal{L}(\mathcal{H})$. In later work Piron made a stronger attempt to found operationally part of the axioms [8], and this attempt was worked out further in [9, 10, 11], to arrive at a full operational foundation only recently [12, 13, 14, 15].

Also mathematically the circle was closed only recently. At the time when Piron gave his five axioms that lead to the representation within a generalized Hilbert space, there only existed three examples of generalized Hilbert spaces that fitted all the axioms, namely real, complex and quaternionic Hilbert space, also referred to as the three standard Hilbert spaces[1]. Years later Hans Keller constructed the first counterexample, more specifically an example of an infinite dimensional generalized Hilbert space that is not isomorphic to one of the three standard Hilbert spaces [16]. The study of generalized Hilbert spaces, nowadays also called orthomodular spaces, developed into a research subject of its own, and recently Maria Pia Solèr proved a groundbreaking theorem in this field. She proved that an infinite dimensional generalized Hilbert space that contains an orthonormal base is isomorphic with one of the three standard Hilbert spaces [17]. It has meanwhile also been possible to formulate an operational axiom, called 'plane transitivity' on the set of operational propositions that implies Solèr's condition [18], which completes the axiomatics for standard quantum mechanics by means of six axioms, the original five axioms of Piron and plane transitivity as sixth axiom.

2 State property spaces

In this section we introduce the basic notions and basic axioms for quantum axiomatics. We introduce notions and axioms that are as simple as possible, but each time show how the more traditional axioms of quantum axiomatics are related and/or derived from our set of axioms.

[1]There do exist a lot of finite dimensional generalized Hilbert spaces that are different from the three standard examples. But since a physical entity has to have at least a position observable, it follows that the generalized Hilbert space must be infinite dimensional. At the time of Piron's representation theorem, the only infinite dimensional cases that were known are the three standard Hilbert spaces, over the real, complex or quaternionic numbers.

2.1 States and properties

With each entity S corresponds a well defined set of states Σ of the entity. These are the modes of being of the entity. This means that at each moment the entity S 'is' in a specific state $p \in \Sigma$. Historically quantum axiomatics has been elaborated mainly by considering the set of properties[2]. With each entity S corresponds a well defined set of properties $\mathcal{L}$. A property $a \in \mathcal{L}$ is 'actual' or is 'potential' for the entity S. To be able to present the axiomatization of the set of states and the set of properties of an entity S in a mathematical way, we introduce some additional notions.

Suppose that the entity S is in a specific state $p \in \Sigma$. Then some of the properties of S are actual and some are not, hence they are potential. This means that with each state $p \in \Sigma$ corresponds a set of actual properties, subset of $\mathcal{L}$. This defines a function $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$, which makes each state $p \in \Sigma$ correspond to the set $\xi(p)$ of properties that are actual in this state. With the notation $\mathcal{P}(\mathcal{L})$ we mean the 'powerset' of $\mathcal{L}$, i.e. the set of all subsets of $\mathcal{L}$. From now on we can replace the statement 'property $a \in \mathcal{L}$ is actual for the entity S in state $p \in \Sigma$' by '$a \in \xi(p)$'.

Suppose that for the entity S a specific property $a \in \mathcal{L}$ is actual. Then this entity is in a certain state $p \in \Sigma$ that makes a actual. With each property $a \in \mathcal{L}$ we can associate the set of states that make this property actual, i.e. a subset of Σ. This defines a function $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$, which makes each property $a \in \mathcal{L}$ correspond to the set of states $\kappa(a)$ that make this property actual. We can replace the statement 'property $a \in \mathcal{L}$ is actual if the entity S is in state $p \in \Sigma$' by the expression '$p \in \kappa(a)$'.

Summarizing the foregoing we have:

$$\begin{aligned}
&\text{property } a \in \mathcal{L} \text{ is actual for the entity } S \text{ in state } p \in \Sigma \\
&\Leftrightarrow a \in \xi(p) \\
&\Leftrightarrow p \in \kappa(a)
\end{aligned} \tag{2}$$

This expresses a fundamental 'duality' between states and properties. We introduce a specific mathematical structure to represent an entity S, its states and its properties, taking into account this duality. First we remark that if Σ and $\mathcal{L}$ are given, and one of the two functions ξ or κ is given, then the other function can be derived. Let us show this explicitly. Hence suppose that Σ, $\mathcal{L}$ and ξ are given, and define $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$ such that $\kappa(a) = \{p \mid p \in \Sigma, a \in \xi(p)\}$. Similarly, if Σ, $\mathcal{L}$ and $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$ are

[2] In the original paper of Birkhoff and Von Neumann [1], the basic notion is the one of 'operational proposition'. An operational proposition is not the same as a property [19, 20], but it points at the same structural part of quantum axiomatics.

given, we can derive ξ in an analogous way. This means that to define the mathematical structure which carries our notions and relations it is enough to introduce Σ, $\mathcal{L}$ and one of the two functions ξ or κ.

Definition 1 (State property space)**.** *Consider two sets Σ and $\mathcal{L}$ and a function*

$$\xi : \Sigma \leftarrow \mathcal{P}(\mathcal{L}) \quad p \mapsto \xi(p) \tag{3}$$

then we say that $(\Sigma, \mathcal{L}, \xi)$ is a state property space. The elements of Σ are interpreted as states and the elements of $\mathcal{L}$ as properties of the entity S. For $p \in \Sigma$ we have that $\xi(p)$ is the set of properties of S which are actual if S is in state p. For a state property space $(\Sigma, \mathcal{L}, \xi)$ we define:

$$\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma) \quad a \mapsto \kappa(a) = \{p \mid p \in \Sigma, a \in \xi(p)\} \tag{4}$$

and hence for $a \in \mathcal{L}$ we have that $\kappa(a)$ is the set of states of the entity S which make the property a actual. The function κ is called the Cartan map of the state property space $(\Sigma, \mathcal{L}, \xi)$.

Proposition 1. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$, and κ defined as in (4). We have:*

$$a \in \xi(p) \Leftrightarrow p \in \kappa(a) \tag{5}$$

There are two natural 'implication relations' on a state property space. If the situation is such that if '$a \in \mathcal{L}$ is actual for S in state $p \in \Sigma$' implies that '$b \in \mathcal{L}$ is actual for S in state $p \in \Sigma$' we say that the property a implies the property b. If the situation is such that '$a \in \mathcal{L}$ is actual for S in state $q \in \Sigma$' implies that '$a \in \mathcal{L}$ is actual for S in state $p \in \Sigma$' we say that the state p implies the state q.

Definition 2 (Property implication and state implication)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$. For $a, b \in \mathcal{L}$ we introduce:*

$$a \leq b \Leftrightarrow \kappa(a) \subseteq \kappa(b) \tag{6}$$

and we say that a 'implies' b. For $p, q \in \Sigma$ we introduce:

$$p \leq q \Leftrightarrow \xi(q) \subseteq \xi(p) \tag{7}$$

and we say that p 'implies' q[3]*.*

[3] The state implication and property implication are not defined in an analogous way. Indeed, then we should for example have written $p \leq q \Leftrightarrow \xi(p) \subseteq \xi(q)$. That we have chosen to define the state implication the other way around is because historically this is how intuitively is thought about states implying one another.

Definition 3 (Equivalent properties and equivalent states)**.** *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$*. We call properties* $a, b \in \mathcal{L}$ *equivalent, and denote* $a \approx b$ *iff* $\kappa(a) = \kappa(b)$*. We call states* $p, q \in \Sigma$ *equivalent and denote* $p \approx q$ *iff* $\xi(p) = \xi(q)$*.*

Let us give two important examples of state property spaces. First, consider a set Ω and let $\mathcal{P}(\Omega)$ be the set of all subsets of Ω, and consider the function $\xi_\Omega : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$, such that for $p \in \Omega$

$$\xi_\Omega(p) = \{A \mid A \in \mathcal{P}(\Omega), p \in A\} \tag{8}$$

The triple $(\Omega, \mathcal{P}(\Omega), \xi_\Omega)$ is a state property space. For $A \in \mathcal{P}(\Omega)$ we have $\kappa_\Omega(A) = \{p \mid p \in \Omega, A \in \xi(p)\} = \{p \mid p \in \Omega, p \in A\} = A$. This shows that $\kappa_\Omega : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is the identity.

Second, consider a complex Hilbert space $\mathcal{H}$, and let $\Sigma(\mathcal{H})$ be the set of unit vectors of $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$ the set of orthogonal projection operators of $\mathcal{H}$. Consider the function $\xi_\mathcal{H} : \Sigma(\mathcal{H}) \to \mathcal{P}(\mathcal{L}(\mathcal{H}))$, such that for $x \in \Sigma(\mathcal{H})$

$$\xi_\mathcal{H}(x) = \{A \mid A \in \mathcal{L}(\mathcal{H}), Ax = x\} \tag{9}$$

The triple $(\Sigma(\mathcal{H}), \mathcal{L}(\mathcal{H}), \xi_\mathcal{H})$ is a state property space. For $A \in \mathcal{L}(\mathcal{H})$ we have $\kappa_\mathcal{H}(A) = \{x \mid x \in \Sigma(\mathcal{H}), Ax = x\}$.

The two examples that we propose here are the archetypical physics examples. The first example is the state property space of a classical physical system, where Ω corresponds with its state space. The second example is the state property space of a quantum physical system, where $\mathcal{H}$ is the complex Hilbert space connected to the quantum system.

Definition 4 (Pre-order relation)**.** *Suppose that we have a set* Z*. We say that* $\leq$ *is a pre-order relation on* Z *iff for* $x, y, z \in Z$ *we have:*

$$\begin{aligned} & x \leq x \\ & x \leq y \text{ and } y \leq z \Rightarrow x \leq z \end{aligned} \tag{10}$$

For two elements $x, y \in Z$ *such that* $x \leq y$ *and* $y \leq x$ *we denote* $x \approx y$ *and we say that* x *is equivalent to* y*.*

It is easy to verify that the implication relations that we have introduced are pre-order relations.

Proposition 2. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$*, then* $\Sigma, \leq$ *and* $\mathcal{L}, \leq$ *are pre-ordered sets.*

We can show the following for a state property space:

Proposition 3. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$. *(1) Suppose that* $a, b \in \mathcal{L}$ *and* $p \in \Sigma$. *If* $a \in \xi(p)$ *and* $a \leq b$, *then* $b \in \xi(p)$. *(2) Suppose that* $p, q \in \Sigma$ *and* $a \in \mathcal{L}$. *If* $q \in \kappa(a)$ *and* $p \leq q$ *then* $p \in \kappa(a)$.

Proof: (1) We have $p \in \kappa(a)$ and $\kappa(a) \subseteq \kappa(b)$. This proves that $p \in \kappa(b)$ and hence $b \in \xi(p)$. (2) We have $a \in \xi(q)$ and $\xi(q) \subseteq \xi(p)$ and hence $a \in \xi(p)$. This shows that $p \in \kappa(a)$. $\square$

Suppose we consider a set of properties $(a_i)_i \subseteq \mathcal{L}$. It is very well possible that there exist states of the entity S in which all the properties a_i are actual. This is in fact always the case if $\cap_i \kappa(a_i) \neq \emptyset$. Indeed, if we consider $p \in \cap_i \kappa(a_i)$ and S in state p, then all the properties a_i are actual. If it is such that the situation where all properties a_i of a set $(a_i)_i$ and no other are actual is again a property of the entity S, we will denote this new property by $\wedge_i a_i$, and call it a 'meet property' of $(a_i)_i$. Clearly we have $\wedge_i a_i$ is actual for S in state $p \in \Sigma$ iff a_i is actual for all i for S in state p. This means that we have $\wedge_i a_i \in \xi(p)$ iff $a_i \in \xi(p)\ \forall i$.

Definition 5 (Meet property)**.** *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *and a set* $(a_i)_i \subseteq \mathcal{L}$ *of properties. If there exists a property, which we denote by* $\wedge_i a_i$, *such that*

$$\kappa(\wedge_i a_i) = \cap_i \kappa(a_i) \tag{11}$$

we call $\wedge_i a_i$ *the 'meet property' of the set of properties* $(a_i)_i$.

If we have the structure of a pre-ordered set, we can wonder about the existence of meets and joins with respect to this pre-order, or conjunctions and disjunctions with respect to the implication related to this pre-order. In relation with the meet property we can prove the following

Proposition 4. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *and a set* $(a_i)_i \subseteq \mathcal{L}$ *of properties. The property* $\wedge_i a_i$, *if it exists, is an infimum*[4] *for the pre-order relation* $\leq$ *on* $\mathcal{L}$.

Proof: We have $\kappa(\wedge_i a_i) = \cap_i \kappa(a_i) \subseteq \kappa(a_j)\ \forall\ j$, and hence $\wedge_i a_i \leq a_j\ \forall\ j$. Suppose that $x \in \mathcal{L}$ is such that $x \leq a_j \forall j$, then we have $\kappa(x) \subseteq \kappa(a_j) \forall\ j$, and hence $\kappa(x) \subseteq \cap_i \kappa(a_i) = \kappa(\wedge_i a_i)$. As a consequence we have $x \leq \wedge_i a_i$. This proves that $\wedge_i a_i$ is an infimum. $\square$

[4] An infimum of a subset $(x_i)_i$ of a pre-ordered set Z is an element of Z that is smaller than all the x_i and greater than any element that is smaller than all x_i.

2.2 Tests

For the operational foundations of the state property space, we need to make explicit how we test whether for a physical entity a specific property is actual.

A test is an experiment we can perform on the physical entity under investigation with the aim of knowing whether a specific property of this physical entity is actual or not. We identify for each test two outcomes, one which we call 'yes' corresponding to the occurrence of the expected outcome, and another one which we call 'no' corresponding to the non occurrence of the expected outcome. However, if for a test the outcome 'yes' occurs, this does not mean that the property which is tested is actual. It is only when we can predict with certainty, i.e. with probability equal to 1, that the test would have an outcome 'yes', if we would perform it, that the property a is actual.

Let us consider the example of an entity which is a piece of wood. We have in mind the property of 'burning well'. A possible test for this property consists of taking the piece of wood and setting it on fire. In general, when we perform the test on a piece of dry wood, the piece of wood will be destroyed by the test. So the property of 'burning well' is a property that the piece of wood eventually has before we make the test. Of course it is after having done a number of tests with pieces of wood and having got always the outcome yes, we decide that the one new piece of wood, prepared under equivalent conditions, whereon we never performed the test, has actually the property of burning well. We will say that the test is 'true' if this is the case.

Definition 6 (Testing a property)**.** *Consider a physical entity with corresponding state property space* $(\Sigma, \mathcal{L}, \xi)$*.* α *is a test of the property* $a \in \mathcal{L}$ *if we have*

$$a \in \xi(p) \quad \Leftrightarrow \quad \text{`yes}' \text{ can be predicted with certainty for } \alpha \text{ when } S \text{ is in state } p \tag{12}$$

Similarly with the pre-order relations on the sets of properties we have pre-order relations on the sets of tests.

Definition 7 (Test implication)**.** *We say that a test* α *is stronger than a test* β *and denote* $\alpha \leq \beta$ *iff whenever the physical entity is in a state such that* α *is true then also* β *is true.*

Proposition 5. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$*. If the test* α *tests property* a*, and the test* β *tests property* b*, we have*

$$\alpha \leq \beta \Leftrightarrow a \leq b \tag{13}$$

Proof: Suppose that $\alpha \leq \beta$, and consider $p \in \Sigma$ such that $a \in \xi(p)$. This means that the test α gives with certainty outcome 'yes' if the entity is in state p. Hence also β gives with certainty 'yes' if the entity is in state p. This means that $b \in \xi(p)$. Hence we have proven that $a \leq b$. Suppose now that $a \leq b$, and suppose that the entity is in a state p such that α gives with certainty outcome 'yes'. This means that $a \in \xi(p)$. Then we have $b \in \xi(p)$. Since β tests b we have that β gives with certainty the outcome 'yes'. Hence we have proven that $\alpha \leq \beta$. □

Definition 8 (Equivalent tests)**.** *We say that two tests α and β are equivalent, and denote $\alpha \approx \beta$ iff $\alpha \leq \beta$ and $\beta \leq \alpha$.*

Proposition 6. *Equivalent tests test equivalent properties, and tests that test equivalent properties are equivalent tests.*

Proof: Consider two equivalent tests $\alpha \approx \beta$ testing respectively properties a and b. Since we have $\alpha \leq \beta$ and $\beta \leq \alpha$ this implies that $a \leq b$ and $b \leq a$, and hence $a \approx b$. Consider two equivalent properties $a \approx b$ being tested respectively by tests α and β. Since we have $a \leq b$ and $b \leq a$ this implies that $\alpha \leq \beta$ and $\beta \leq \alpha$, and hence $\alpha \approx \beta$. □

In general the outcomes of a test of one property are profoundly influenced by the testing of another property. In most cases it makes even no sense to perform two tests at once or one after the other on the entity. But still it is so that every entity can have several properties which are actual at once. There is indeed a way to construct a test that makes it possible to test the actuality of several properties at once, even if the tests corresponding to the different properties disturb each other profoundly. Let us illustrate this by means of the example of the piece of wood.

Consider the following two properties of the piece of wood: Property c 'the piece of wood burns well' and property d 'the piece of wood floats on water'. Suppose that γ is a test of property c which consists of setting the wood on fire and giving the outcome 'yes' if it burns well. The test δ consists of putting the wood on water and giving the outcome 'yes' if it floats, hence it is a test of property d. If we perform first the test δ, and put the piece of wood on water, we have changed the state of the wood in a state of 'wet wood' and as a result the wood will not burn well. On the other hand if we perform the test γ and burn the wood, it will no longer float on water. However we all know plenty of pieces of wood for which both properties c and d are actual at once. This means that the way in which we decide both properties to be actual for a specific piece of wood is not related to performing both tests one after the other. If we analyze carefully this situation we see that we agree for a piece of wood both properties c and d to be actual if which ever of the tests γ or δ is performed, the outcome 'yes' can be predicted with certainty for this test. Hence, to state this in a slightly more formal way: 'If we choose, or if some process external to us produces a choice, between one of the two tests γ or δ, and it is certain to obtain the outcome 'yes' no matter what is this choice, then we agree that both properties c and d are actual for the piece of wood'.

This leads us to the following. Given two tests γ and δ we define a new test which we denote $\gamma \cdot \delta$ and call the product test of γ and δ. The performance of $\gamma \cdot \delta$ consists of a choice being made between γ and δ, and then the performance of this chosen test, and the attribution of the outcome obtained in this way. As a consequence, we have $\gamma \cdot \delta$ is true iff γ is true 'and' δ is true, which shows that $\gamma \cdot \delta$ tests both properties c and d, or, it tests the conjunction of properties c and d. Remark that for the performance of the test $\gamma \cdot \delta$ only one test γ or δ has to be performed, and hence the definition of the product test is valid independent of the way in which tests disturb each other. The definition of product test is valid for any number of tests, which means that we have found a way to test any number of properties at once, or, to test the conjunction of any number of properties. Let us formally introduce the product test for an arbitrary number of properties.

Consider a family $(a_i)_i$ of properties a_i and tests α_i, such that α_i tests property a_i. A test which tests the actuality of all the properties a_i, and which we denote $\Pi_i \alpha_i$ and call the product of the α_i is the following:

Definition 9 (Product test)**.** *The performance of $\Pi_i \alpha_i$ consists of a choice between one of the tests α_i followed by the performance of this chosen test.*

Proposition 7. *For a set of tests $(\alpha_i)_i$ we have*

$$\Pi_i \alpha_i \leq \alpha_j \ \forall \ j \tag{14}$$

Let us prove that the product test tests the meet of a set of properties.

Proposition 8. *Consider an entity with corresponding state property space $(\Sigma, \mathcal{L}, \xi)$ and a set of properties $(a_i)_i \subseteq \mathcal{L}$. Suppose that we have tests $(\alpha_i)_i$ available for the properties $(a_i)_i$, then the product test $\Pi_i \alpha_i$ tests a meet property $\wedge_i a_i$.*

Proof: Following the definition of 'meet property' given in Definition 5, to prove that $\Pi_i \alpha_i$ tests the meet property $\wedge_i a_i$ of the set of properties $(a_i)_i$, where α_i tests a_i, we need to show that 'yes can be predicted with certainty for $\Pi_i\alpha$ the entity being in state p' is equivalent to '$a_i \in \xi(p)\ \forall\ i$'. This follows from the definition of the product test. Indeed 'yes can be predicted with certainty for $\Pi_i\alpha_i$ the entity being in state p' is equivalent to 'yes can be predicted with certainty for $\alpha_i\ \forall\ i$ the entity being in state p'. $\square$

2.3 Orthogonality

Let us investigate the operational foundation of orthogonality.

Definition 10 (Inverse test)**.** *For a test α we consider the test that consists of performing the same experiment and changing the role of 'yes' and 'no'. We denote this new test by $\widetilde{\alpha}$, and call it the inverse test of α.*

Proposition 9. *Consider a test α and a set of tests $(\alpha_i)_i$, then we have*

$$\widetilde{\widetilde{\alpha}} = \alpha \tag{15}$$

$$\widetilde{\Pi_i\alpha_i} = \Pi_i\widetilde{\alpha_i} \tag{16}$$

Proof: Obviously if we exchange 'yes' and 'no' for an experiment corresponding to the test α, and then exchange 'yes' and 'no' again, we get the same test. The test $\widetilde{\Pi_i\alpha_i}$ consists of exchanging 'yes' and 'no' for the experiment corresponding to the test $\Pi_i\alpha_i$. This comes to exchanging 'yes' and 'no' after the choice of one of the α_j is made. The test $\Pi_i\widetilde{\alpha_i}$ on the contrary consists of exchanging the 'yes' and 'no' of each of the tests α_i, hence before the choice of one of the α_j is made. These are the same tests. $\square$

There is a fundamental problem with the inverse test which is the following. Suppose that $\alpha(a)$ tests the property a, and $\beta(a)$ also tests the property a, then $\widetilde{\alpha}$ and $\widetilde{\beta}$ in general test completely different properties. Let us show this by means of a concrete example. We introduce the test τ, which is the unit test, in the sense that it is a test which gives 'yes' as outcome with certainty for each state of the entity. The test τ tests the maximal property I. Obviously $\widetilde{\tau}$ is a test which gives with certainty outcome 'no' for each state of the entity, which means that it tests a property which is never actual. It can be shown that this property which is never actual can only be represented by the minimal property 0, hence $\widetilde{\tau}$ tests 0. Consider now an arbitrary property $a \in \mathcal{L}$, and a test α which tests property a. Let us suppose that $\widetilde{\alpha}$ tests a property b, and that both properties a and b can be actual. Consider the product test $\alpha \cdot \tau$. This is a test which also tests the property a, because indeed $\alpha \cdot \tau$ gives with certainty the outcome 'yes' iff α gives with certainty the outcome 'yes'. However $\widetilde{\alpha \cdot \tau} = \widetilde{\alpha} \cdot \widetilde{\tau}$ tests the property 0, and not the property b. Indeed $\widetilde{\alpha} \cdot \widetilde{\tau}$ gives with certainty the outcome 'yes' iff $\widetilde{\alpha}$ gives with certainty 'yes' and $\widetilde{\tau}$ gives with certainty 'yes'. This is never the case, which proves that it tests the property 0.

Definition 11 (Orthogonal states)**.** *If p and q are two states of S we will say that p is orthogonal to q, iff there exists a test γ such that γ is true if S is in the state p and $\widetilde{\gamma}$ is true if S is in the state q. We will denote then $p \perp q$*

Proposition 10. *For $p, q, r, s \in \Sigma$ we have*

$$p \perp q \quad \Rightarrow \quad q \perp p \tag{17}$$

$$p \perp q,\ r \leq p,\ s \leq q \quad \Rightarrow \quad r \perp s \tag{18}$$

$$p \perp q \quad \Rightarrow \quad p \wedge q = 0 \tag{19}$$

Proof: The orthogonality relation is obviously symmetric. If $r \leq p$ and $s \leq q$, and $p \perp q$, and γ is a test such that γ is true if the entity is in state p and $\widetilde{\gamma}$ is true if the entity is in state q, then we have that γ is true if the entity is in state r and $\widetilde{\gamma}$ is true if the entity is in state s. This proves that $r \perp s$.

Definition 12 (Orthogonal properties and states)**.** *We say that a state $p \in \Sigma$ is orthogonal to a property $a \in \mathcal{L}$ iff for every $q \in \Sigma$ such that $a \in \xi(q)$ we have $p \perp q$. We denote $p \perp a$. We say that two properties $a, b \in \mathcal{L}$ are orthogonal iff for every $p, q \in \Sigma$ such that $a \in \xi(p)$ and $b \in \xi(q)$ we have $p \perp q$. We denote $a \perp b$.*

Proposition 11. *For $a, b, c, d \in \mathcal{L}$ and $p, r \in \Sigma$ we have*

$$p \perp a,\ r \leq p,\ c \leq a \quad \Rightarrow \quad r \perp c \tag{20}$$

$$a \perp b \quad \Rightarrow \quad b \perp a \tag{21}$$

$$a \perp b,\ c \leq a,\ d \leq b \quad \Rightarrow \quad c \perp d \tag{22}$$

$$a \perp b \quad \Rightarrow \quad a \wedge b = 0 \tag{23}$$

3 A set of axioms

In this section we put forward a set of axioms and derive the consequences for the structure of the state property space of this set of axioms. We make an attempt to introduce all the axioms in a way which is as operational as possible.

3.1 The axiom of property determination

The first axiom expresses a relation between the states and the properties. We consider two properties $a, b \in \mathcal{L}$ of the entity S, and suppose that $\kappa(a) = \kappa(b)$. This means that each state which make property a actual also makes property b actual, and vice versa. It also means that we cannot distinguish between property a and property b by means of the states of the entity S. Hence, this means that for entity S, property a and property b are equivalent.

Axiom 1 (Property determination)**.** *We say that the axiom of property determination is satisfied for a state property space* $(\Sigma, \mathcal{L}, \xi)$ *iff for* $a, b \in \mathcal{L}$ *we have:*

$$\kappa(a) = \kappa(b) \Rightarrow a = b \tag{24}$$

Definition 13 (Partial order relation)**.** *Suppose that we have a set* Z*. We say that* $\leq$ *is a partial order relation on* Z *iff* $\leq$ *is a pre-order relation for which equivalent elements are equal.*

If axiom 1 is satisfied for a state property space $(\Sigma, \mathcal{L}, \xi)$, the pre-order relation on the set of properties $\mathcal{L}$ is then a partial order relation.

Theorem 1. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for an entity* S *for which axiom 1 is satisfied. The 'property implication' on* $\mathcal{L}$ *is then a partial order relation on* $\mathcal{L}$*.*

Proof: Suppose that axiom 1 is satisfied for $(\Sigma, \mathcal{L}, \xi)$, and consider $a, b \in \mathcal{L}$ such that $a \leq b$ and $b \leq a$. Then we have $\kappa(a) \subseteq \kappa(b)$ and $\kappa(b) \subseteq \kappa(a)$, and hence $\kappa(a) = \kappa(b)$. As a consequence, because of axiom 1, we have $a = b$. This proves that $\leq$ is a partial order relation on $\mathcal{L}$. □

The two archetypical examples we have introduced both satisfy the axiom of property determination. Indeed, consider the first example of classical mechanics. Since κ is the identity, we have for $A, B \in \mathcal{P}(\Omega)$ that $\kappa(A) = \kappa(B)$ implies $A = B$. For the second example of quantum mechanics, consider $A, B \in \mathcal{L}(\mathcal{H})$, and suppose that $\kappa_{\mathcal{H}}(A) = \kappa_{\mathcal{H}}(B)$. Consider the vector $x \in \mathcal{H}$ such that $Ax = x$. Since $\kappa_{\mathcal{H}}(A) = \kappa_{\mathcal{H}}(B)$ this implies that $Bx = x$. This proves that $AB = A$. In an analogous way we prove that $AB = B$, and hence $A = B$.

3.2 The axiom of completeness

We want to be able to distinguish between properties that are not necessarily of the type that they are meet properties, and between properties which are meet properties. In [9, 10] we have introduced in this way a subset of properties $\mathcal{T} \subseteq \mathcal{L}$, and called it a 'generating set of properties' for the state property space $(\Sigma, \mathcal{L}, \xi)$.

Axiom 2 (Property completeness)**.** *We say that the axiom of property completeness is satisfied for a state property space $(\Sigma, \mathcal{L}, \xi)$ iff there exists a subset $\mathcal{T} \subseteq \mathcal{L}$ such that for each $(a_i)_i \subseteq \mathcal{T}$ there exists $a \in \mathcal{L}$ such that*

$$\kappa(a) = \bigcap_i \kappa(a_i) \tag{25}$$

and, each property $a \in \mathcal{L}$ is of this form, i.e. for $a \in \mathcal{L}$ there exists a subset $(a_i)_i \subseteq T$ such that (25) is satisfied. We call $\mathcal{T} \subseteq \mathcal{L}$ a generating set of properties of the state property space $(\Sigma, \mathcal{L}, \xi)$, and call the property a of (25) a meet of the set of properties $(a_i)_i$, and denote it by

$$a = \bigwedge_i a_i \tag{26}$$

The following definition and proposition explain why we have chosen to call axiom 2 the axiom of completeness.

Definition 14 (Complete pre-ordered set)**.** *Suppose that $Z, \leq$ is a pre-ordered set. We say that Z is a complete pre-ordered set iff for each subset of elements of Z there exists an infimum and a supremum in Z.*

Proposition 12. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axiom 2 is satisfied. Then $\mathcal{L}, \leq$ is a complete pre-ordered set, and if for a subset $(a_i)_i \subseteq \mathcal{L}$ we denote an infimum of $(a_i)_i$ by $\wedge_i a_i$ we have*

$$\kappa(\bigwedge_i a_i) = \bigcap_i \kappa(a_i) \tag{27}$$

Proof: Consider an arbitrary set $(a_i)_i \subseteq \mathcal{L}$ of properties. We need to prove that there exists an infimum and a supremum in $\mathcal{L}$ for this set of properties $(a_i)_i$. From axiom 2 we know that for each a_i there is a set $(b^i_{j_i})_{j_i} \subseteq \mathcal{T}$, such that $a_i = \wedge_{j_i} b^i_{j_i}$, and $\kappa(a_i) = \cap_{j_i} \kappa(b^i_{j_i})$. From the same axiom 2 follows that for the subset $(b^i_{j_i})^i_{j_i} \subseteq \mathcal{T}$ there exists a property $a \in \mathcal{L}$ such that $\kappa(a) = \cap_i \cap_{j_i} \kappa(b^i_{j_i}) = \cap_i \kappa(a_i)$. Let us prove that a is an infimum for the set $(a_i)_i \subseteq \mathcal{L}$. Since $\kappa(a) = \cap_i \kappa(a_i)$ we have $\kappa(a) \subseteq \kappa(a_j)\ \forall j$, and hence $a \leq a_j\ \forall\ j$, which proves that a is a lower bound for $(a_i)_i$. Consider $x \in \mathcal{L}$ such that $x \leq a_j\ \forall j$. This implies that $\kappa(x) \subseteq \kappa(a_j)\ \forall j$, and hence $\kappa(x) \subseteq \cap_i \kappa(a_i) = \kappa(a)$. From this follows that $x \leq a$, which proves that a is a greatest lower bound or infimum. It is a consequence that for each subset $(a_i)_i \subseteq \mathcal{L}$, there exists also a supremum in $\mathcal{L}$, let us denote it by $\vee_i a_i$. It is given by

$$\bigvee_i a_i = \bigwedge_{x \in \mathcal{L}, a_i \leq x \forall i} x \tag{28}$$

This proves that $\mathcal{L}, \leq$ is a complete pre-ordered set. □

Remark that the supremum for elements of $\mathcal{L}$, although it exists, has no simple operational meaning.

Definition 15 (Complete lattice)**.** *Suppose that $Z, \leq$ is a partially ordered set. We say that $Z, \leq, \wedge, \vee$ is a complete lattice iff for each subset $(x_i)_i \subseteq Z$ of elements of Z there exists an infimum $\wedge_i x_i \in Z$ and a supremum $\vee_i x_i \in Z$ in Z. A complete lattice has a minimal element which we denote 0, and which is the infimum of all elements of Z, and a maximal element, which we denote I, and which is the supremum of all elements of Z.*

Theorem 2. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1 and 2 are satisfied. Then $\mathcal{L}, \leq, \wedge, \vee$ is a complete lattice. For I the maximum of $\mathcal{L}$, $(a_i)_i \subseteq \mathcal{L}$ and $p \in \Sigma$ we have:*

$$\kappa(I) = \Sigma \tag{29}$$

$$\bigcap_i \kappa(a_i) = \kappa(\bigwedge_i a_i) \tag{30}$$

$$\bigcup_i \kappa(a_i) \subseteq \kappa(\bigvee_i a_i) \tag{31}$$

$$a_i \in \xi(p) \ \forall i \Leftrightarrow \bigwedge_i a_i \in \xi(p) \tag{32}$$

Proof: From proposition 1 follows that $\mathcal{L}, \leq$ is a partially ordered set, and from proposition 12 follows that $\mathcal{L}, \leq, \wedge, \vee$ is a complete lattice. We have $\kappa(I) \subseteq \Sigma$. For an arbitrary $p \in \Sigma$ consider $\xi(p)$. Since $a \leq I \ \forall a \in \xi(p)$, we have $I \in \xi(p)$, and hence $p \in \kappa(I)$. This proves that $\Sigma \subseteq \kappa(I)$. As a consequence we have $\kappa(I) = \Sigma$. From (27) of proposition 12 follows (30). Let us prove (31). Since $\vee_i a_i$ is a supremum of $(a_i)_i$ we have $a_j \leq \vee_i a_i \ \forall \ j$. Hence $\kappa(a_j) \subseteq \kappa(\vee_i a_i) \ \forall \ j$. This proves that $\cup_i \kappa(a_i) \subseteq \kappa(\vee_i a_i)$. Suppose that $a_i \in \xi(p) \ \forall i$, then $p \in \kappa(a_i) \ \forall i$, and hence $p \in \cap_i \kappa(a_i) = \kappa(\wedge_i a_i)$. From this follows that $\wedge_i a_i \in \xi(p)$, and hence we have proven one of the implications of (32). Let us prove the other one, and hence suppose that $\wedge_i a_i \in \xi(p)$. From this follows that $p \in \kappa(\wedge_i a_i) = \cap_i \kappa(a_i)$. As a consequence we have $p \in \kappa(a_i) \ \forall i$, and hence $a_i \in \xi(p) \ \forall i$. □

If axiom 1 and 2 are satisfied for a state property space $(\Sigma, \mathcal{L}, \xi)$, and hence the set of properties $\mathcal{L}$ is a complete lattice, we can represent the states by means of properties.

Definition 16 (Property state). *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1 and 2 are satisfied. For each state* $p \in \Sigma$ *we define the 'property state' corresponding to* p *as the property*

$$s(p) = \bigwedge_{a \in \xi(p)} a \tag{33}$$

Proposition 13. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1 and 2 are satisfied. For* $p, q \in \Sigma$ *and* $a \in \mathcal{L}$ *we have:*

$$s(p) \in \xi(p) \tag{34}$$

$$a \in \xi(p) \Leftrightarrow s(p) \leq a \tag{35}$$

$$p \leq q \Leftrightarrow s(p) \leq s(q) \tag{36}$$

$$\xi(p) = \{a \mid a \in \mathcal{L}, s(p) \leq a\} = [s(p), I] \tag{37}$$

Proof: That $s(p) \in \xi(p)$ follows directly from (32). Suppose $a \in \xi(p)$ then $\wedge_{a \in \xi(p)} a \leq a$ and hence $s(p) \leq a$. We have that $s(p) \in \xi(p)$, and if $s(p) \leq a$, from proposition 3 follows then that $a \in \xi(p)$. Suppose that $p \leq q$. Then we have $\xi(q) \subseteq \xi(p)$. From this follows that $s(p) = \wedge_{a \in \xi(p)} a \leq \wedge_{a \in \xi(q)} a = s(q)$. Suppose now that $s(p) \leq s(q)$. Take $a \in \xi(q)$, then we have $s(q) \leq a$. Hence also $s(p) \leq a$. But this implies that $a \in \xi(p)$. Hence this shows that $\xi(q) \subseteq \xi(p)$ and as a consequence we have $p \leq q$. Consider $b \in [s(p), I]$. This means that $s(p) \leq b$, and hence $b \in \xi(p)$. Consider now $b \in \xi(p)$. Then $s(p) \leq b$ and hence $b \in [s(p), I]$. □

For a state property space satisfying axioms 1 and 2 we can prove that the set of property states is a full set for the complete lattice $\mathcal{L}$.

Theorem 3 (Full set of property states). *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1 and 2 are satisfied. For* $a \in \mathcal{L}$ *we have*

$$\kappa(a) = \bigcup\nolimits_{a \in \xi(p)} \kappa(s(p)) \tag{38}$$

$$a = \bigvee\nolimits_{a \in \xi(p)} s(p) \tag{39}$$

Proof: From (35) follows that if $a \in \xi(p)$ we have $s(p) \leq a$ and hence $\kappa(s(p)) \subseteq \kappa(a)$. This proves that $\cup_{a \in \xi(p)} \kappa(s(p)) \subseteq \kappa(a)$. From (34) follows that for $p \in \Sigma$ we have $p \in \kappa(s(p))$ and hence $\{p\} \subseteq \kappa(s(p))$. This proves that $\kappa(a) = \cup_{p \in \kappa(a)} \{p\} \subseteq \cup_{p \in \kappa(a)} \kappa(s(p)) = \cup_{a \in \xi(p)} \kappa(s(p))$. From (35) we have $a \in \xi(p)$ then $s(p) \leq a$. This proves that $\vee_{a \in \xi(p)} s(p) \leq a$. Using (31) we have $\kappa(a) = \cup_{a \in \xi(p)} \kappa(s(p)) \subseteq \kappa(\vee_{a \in \xi(p)} s(p))$. This proves that $a \leq \vee_{a \in \xi(p)} s(p)$. Hence we have proven that $a = \vee_{a \in \xi(p)} s(p)$. □

The two archetypical examples of classical mechanics and quantum mechanics satisfy the axiom of completeness. Consider the state property space $(\Omega, \mathcal{P}(\Omega), \xi_\Omega)$ of a classical mechanical physical system with state space Ω. Consider a set of properties $(A_i)_i \subseteq \mathcal{P}(\Omega)$ of the classical mechanical system. The property $A = \cap_i A_i$ makes axiom 2 to be satisfied. Indeed, consider an arbitrary state $p \in \Omega$. We have $\cap_i A_i \in \xi_\Omega(p) \Leftrightarrow p \in \cap_i A_i \Leftrightarrow p \in A_i \ \forall i \Leftrightarrow A_i \in \xi_\Omega(p) \ \forall i$. From (30) follows that axiom 2 is satisfied.

Next, consider the state property space $(\Sigma(\mathcal{H}), \mathcal{L}(\mathcal{H}), \xi_\mathcal{H})$ corresponding to a quantum mechanical physical system described by means of a complex Hilbert space $\mathcal{H}$. Consider a set of properties $(A_i)_i \subseteq \mathcal{L}(\mathcal{H})$ of the quantum mechanical physical system. The property $\cap_i A_i \in \mathcal{L}(\mathcal{H})$ makes axiom 2 to be satisfied. Indeed, consider an arbitrary state $x \in \Sigma(\mathcal{H})$. We have $\cap_i A_i \in \xi_\mathcal{H}(P) \Leftrightarrow (\cap_i A_i)x = x \Leftrightarrow A_i x = x \ \forall i \Leftrightarrow A_i \in \xi_\mathcal{H}(P) \ \forall i$. From (30) follows that axiom 2 is satisfied.

3.3 Ortho tests

We have come to the point where we will introduce the first operational element which is specifically quantum, in the sense that it does not necessarily correspond with our intuition about reality. We will suppose that a special type test exists, which we call an ortho test.

Definition 17 (Ortho test)**.** *A test α is called an ortho test if it is such that if the physical entity is in a state $p \perp a$, where a is a property tested by α, then $\widetilde{\alpha}$ is true, and if the physical entity is in state $q \perp b$, where b is a property tested by $\widetilde{\alpha}$, then α is true.*

Proposition 14. *Consider a test α. If α is an ortho test then $\widetilde{\alpha}$ is an ortho test.*

Proof: Follows directly from the definition. □

We can see immediately that ortho tests are special types of test because of the next proposition, where we prove that a product test is never an ortho test, except when it is a trivial product test of equivalent tests.

Proposition 15. *Consider a set of tests $(\alpha_i)_i$. The product test $\Pi_i \alpha_i$ is an ortho test iff α_j is an ortho test for each j, and $\alpha_j \approx \alpha_k$ for each j, k.*

Proof: Suppose that $\Pi_i \alpha_i$ is an ortho test, and let us call a_i a property tested by α_i, and hence $\wedge_i a_i$ a property tested by $\Pi_i \alpha_i$. Consider an arbitrary α_j of the set $(\alpha_i)_i$, and a state p such that $p \perp a_j$. From (20) follows that

$p \perp \wedge_i a_i$. Since $\Pi_i \alpha_i$ is an ortho test, we have that whenever $p \perp \wedge_i a_i$ the test $\widetilde{\Pi_i \alpha_i} = \Pi_i \widetilde{\alpha}_i$ is true. This means that $\widetilde{\alpha}_k$ is true for all k. Hence $\widetilde{\alpha}_j$ is true. Hence, we have proven that if p is a state orthogonal to a_j, then $\widetilde{\alpha}_j$ is true, which is one of the necessary conditions for α_j to be an ortho test. Let us proceed proving the other. Suppose that $(b_i)_i$ is a set of properties such that each b_i is a property tested by $\widetilde{\alpha}_i$, and hence $\wedge_i b_i$ is a property tested by $\Pi_i \widetilde{\alpha}_i$. Let us consider $\widetilde{\alpha}_j$ which tests b_j. Consider a state q such that $q \perp b_j$. From (20) follows that $q \perp \wedge_i b_i$, and since $\Pi_i \widetilde{\alpha}_i$ is an ortho test, we have that $\widetilde{\Pi_i \widetilde{\alpha}_i} = \Pi \alpha_i$ is true. This implies that α_i is true for all i, and hence α_j is true. This proves that α_j is an ortho test. Since we had chosen j arbitrary, this proves that all tests α_i are ortho tests. Let us next prove that all test are equivalent. Consider α_j and suppose that the entity is in state p such that α_j is true. From this follows that $p \perp b_j$ and hence $p \perp \wedge_i b_i$. Since $\Pi_i \widetilde{\alpha}_i$ is an ortho test, this implies that $\Pi_i \alpha_i$ is true, and hence α_k is true for all k. Hence we have proven that $\alpha_j \leq \alpha_k$ for all j and k, and as a consequence all the tests are equivalent. □

Proposition 15 tells us something important. Ortho tests are the test which exist commonly in quantum mechanics. This proposition proves that such an ortho test cannot be generated in a non trivial way by the product test mechanism. This means that if a property can be tested by an ortho test, hence this ortho test is specific for this property, even if this property is a meet property. It is not a product of other tests, except in a trivial way, when these other tests are also ortho tests testing this same property. But ortho test and the corresponding ortho properties have other unsuspected features.

Proposition 16. *Consider two tests α, β such that α is an ortho test. We have*

$$\alpha \leq \beta \Rightarrow \widetilde{\beta} \leq \widetilde{\alpha} \tag{40}$$

Proof: Suppose that $\alpha \leq \beta$ and that the state p of the entity is such that $\widetilde{\beta}$ is true. This means that $p \perp b$ where b is a property tested by β. Suppose that a is a property tested by α, then we have $a \leq b$, and hence $p \perp a$. Since α is an ortho test, it follows that $\widetilde{\alpha}$ is true. Hence we have proven that $\widetilde{\beta} \leq \widetilde{\alpha}$. □

3.4 The axiom of orthocomplementation

There is a specific structure, namely the structure of an orthocomplementation, which has been identified mathematically in the formalism of quantum mechanics, and this structure has played an important role in the mathematical axiomatization, for example the one worked out in [6, 8]. With the notion of ortho test we introduce an orthocomplementation in an operational way. Lets first define what an orthocomplementation on a partially ordered set with minimum is.

Definition 18 (Orthocomplementation)**.** *Suppose that we have a set Z with a partial-order $\leq$ and a smallest element 0. A bijective map $^\perp : Z \to Z$ is an orthocomplementation if for $x, y \in Z$ we have*

$$(x^\perp)^\perp = x \tag{41}$$

$$x \leq y \Rightarrow y^\perp \leq x^\perp \tag{42}$$

$$0 \text{ is the infimum of } x \text{ and } x^\perp \tag{43}$$

The ortho test satisfies (41), (42) and (43). Of course, the set of tests corresponding to a physical entity is not a partially ordered set, but only a pre-ordered set. But the set of properties, if axiom 1 of property determination is satisfied, is a partially ordered set. Hence our aim is to make operational steps such that on the set of properties an orthocomplementation arises. This makes us introduce the following definition for an ortho property.

Definition 19 (Ortho property)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$. We say that $a \in \mathcal{L}$ is an ortho property if there exists an ortho test testing a. If α is the ortho test testing a, we denote by $a^\perp$ the property tested by $\widetilde{\alpha}$.*

Let us introduce the following definition.

Definition 20 (Orthogonal set)**.** *For a subset of states $A \subseteq \Sigma$ we define the orthogonal $A^\perp$ of this subset*

$$A^\perp = \{p \mid p \perp q \ \forall\ q \in A\} \tag{44}$$

Proposition 17. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$. If $a \in \mathcal{L}$ is an ortho property then we have*

$$\kappa(a)^\perp = \kappa(a^\perp) \tag{45}$$

Proof: Suppose that $a \in \mathcal{L}$ is an ortho property. This means that $p \perp a \Leftrightarrow a^\perp \in \xi(p)$. Hence $p \in \kappa(a)^\perp \Leftrightarrow p \in \kappa(a^\perp)$. And as a consequence we have $\kappa(a)^\perp = \kappa(a^\perp)$. □

Axiom 3 (Orthocomplementation)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$. For each property $a \in \mathcal{L}$ there exists an ortho test α testing this property.*

Theorem 4. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *and suppose that axiom 1, axiom 2 and axiom 3 are satisfied. For* $a \in \mathcal{L}$ *and* α *an ortho test testing* a *let us denote the property tested by* $\widetilde{\alpha}$ *by* $a^\perp$*. For* $a, b \in \mathcal{L}$ *and* $p, q \in \Sigma$ *we then have*

$$(a^\perp)^\perp = a \tag{46}$$

$$a \leq b \Rightarrow b^\perp \leq a^\perp \tag{47}$$

$$a \wedge a^\perp = 0 \tag{48}$$

which proves that $^\perp : \mathcal{L} \to \mathcal{L}$ *is an orthocomplementation.*

Proof: First we remark that if α and β are ortho tests testing property a, and hence $\alpha \approx \beta$, we have that $\widetilde{\alpha} \approx \widetilde{\beta}$. This shows that $^\perp$ is a function. Consider now $a \in \mathcal{L}$ and α an ortho test testing a. Then $\widetilde{\alpha}$ is an ortho test testing $a^\perp$. We have that $(a^\perp)^\perp$ is the property tested by $\widetilde{\widetilde{\alpha}} = \alpha$. Hence $(a^\perp)^\perp = a$. Consider $a, b \in \mathcal{L}$ such that $a \leq b$, and α and β ortho tests testing respectively a and b. Hence $\alpha \leq \beta$. From this follows that $\widetilde{\beta} \leq \widetilde{\alpha}$, and hence $b^\perp \leq a^\perp$. Consider $a \in \mathcal{L}$ and α an ortho test testing a. Hence $\widetilde{\alpha}$ tests $a^\perp$. The infimum property $a \wedge a^\perp$ in $\mathcal{L}$ is tested by $\alpha \cdot \widetilde{\alpha}$ and hence $a \wedge a^\perp = 0$. □

Proposition 18. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2 and 3 are satisfied. We have for* $a, b \in \mathcal{L}$, $(a_i)_i \subseteq \mathcal{L}$ *and* $p, q \in \mathcal{L}$

$$(\bigvee_i a_i)^\perp = \bigwedge_i a_i^\perp \tag{49}$$

$$(\bigwedge_i a_i)^\perp = \bigvee_i a_i^\perp \tag{50}$$

$$0^\perp = I \quad I^\perp = 0 \tag{51}$$

$$a \vee a^\perp = I \tag{52}$$

$$p \perp q \Leftrightarrow \exists\ c \in \mathcal{L} \text{ such that } c \in \xi(p) \text{ and } c^\perp \in \xi(q) \tag{53}$$

Proof: Let us prove (49) and (50). We have $\wedge_i a_i \leq a_j\ \forall j$, which implies that $a_j^\perp \leq (\wedge_i a_i)^\perp\ \forall j$, and hence $\vee_i a_i^\perp \leq (\wedge_i a_i)^\perp$. This also implies $\vee_i (a_i^\perp)^\perp = \vee_i a_i \leq (\wedge_i a_i^\perp)^\perp$. Hence $((\wedge_i a_i^\perp)^\perp)^\perp = \wedge_i a_i^\perp \leq (\vee_i a_i)^\perp$. We also have $a_j \leq \vee_i a_i\ \forall j$, which implies that $(\vee_i a_i)^\perp \leq a_j^\perp\ \forall j$. Hence $(\vee_i a_i)^\perp \leq \wedge_i a_i^\perp$. This also implies that $(\vee_i a_i^\perp)^\perp \leq \wedge_i (a_i^\perp)^\perp = \wedge_i a_i$. Hence $(\wedge_i a_i)^\perp \leq ((\vee_i a_i^\perp)^\perp)^\perp = \vee_i a_i^\perp$. Consider $a \in \mathcal{L}$, then $0 \leq a^\perp$, and hence $a \leq 0^\perp$. This proves that $0^\perp$ is a maximal element of $\mathcal{L}$, and hence $0^\perp = I$. In a analogous way we prove that $I^\perp = 0$. We have $I = 0^\perp = (a \wedge a^\perp)^\perp = a^\perp \vee a$ which proves (52). To prove (53) we remark that if $c \in \mathcal{L}$ is such that $c \in \xi(p)$ and $c^\perp \in \xi(q)$, we have $p \in \kappa(c)$ and $q \in \kappa(c^\perp)$. Since $\kappa(c^\perp) = \kappa(c)^\perp$ we have $p \perp q$. □

In foregoing work on quantum axiomatics we have worked most of the time with state property systems [14, 18, 21, 22, 23, 24, 25, 26].

Definition 21 (State property system)**.** *We say that* $(\Sigma, \mathcal{L}, \xi)$ *is a state-property system if* $(\Sigma, \leq)$ *is a pre-ordered set,* $(\mathcal{L}, \leq, \wedge, \vee)$ *is a complete lattice with the greatest element* I *and the smallest element* 0*, and* ξ *is a function*

$$\xi : \Sigma \to \mathcal{P}(\mathcal{L}) \tag{54}$$

such that for $p \in \Sigma$ *and* $(a_i)_i \subseteq \mathcal{L}$*, we have*

$$I \in \xi(p), \tag{55}$$

$$0 \notin \xi(p), \tag{56}$$

$$a_i \in \xi(p)\ \forall i \Leftrightarrow \wedge_i a_i \in \xi(p) \text{ (for an arbitrary set of indices)} \tag{57}$$

and for $p, q \in \Sigma$ *and* $a, b \in \mathcal{L}$ *we have*

$$p \leq q \Leftrightarrow \xi(q) \subseteq \xi(p) \tag{58}$$

$$a \leq b \Leftrightarrow \forall r \in \Sigma : a \in \xi(r) \Rightarrow b \in \xi(r) \tag{59}$$

Elements of Σ *are called states, elements of* $\mathcal{L}$ *are called properties.*

A state property space for which the three axioms which we have formulated are satisfied is a state property system.

Theorem 5. *A state property space for which axioms 1, 2 and 3 are satisfied is a state property system.*

Proof: Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1, 2 and 3 are satisfied. From (7) follows that $(\Sigma, \leq)$ is a pre-ordered set. In theorem 2 we prove that $\mathcal{L}, \leq, \wedge, \vee$ is a complete lattice, and from (29) follows that $I \in \xi(p)\ \forall\ p \in \Sigma$. We have $a \wedge a^\perp = 0$ and hence $\kappa(0) = \kappa(a) \cap \kappa(a)^\perp = \emptyset$. This proves $0 \notin \xi(p)\ \forall\ p \in \Sigma$. From (25) and (26) of axiom 2 follows (57), and (58) and (59) follows respectively from (7) and (6). $\square$

4 Morphisms

We derive the notion of morphism from a covariance situation. Consider two state property spaces $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$, describing respectively entities S and S'. We will arrive at the notion of morphism by analyzing the situation where the entity S is a sub-entity of the entity S'. In that case, the following three natural requirements should be satisfied:

i) If the entity S' is in a state p' then the state $m(p')$ of S is determined. This defines a function m from the set of states of S' to the set of states of S;

ii) If we consider a property a of the entity S, then to a corresponds a property $n(a)$ of the 'bigger' entity S'. This defines a function n from the set of properties of S to the set of properties of S';

iii) We want a and $n(a)$ to be two descriptions of the 'same' property of S, once considered as an entity on itself, once as a sub-entity of S'. In other words we want a and $n(a)$ to be actual at once. This means that for a state p' of S' (and a corresponding state $m(p')$ of S) we want the following 'covariance principle' to hold:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{60}$$

We are now ready to present a formal definition of a morphism of state property spaces.

Definition 22. *Consider two state property spaces $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$. We say that*

$$(m, n) : (\Sigma', \mathcal{L}', \xi') \longrightarrow (\Sigma, \mathcal{L}, \xi) \tag{61}$$

is a 'morphism' (of state property spaces) if m is a function:

$$m : \Sigma' \to \Sigma \tag{62}$$

and n is a function:

$$n : \mathcal{L} \to \mathcal{L}' \tag{63}$$

such that for $a \in \mathcal{L}$ and $p' \in \Sigma'$ the following holds:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{64}$$

Proposition 19. *Consider two state property spaces $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$, and functions*

$$m : \Sigma' \to \Sigma \quad n : \mathcal{L} \to \mathcal{L}' \tag{65}$$

We have that

$$(m, n) : (\Sigma', \mathcal{L}', \xi') \longrightarrow (\Sigma, \mathcal{L}, \xi) \tag{66}$$

is a morphism iff for $a \in \mathcal{L}$, and $p' \in \Sigma'$

$$m(p') \in \kappa(a) \Leftrightarrow p' \in \kappa'(n(a)) \tag{67}$$

Proof: Let us prove (64) to show that (m, n) is a morphism. We have $a \in \xi(m(p')) \Leftrightarrow m(p') \in \kappa(a) \Leftrightarrow p' \in \kappa'(n(a)) \Leftrightarrow n(a) \in \xi'(p')$.

The next theorem gives some properties of morphisms.

Theorem 6. *Consider two state property spaces* $(\Sigma, \mathcal{L}, \xi)$ *and* $(\Sigma', \mathcal{L}', \xi')$ *connected by a morphism* $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$*. For* $p', q' \in \Sigma'$ *and* $a, b \in \mathcal{L}$ *we have:*

$$p' \leq q' \Rightarrow m(p') \leq m(q') \tag{68}$$

$$a \leq b \Rightarrow n(a) \leq n(b) \tag{69}$$

Proof: Suppose that $p' \leq q'$. We then have $\xi'(q') \subseteq \xi'(p')$. Consider $a \in \xi(m(q'))$, then (64) implies that $n(a) \in \xi'(q')$, and hence $n(a) \in \xi'(p')$, which means that $a \in \xi(m(p'))$. As a consequence we have $\xi(m(q')) \subseteq \xi(m(p'))$, whence $m(p') \leq m(q')$. Next consider $a \leq b$. We then have $\kappa(a) \subseteq \kappa(b)$. Let $r' \in \Sigma'$ be such that $n(a) \in \xi'(r')$. Then we have $a \in \xi(m(r'))$ and hence $m(r') \in \kappa(a) \subseteq \kappa(b)$. This yields $b \in \xi(m(r'))$. From this follows that $n(b) \in \xi'(r')$. So we have shown that $n(a) \leq n(b)$. $\square$

Theorem 7. *Consider two state property spaces* $(\Sigma, \mathcal{L}, \xi)$ *and* $(\Sigma', \mathcal{L}', \xi')$ *connected by a morphism* $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ *for which the axioms 1, 2 are satisfied. For* I *and* I' *the maximum of respectively* $\mathcal{L}$ *and* $\mathcal{L}'$ *and* $(a_i)_i \subseteq \mathcal{L}$ *we have:*

$$n(I) = I' \tag{70}$$

$$n(\bigwedge_i a_i) = \bigwedge_i n(a_i) \tag{71}$$

Proof: We clearly have $n(I) \leq I'$. Hence remains to show that $I' \leq n(I)$. Consider $r' \in \Sigma' = \kappa'(I')$, then $m(r') \in \Sigma = \kappa(I)$. From (67) follows that $r' \in \kappa'(n(I))$. This proves that $\kappa'(I') \subseteq \kappa'(n(I))$, and hence $I' \leq n(I)$. Hence we have proven that $n(I) = I'$.
From $\wedge_i a_i \leq a_j \ \forall j$ we obtain $n(\wedge_i a_i) \leq n(a_j) \ \forall j$. This yields $n(\wedge_i a_i) \leq \wedge_i n(a_i)$. We still have to show that $\wedge_i n(a_i) \leq n(\wedge_i a_i)$. Let $r' \in \Sigma'$ be such that $r' \in \kappa'(\wedge_i n(a_i))$. Using (30) we have $r' \in \cap_i \kappa'(n(a_i))$, and hence $r' \in \kappa'(n(a_i)) \ \forall i$. From (67) follows that this implies that $m(r') \in \kappa(a_i) \ \forall i$, and hence $m(r') \in \cap_i \kappa(a_i) = \kappa(\wedge_i a_i)$ using again (30). From (67) this implies that $r' \in \kappa'(n(\wedge_i a_i))$. Hence we have shown that $\kappa'(\wedge_i n(a_i)) \subseteq \kappa'(n(\wedge_i a_i))$, and it follows that $\wedge_i n(a_i) \leq n(\wedge_i a_i)$. Hence we have proven that $n(\wedge_i a_i) = \wedge_i n(a_i)$. $\square$

Definition 23. *Consider two state property spaces* $(\Sigma, \mathcal{L}, \xi)$ *and* $(\Sigma', \mathcal{L}', \xi')$ *for which the axioms 1, 2, 3 are satisfied. A morphism* $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ *is an orthomorphism iff*

$$m(p') \in \kappa(a^\perp) \Leftrightarrow p' \in \kappa'(n(a)^\perp) \tag{72}$$

Theorem 8. *Consider two state property spaces* $(\Sigma, \mathcal{L}, \xi)$ *and* $(\Sigma', \mathcal{L}', \xi')$ *for which the axioms 1, 2, 3 are satisfied and connected by an orthomorphism* $(m, n) : (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$*. For* 0 *and* $0'$ *being the minimal elements of* $\mathcal{L}$ *and* $\mathcal{L}'$ *respectively,* $a \subseteq \mathcal{T}$ *and* $p', q' \in \Sigma'$ *we have:*

$$n(a^\perp) = n(a)^\perp \tag{73}$$

$$n(0) = 0' \tag{74}$$

$$m(p') \perp m(q') \Rightarrow p' \perp q' \tag{75}$$

Proof: $p' \in \kappa'(n(a)^\perp) \Leftrightarrow m(p') \in \kappa(a^\perp) \Leftrightarrow p' \in \kappa'(n(a^\perp))$. This proves that $\kappa'(n(a)^\perp) = \kappa'(n(a^\perp))$, and hence $n(a)^\perp = n(a^\perp)$. We have $0 = a \wedge a^\perp$, and hence $n(0) = n(a \wedge a^\perp) = n(a) \wedge n(a^\perp) = n(a) \wedge n(a)^\perp = 0'$. Consider $p', q' \in \Sigma'$ such that $m(p') \perp m(q')$. This means that there exists $a \in \mathcal{L}$ such that $m(p') \in \kappa(a)$ and $m(q') \in \kappa(a^\perp)$. Hence $p' \in \kappa'(n(a))$ and $q' \in \kappa'(n(a^\perp)) = \kappa'(n(a)^\perp)$. This proves that $p' \perp q'$.

5 Decomposition of a state property space

In this section we introduce the notion of classical test, classical property and classical state. This will lead us to elaborate a decomposition theorem for a state property space into non classical components over a classical state space.

5.1 The classical state property space

In this section we identify the classical part of an entity S. We start by introducing the notion of classical test. The basic idea for a classical test is that it is a test which does not contain any indeterminism. This means that for such a test the outcome 'yes' is certain or the outcome 'no' is certain for each state of the physical entity. Hence we put forward the following definition

Definition 24 (Classical test)**.** *A test* α *is a classical test if for any arbitrary state* p *of the physical entity we have* α *is true or* $\widetilde{\alpha}$ *is true.*

For the product of classical tests we can prove the following

Proposition 20. *If $(\alpha_i)_i$ is a set of tests, then $\Pi_i \alpha_i$ is a classical test iff each of the α_j are classical tests and $\alpha_j \approx \alpha_k$ for all j, k.*

Proof: Suppose that $\Pi_i \alpha_i$ is a classical test. Consider α_j and a state p such that α_j is not true if the entity is in state p. Then $\Pi_i \alpha$ is not true and since $\Pi_i \alpha_i$ is a classical test, we have that $\widetilde{\Pi}_i \alpha_i = \Pi \widetilde{\alpha}_i$ is true. But then $\widetilde{\alpha}_i$ is true for all i, and hence $\widetilde{\alpha}_j$ is true. This proves that α_j is a classical test. Since we had chosen j arbitrary, we can conclude that α_i is a classical test for all i. Suppose now that α_j is true. Then $\widetilde{\alpha}_j$ is not true, and hence $\Pi \widetilde{\alpha}_i = \widetilde{\Pi} \alpha_i$ is not true. But then, since $\widetilde{\Pi} \alpha_i$ is a classical test, we have that $\Pi_i \alpha_i$ is true, and hence α_k is true for all k. Hence we have proven that $\alpha_j \leq \alpha_k$ for all k. Hence $\alpha_j \approx \alpha_k$ for all j, k. □

It is easy to see that a classical test is always an ortho test.

Proposition 21. *If α is a classical test then α is an ortho test.*

Proof: Suppose that α is a classical test, and consider a state p such that $p \perp a$ where a is a property tested by α. This means that $a \notin \xi(p)$, and hence $\widetilde{\alpha}$ is true for the physical entity in state p. In an analogous way we prove that for $q \perp a^\perp$ when $a^\perp$ is a property tested by $\widetilde{\alpha}$ and the physical entity in state q we have α is true. This proves that α is an ortho test. □

Definition 25 (Classical property)**.** *A classical property $a \in \mathcal{L}$ is a property such that there exists a set $(\alpha_i)_i$ of classical tests α_i such that $\Pi_i \alpha_i$ tests this property. We denote $\mathcal{C}$ the set of all classical properties. A basic classical property $a \in \mathcal{L}$ is a property such that there exists a classical test α testing this property. We denote $\mathcal{K}$ the set of basic classical properties.*

Definition 26 (Classical elements)**.** *Consider a state property space for which axioms 1, 2 and 3 are satisfied. For $p \in \Sigma$, we introduce*

$$\omega(p) = \bigwedge_{a \in \xi(p) \cap \mathcal{C}} a \tag{76}$$

and call $\omega(p)$ the classical state of the entity S whenever S is in a state $p \in \Sigma$. The set of all classical states is denoted by Ω. We introduce

$$\xi_c : \Omega \;\; \rightarrow \;\; \mathcal{C} \quad \omega(p) \mapsto \xi(p) \cap \mathcal{C} \tag{77}$$

$$\kappa_c : \mathcal{C} \;\; \rightarrow \;\; \mathcal{P}(\Omega) \quad a \mapsto \{\omega(p) \; | a \in \xi(p)\} \tag{78}$$

and call κ_c the classical Cartan map of the state property space $(\Sigma, \mathcal{L}, \xi)$.

Proposition 22. *Consider a state property space for which axioms 1 and 2 are satisfied. For classical states $\omega(p), \omega(q) \in \Omega$, classical property $a \in \mathcal{C}$, and states $p, q \in \Sigma$ we have*

$$a \in \xi(p) \quad \Leftrightarrow \quad \omega(p) \leq a \tag{79}$$

$$\omega(p) \leq \omega(q) \quad \Leftrightarrow \quad \xi_c(q) \subseteq \xi_c(p) \tag{80}$$

$$p \leq q \quad \Rightarrow \quad \omega(p) \leq \omega(q) \tag{81}$$

Proof: Suppose that $a \in \mathcal{C}$ and $a \in \xi(p)$. Since $\omega(p) = \wedge_{a \in \xi(p) \cap \mathcal{C}} a$ we have $\omega(p) \leq a$. Suppose now that $\omega(p) \leq a$. Since $\omega(p) \in \xi(p)$ we have $a \in \xi(p)$. Consider $a \in \xi_c(q) = \xi(q) \cap \mathcal{C}$ and $\omega(p) \leq \omega(q)$. This implies that $\omega(q) \leq a$ and hence $\omega(p) \leq a$. From this follows that $a \in \xi(p)$ and hence $a \in \xi(p) \cap \mathcal{C} = \xi_c(p)$. Hence we have proven that $\xi_c(q) \subseteq \xi_c(p)$. Suppose now that $\xi_c(q) \subseteq \xi_c(p)$, then $\omega(p) = \wedge_{a \in \xi_c(p)} \leq \wedge_{a \in \xi_c(q)} = \omega(q)$. Suppose that $p \leq q$ and hence $\xi(q) \subseteq \xi(p)$. We then have $\xi_c(q) = \xi(q) \cap \mathcal{C} \subseteq \xi(p) \cap \mathcal{C} = \xi_c(p)$, and hence $\omega(p) \leq \omega(q)$. □

Let us consider our two physics examples, and see what the notion of classical property and classical state means in these cases. Consider first the state property space $(\Omega, \mathcal{P}(\Omega), \xi_\Omega)$ for a classical physical system, and consider a property $A \in \mathcal{P}(\Omega)$. Take $p \in \Omega$, then we have $p \in A$ or $p \in A^C$. This proves that any arbitrary property A is a classical property for the state property system $(\Omega, \mathcal{P}(\Omega), \xi_\Omega)$ corresponding to a classical physical system. Clearly, for such a state property system the states coincide with the classical states, which proves that any state is a classical state.

Consider now the state property system $(\Sigma(\mathcal{H}), \mathcal{L}(\mathcal{H}), \xi_\mathcal{H})$ corresponding to a quantum physical system, and consider a property $A \in \mathcal{L}(\mathcal{H})$ such that $A \neq 0$ and $A \neq \mathcal{H}$. In this case we have $A^{orth} \neq \mathcal{H}$ and $A^{orth} \neq 0$. Take $x \in A, x \neq 0$ and $y \in A^{orth}, y \neq 0$, and consider the vector $z = x + y$. Then $z \notin A$ and $z \notin A^{orth}$, and as a consequence $\bar{z} \notin A$ and $\bar{z} \notin A^{orth}$, which proves that A is not a classical property. We have proven that for the state property system corresponding to a quantum physical system the only classical properties are the minimal property and the maximal property. Moreover, the only classical state of the state property system corresponding to a quantum physical system is the classical state corresponding to $\mathcal{H}$ itself. This is the state describing the situation 'the entity is present'.

Definition 27 (Classical orthogonality relation)**.** *Consider a state property space describing a physical entity S for which axioms 1 and 2 are satisfied, and classical states $\omega(p), \omega(q) \in \Omega$ of this physical entity. We say that*

$\omega(p) \perp_c \omega(q)$ if there exists a classical test γ such that γ is true if $\omega(p)$ is actual, hence if the entity is in classical state $\omega(p)$, and $\widetilde{\gamma}$ is true if $\omega(q)$ is actual, hence if the entity is in classical state $\omega(q)$.

Definition 28 (Classical ortho test)**.** *Consider a state property space describing a physical entity S for which axioms 1 and 2 are satisfied. A classical test α is a classical ortho test if it is such that if the physical entity is in classical state $\omega(p) \perp_c a$, where a is the property tested by α, then $\widetilde{\alpha}$ is true, and if the physical entity is in state $\omega(q) \perp_c b$, where b is the property tested by $\widetilde{\alpha}$, then α is true.*

Proposition 23. *Consider a state property space for which axioms 1 and 2 are satisfied, and classical states $\omega(p), \omega(q) \in \Omega$ of this physical entity. We have*

$$\omega(p) \neq \omega(q) \Leftrightarrow \omega(p) \perp_c \omega(q) \Leftrightarrow \omega(p) \perp \omega(q) \Leftrightarrow \omega(p) \notin \xi(q) \Leftrightarrow q \perp \omega(p) \tag{82}$$

Proof: If $\omega(p) \perp_c \omega(q)$ then obviously $\omega(p) \neq \omega(q)$. Suppose now that $\omega(p) \neq \omega(q)$. Since $\omega(p)$ and $\omega(q)$ as classical states are also both classical properties there exist $(\alpha_i)_i$ and $(\beta_j)_j$ where α_i and β_j are classical tests for all i, j and such that $\Pi_i\alpha_i$ tests $\omega(p)$ and $\Pi_j\beta_j$ tests $\omega(q)$. If $\omega(p) \neq \omega(q)$ this can mean that $\omega(p) \not\leq \omega(q)$ or that $\omega(q) \not\leq \omega(p)$. Suppose we have that $\omega(p) \not\leq \omega(q)$, and suppose that $\omega(p)$ is actual. Since in this case $\omega(q)$ is not actual there is at least one β_j which is not true. But then $\widetilde{\beta_j}$ is true. If $\omega(q)$ is actual we have that β_j is true. Hence we have proven that $\omega(p) \perp_c \omega(q)$. Analogously we prove that $\omega(p) \perp_c \omega(q)$ if $\omega(q) \not\leq \omega(p)$. Suppose that $\omega(p) \perp_c \omega(q)$ and let γ be the test which is true if $\omega(p)$ is actual such that $\widetilde{\gamma}$ is true if $\omega(q)$ is actual. Consider states $r, s \in \Sigma$ such that $\omega(p) \in \xi(r)$ and $\omega(q) \in \xi(s)$. If c is the property tested by γ and d the property tested by $\widetilde{\gamma}$ we have $\omega(p) \leq c$ and $\omega(q) \leq d$. Hence $c \in \xi(r)$ and $d \in \xi(s)$. This proves that $r \perp s$, and hence $\omega(p) \perp \omega(q)$. Suppose now that $\omega(p) \perp \omega(q)$. Then certainly $\omega(p) \neq \omega(q)$ and hence $\omega(p) \perp_c \omega(q)$. Suppose that $\omega(p) \notin \xi(q)$. Then $\omega(q) \not\leq \omega(p)$, and as a consequence we have $\omega(q) \neq \omega(p)$. Hence $\omega(p) \perp \omega(q)$. If $\omega(p) \perp \omega(q)$ then $\omega(q) \not\leq \omega(p)$ and hence $\omega(p) \notin \xi(q)$. $\square$

Proposition 24. *Consider a state property space for which axioms 1 and 2 are satisfied. A classical test α is a classical ortho test.*

Proof: Consider a classical test α such that a is the property tested by α and b is the property tested by $\widetilde{\alpha}$, and suppose we have $\omega(p) \perp_c a$. Consider $q \in \Sigma$ such that $a \in \xi(q)$. Then we have $\omega(q) \leq a$, and hence $\omega(p) \perp_c \omega(q)$.

As a consequence we have $\omega(p) \perp q$ and hence $p \perp q$. This implies that $b \in \xi(p)$, and hence $\omega(p) \leq b$. In an analogous way we show that $\omega(q) \leq a$ if $\omega(q) \perp_c b$. This proves that α is a classical ortho test. $\square$

Suppose we consider a state property space $(\Sigma, \mathcal{L}, \xi)$ describing a physical entity S for which axioms 1, 2 and 3 are satisfied. We wonder whether $(\Omega, \mathcal{C}, \xi_c)$ is a state property space satisfying 1, 2 and 3. If this is the case we can consider $(\Omega, \mathcal{C}, \xi_c)$ as the state property space describing the classical aspects of the physical entity S.

Theorem 9. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ describing a physical entity S for which axioms 1, 2 and 3 are satisfied. Consider $\omega(p), \omega(q) \in \Omega$ and $a, b \in \mathcal{C}$. We have*

$$a \in \xi(p) \Leftrightarrow \omega(p) \leq a \tag{83}$$

$$\omega(p) \leq \omega(q) \Leftrightarrow \xi_c(\omega(q)) \subseteq \xi_c(\omega(p)) \tag{84}$$

$$\omega(p) \in \kappa_c(a) \Leftrightarrow a \in \xi_c(\omega(p)) \tag{85}$$

$$a \leq b \Leftrightarrow \kappa_c(a) \subseteq \kappa_c(b) \tag{86}$$

$$\kappa_c(a) = \kappa_c(b) \Rightarrow a = b \tag{87}$$

$$\kappa_c(\wedge_i a_i) = \cap_i \kappa_c(a_i) \tag{88}$$

$$\kappa(a^\perp) = \Sigma \setminus \kappa(a) \tag{89}$$

$$\kappa_c(a^{\perp_c}) = \Omega \setminus \kappa_c(a) \tag{90}$$

$$\text{There exists a classical test } \alpha \text{ testing } a, \text{ hence } \mathcal{C} = \mathcal{K} \tag{91}$$

and $(\Omega, \mathcal{C}, \xi_c)$ is a state property space satisfying axioms 1, 2 and 3.

Proof: Suppose that $a \in \xi(p)$. Since $\omega(p) = \wedge_{a \in \xi(p) \cap \mathcal{C}} a$ we have $\omega(p) \leq a$. Suppose now that $\omega(p) \leq a$. Since $\omega(p) \in \xi(p)$ we have $a \in \xi(p)$. This proves (83). Suppose that $\omega(p) \leq \omega(q)$ and consider $a \in \xi_c(\omega(q))$ and hence $\omega(q) \leq a$. Then we have $\omega(p) \leq a$ and hence $a \in \xi_c(\omega(p))$. This proves that $\xi_c(\omega(q)) \subseteq \xi_c(\omega(p))$. Suppose now that $\xi_c(\omega(q)) \subseteq \xi_c(\omega(p))$, and hence $\xi(q) \cap \mathcal{C} \subseteq \xi(p) \cap \mathcal{C}$. Then we have $\omega(p) = \wedge_{a \in \xi(p) \cap \mathcal{C}} a \leq \wedge_{a \in \xi(q) \cap \mathcal{C}} a = \omega(q)$. This proves (84). Suppose that $\omega(p) \in \kappa_c(a)$, then $a \in \xi(p)$ and hence $\omega(p) \leq a$ which shows that $a \in \xi_c(\omega(p))$. Contrary, suppose that $a \in \xi_c(\omega(p)) = \xi(p) \cap \mathcal{C}$. Then we have $a \in \xi(p)$, and hence $\omega(p) \in \kappa_c(a)$. This proves (85). Suppose that $a \leq b$ and consider $\omega(p) \in \kappa_c(a)$. We then have $a \in \xi(p)$ and hence $b \in \xi(p)$. From this follows that $\omega(p) \in \kappa_c(b)$. Hence we have proven that $\kappa_c(a) \subseteq \kappa_c(b)$. Suppose now that $\kappa_c(a) \subseteq \kappa_c(b)$ and suppose we have $p \in \Sigma$ such that $a \in \xi(p)$. This means that $\omega(p) \in \kappa_c(a)$, and hence $\omega(p) \in \kappa_c(b)$, which implies that $b \in \xi(p)$. Hence we have proven that $a \leq b$. This proves (86). Suppose that $\kappa_c(a) = \kappa_c(b)$ and consider $p \in \kappa(a)$. Then we have $a \in \xi(p)$ and hence $\omega(p) \in \kappa_c(a)$. From this follows that $\omega(p) \in \kappa_c(b)$, and hence $b \in \xi(p)$. As a consequence we have $p \in \kappa(b)$. This means that we have proven that $\kappa(a) \subseteq \kappa(b)$. Analogously we prove that $\kappa(b) \subseteq \kappa(a)$. Since axiom 1 is satisfied for the state property space $(\Sigma, \mathcal{L}, \xi)$ we have $a = b$. This proves (87). Consider $(a_i)_i \subseteq \mathcal{K} \subset \mathcal{T}$. Since axiom 2 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ we have a property $\wedge_i a_i \in \mathcal{L}$ such that $\kappa(\wedge_i a_i) = \wedge_i \kappa(a_i)$. We have $\wedge_i a_i \in \mathcal{C}$ since $a_i \in \mathcal{C}\ \forall\ i$. This means that we have the following $\omega(p) \in \kappa_c(\wedge_i a_i) \Leftrightarrow p \in \kappa(\wedge_i a_i) \Leftrightarrow p \in \cap_i \kappa(a_i) \Leftrightarrow p \in \kappa(a_i)\ \forall\ i \Leftrightarrow \omega(p) \in \kappa_c(a_i)\ \forall\ i \Leftrightarrow \omega(p) \in \cap_i \kappa_c(a_i)$. This means that $\kappa_c(\wedge_i a_i) = \cap_i \kappa_c(a_i)$, and hence we have proven (88). Consider $p \in \Sigma$ and $p \notin \kappa(a)$. Then we have $\omega(p) \not\leq a$ and hence $\omega(p) \neq \omega(q)\ \forall\ \omega(q) \leq a$. From this follows that $\omega(p) \perp_c \omega(q)\ \forall\, \omega(q) \leq a$, and hence $\omega(p) \perp \omega(q)\ \forall\, \omega(q) \leq a$. As a consequence we have that $p \perp q\ \forall\ q \in \kappa(a)$, and hence $p \in \kappa(a^\perp)$. This proves that $\Sigma \setminus \kappa(a) \subseteq \kappa(a^\perp)$. We obviously have that $\kappa(a^\perp) \subseteq \Sigma \setminus \kappa(a)$. Hence we have proven (89). Consider $\omega(p) \in \Omega$ and $\omega(p) \notin \kappa_c(a)$. This means that $\omega(p) \neq \omega(q)\ \forall\ \omega(q) \leq \kappa_c(a)$, and hence $\omega(p) \perp_c \omega(q)\ \forall\ \omega(q) \leq \kappa_c(a)$. As a consequence we have $\omega(p) \leq \kappa_c(a)^{\perp_c}$. This proves that $\Omega \setminus \kappa_c(a) \subseteq \kappa_c(a)^{\perp_c}$. We also have that $\kappa_c(a)^{\perp_c} \subseteq \Omega \setminus \kappa_c(a)$. Hence we have proven (90). Remark that (84) shows that ξ_c defines the pre-order relation on the set of states Ω in a way which is necessary for $(\Omega, \mathcal{C}, \xi_c)$ to be a state property space, while (85) shows that the classical Cartan map is indeed the Galois inverse of the function ξ_c which has to be the case if $(\Omega, \mathcal{C}, \xi_c)$ is a state property space. With (86) we prove that the classical Cartan map indeed defines the pre-order relation on the set of properties. Hence we have proven that $(\Omega, \mathcal{C}, \xi_c)$ is a state property space. That axiom 1 is satisfied for the state property space $(\Omega, \mathcal{C}, \xi_c)$ is proven by (87). Consider now an arbitrary $a \in \mathcal{C}$. From the definition of $\mathcal{C}$ follows that there exists $(a_i)_i \subseteq \mathcal{K}$ such that for an arbitrary $p \in \Sigma$ we have $a_i \in \xi(p)\ \forall\ i \Leftrightarrow a \in \xi(p)$. Hence $p \in \kappa(a_i)\ \forall\ i \Leftrightarrow p \in \kappa(a)$. This means that $\kappa(a) = \cap_i \kappa(a_i)$. This proves that axiom 2 is satisfied for the state property space $(\Omega, \mathcal{C}, \xi_c)$. Let us prove now that axiom 3 is satisfied for $(\Omega, \mathcal{C}, \xi_c)$. Consider $a \in \mathcal{C}$. From (89) we know that $\kappa(a^\perp) = \Sigma \setminus \kappa(a)$. Since axiom 3 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ there exists an ortho test α, such that α tests a and $\widetilde{\alpha}$ tests $a^\perp$. Since for $p \in \Sigma$ we have $p \in \kappa(a)$, and then α is true, or $p \in \kappa(a^\perp)$ and then $\widetilde{\alpha}$ is true, it follows that α is a classical test. This proves (91). Using proposition 24 it follows that α is a classical ortho test, which proves that axiom 3 is satisfied for $(\Omega, \mathcal{C}, \xi_c)$. □

Proposition 25. *Suppose that $(\Sigma, \mathcal{L}, \xi)$ is a state property space describing* *an entity S for which axioms 1, 2 and 3 are satisfied. For $p \in \Sigma$ and $a \in \mathcal{L}$ we have*

$$a \in \mathcal{C} \Leftrightarrow \kappa(a) \cup \kappa(a^\perp) = \Sigma \tag{92}$$

Proof: Suppose that a is a classical property. From (91) follows that a is a basic classical property, and hence there exists a classical test α such that α tests a. Consider the ortho test β that tests a. We have that $\alpha \approx \beta$ and hence $\widetilde{\alpha} \approx \widetilde{\beta}$. This means that $\widetilde{\alpha}$ tests $a^\perp$. Since α is true or $\widetilde{\alpha}$ is true we have $a \in \xi(p)$ or $a^\perp \in \xi(p)$ for an arbitrary $p \in \Sigma$, and hence $\Sigma = \kappa(a) \cup \kappa(a^\perp)$. Suppose now that $\Sigma = \kappa(a) \cup \kappa(a^\perp)$. This means that for an arbitrary $p \in \Sigma$ we have $a \in \xi(p)$ or $a^\perp \in \xi(p)$, and consider the ortho test α testing a such that $\widetilde{\alpha}$ tests $a^\perp$. For this ortho test we have that α is true or $\widetilde{\alpha}$ is true, which proves that α is a classical test. Hence $a \in \mathcal{K} = \mathcal{C}$. □

We can prove that the classical state property system $(\Omega, \mathcal{C}, \xi_c)$ is isomorphic to the canonical state property system $(\Omega, \mathcal{P}(\Omega), Id)$.

Theorem 10. *$\kappa_c : \mathcal{C} \rightarrow \mathcal{P}(\Omega)$ is an isomorphism.*

Proof: From (86) it follows that κ_c is an injective function. Let us prove that κ_c is a surjective function. Take an arbitrary element $A \in \mathcal{P}(\Omega)$. Consider the property

$$a = \bigwedge_{\kappa_c(\omega(p)) \subseteq \Omega \setminus A} \omega(p)^{\perp_c} \tag{93}$$

We have

$$\kappa_c(a) = \kappa_c(\bigwedge_{\kappa_c(\omega(p)) \subseteq \Omega \setminus A} \omega(p)^{\perp_c}) = \bigcap_{\kappa_c(\omega(p)) \subseteq \Omega \setminus A} \kappa_c(\omega(p)^{\perp_c}) = \bigcap_{\kappa_c(\omega(p)) \subseteq \Omega \setminus A} \kappa_c(\Omega \setminus \omega(p)) \tag{94}$$

$$= \Omega \setminus \bigcup_{\kappa_c(\omega(p)) \subseteq \Omega \setminus A} \kappa_c(\omega(p)) = \Omega \setminus (\Omega \setminus A) = A \tag{95}$$

□

Let us consider again our two archetypical examples. For the state property system $(\Omega, \mathcal{P}(\Omega), Id)$ corresponding to a classical physical system, we have that classical state property system coincides with this state property system. This shows that in the case of a classical physical system our construction comes out as it should be, the classical state property system is the state property system of this classical physics system. For the state property system $(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}), \xi_\mathcal{H})$ of a quantum physical system, the classical state property system $(\Omega, \mathcal{C}, \kappa_\mathcal{C})$ is the following: $\Omega = \{\mathcal{H}\}$, $\mathcal{C} = \{0, \mathcal{H}\}$, $\kappa_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{P}(\Omega)$, such that $\kappa_\mathcal{C}(0) = \emptyset$ and $\kappa_\mathcal{C}(\mathcal{H}) = \{\mathcal{H}\}$. This classical state property system describes the aspect of the quantum physical system which has to do with the properties 'the system is present' and 'the system is not present', properties that even for quantum systems are classical properties. In the next chapter we decompose an arbitrary state property system into its non-classical components and its classical state space. This structure shows us how we can describe a general situation.

5.2 The non classical components of a state property space

In this section we study the description of a physical entity whenever it is in a classical state. This leads to the existence of a non classical property space for each classical state describing the non classical elements of the entity. Consider a classical state $\omega \in \Omega$. Then $\omega \in \mathcal{C}$ is also a classical property. Hence there exists a classical test testing ω.

Definition 29 (ω-test)**.** *A classical test testing the classical state $\omega \in \Omega$ is called a ω-test, and we denote it α_ω.*

Definition 30 (ω-inverse)**.** *Consider a test α and the product test $\alpha \cdot \alpha_\omega$. We define $\widetilde{\alpha \cdot \alpha_\omega}^\omega = \widetilde{\alpha} \cdot \alpha_\omega$ and call $\widetilde{\alpha \cdot \alpha_\omega}^\omega$ the ω-inverse of $\alpha \cdot \alpha_\omega$.*

Proposition 26. *We have*

$$\widetilde{\widetilde{\alpha \cdot \alpha_\omega}^\omega}^\omega = \alpha \cdot \alpha_\omega \tag{96}$$

such that the operation is an inverse operation on the set of tests of the form $\alpha \cdot \alpha_\omega$.

Proof: We have $\widetilde{\widetilde{\alpha \cdot \alpha_\omega}^\omega}^\omega = \widetilde{\widetilde{\alpha} \cdot \alpha_\omega}^\omega = \widetilde{\widetilde{\alpha}} \cdot \alpha_\omega = \alpha \cdot \alpha_\omega$. □

Let us explain the physical meaning of this. Suppose we consider a typical classical property ω in standard quantum mechanics, for example the property 'the neutron is there', in case the entity we are considering is a neutron. The test α_ω consists of verifying whether the neutron is there, for example by absorbing it on a detection screen. In general such a verification of the presence of the neutron destroys the neutron, which means that if we want to test another property, this time a non classical property of the neutron, we need to make recourse to the product test. And hence indeed, when we test the quantum test α, for example the spin of the neutron, then actually we perform the test $\alpha \cdot \alpha_\omega$. We test whether the neutron is there 'and' whether it has spin in a certain direction, by making sure that which ever of the two tests α_ω or α we perform, the outcome will be 'yes'. But we do not have to perform both tests together, it is sufficient to perform one 'or' the other. We are in a similar situation as the one with the piece of wood tested to burn well 'and' float on water, by performing one of the both tests.

Definition 31 (ω-orthogonality)**.** *Consider two states $p, q \in \Sigma$ such that $\omega \in \xi(p) \cap \xi(q)$ where $\omega \in \Omega$. We say that p and q are ω-orthogonal, and denote $p \perp_\omega q$, if there exists a test α such that $\alpha \cdot \alpha_\omega$ is true if the entity is in state p and $\widetilde{\alpha \cdot \alpha_\omega}^\omega$ is true if the entity is in state q.*

Proposition 27. *For $x \in \mathcal{L}$ and $a \in \mathcal{C}$ we have*

$$x = (x \wedge a) \vee (x \wedge a^\perp) \tag{97}$$
$$\kappa(x) = \kappa(x \wedge a) \cup \kappa(x \wedge a^\perp) \tag{98}$$

Proof: Since $x \wedge a \leq x$ and $x \wedge a^\perp \leq x$ we have $(x \wedge a) \vee (x \wedge a^\perp) \leq x$. Since $a \in \mathcal{C}$ we have $\kappa(a) \cup \kappa(a^\perp) = \Sigma$. This gives $\kappa(x) = \kappa(x) \cap (\kappa(a) \cup \kappa(a^\perp)) = (\kappa(x) \cap \kappa(a)) \cup (\kappa(x) \cap \kappa(a^\perp)) = \kappa(x \wedge a) \cup \kappa(x \wedge a^\perp) \subseteq \kappa((x \wedge a) \vee (x \wedge a^\perp))$. This proves (97) and (98). $\square$

Proposition 28. *For $x, y \in \mathcal{L}$ and $a \in \mathcal{C}$ such that $x \leq a$ and $y \leq a^\perp$ we have*

$$(x \vee y)^\perp = (x^\perp \wedge a) \vee (y^\perp \wedge a^\perp) \tag{99}$$
$$(x \vee y) \wedge a = x \tag{100}$$

Proof: We have $a^\perp \leq x^\perp$ and $a \leq y^\perp$. From this it follows that $y^\perp \wedge a^\perp \leq x^\perp$ and $x^\perp \wedge a \leq y^\perp$. This implies that $x^\perp \wedge y^\perp \wedge a^\perp = y^\perp \wedge a^\perp$ and $x^\perp \wedge y^\perp \wedge a = x^\perp \wedge a$. Since $a \in \mathcal{C}$ we have $x^\perp \wedge y^\perp = (x^\perp \wedge y^\perp \wedge a) \vee (x^\perp \wedge y^\perp \wedge a^\perp)$. So $x^\perp \wedge y^\perp = (x^\perp \wedge a) \vee (y^\perp \wedge a^\perp)$. Hence $x \vee y = (x \vee a^\perp) \wedge (y \vee a)$. But then $(x \vee y) \wedge a = (x \vee a^\perp) \wedge a$. We know that $x^\perp = (x^\perp \wedge a) \vee (x^\perp \wedge a^\perp) = (x^\perp \wedge a) \vee a^\perp$. Hence $x = (x \vee a^\perp) \wedge a$. This proves that $(x \vee y) \wedge a = x$. $\square$

Proposition 29. *For $x, x_i \in \mathcal{L}$ and $a \in \mathcal{C}$ we have*

$$a \wedge (\vee_i x_i) = \vee_i (a \wedge x_i) \tag{101}$$
$$a = (a \wedge x) \vee (a \wedge x^\perp) \tag{102}$$

Proof: We have $a \wedge (\vee_i x_i) = a \wedge (\vee_i((x_i \wedge a) \vee (x_i \wedge a^\perp)) = a \wedge (\vee_i(x_i \wedge a) \vee \vee_i(x_i \wedge a^\perp)) = \vee_i(x_i \wedge a)$. We have $a = a \wedge (x \vee x^\perp)$. From (101) it follows that $a \wedge (x \vee x^\perp) = (a \wedge x) \vee (a \wedge x^\perp)$, which proves (102). $\square$

Proposition 30. *For $a \in \mathcal{L}$ we have*

$$a = \bigvee_{\omega \in \Omega} (a \wedge \omega) \tag{103}$$

$$\kappa(a) = \bigcup_{\omega \in \Omega} \kappa(a \wedge \omega) \tag{104}$$

with

$$a \wedge \omega \perp a \wedge \omega' \quad \text{and} \quad \kappa(a \wedge \omega) \cap \kappa(a \wedge \omega') = \emptyset \quad \text{for} \quad \omega \neq \omega' \tag{105}$$

Proof: We have that $a \wedge \omega \leq a \; \forall \omega \in \Omega$, hence $\kappa(a \wedge \omega) \subseteq \kappa(a) \; \forall \; \omega \in \Omega$, and as a consequence $\cup_{\omega \in \Omega} \kappa(a \wedge \omega) \subseteq \kappa(a)$. Consider $p \in \kappa(a)$. We have $p \in \kappa(\omega(p))$, and hence $p \in \kappa(a) \cap \kappa(\omega(p)) = \kappa(a \wedge \omega(p)) \subseteq \cup_{\omega \in \Omega} \kappa(a \wedge \omega)$. So we have shown that $\kappa(a) \subseteq \cup_{\omega \in \Omega} \kappa(a \wedge \omega)$. This proves (104), namely $\kappa(a) = \cup_{\omega \in \Omega} \kappa(a \wedge \omega)$. We have that $a \wedge \omega \leq a \; \forall \omega \in \Omega$, hence $\vee_{\omega \in \Omega}(a \wedge \omega) \leq a$. Consider $p \in \kappa(a)$. We have $p \in \cup_{\omega \in \Omega} \kappa(a \wedge \omega) \subseteq \kappa(\vee_{\omega \in \Omega}(a \wedge \omega))$. So we have shown that $\kappa(a) \subseteq \kappa(\vee_{\omega \in \Omega}(a \wedge \omega))$. From this it follows that $a \leq \vee_{\omega \in \Omega}(a \wedge \omega)$, which proves (103), namely $a = \vee_{\omega \in \Omega}(a \wedge \omega)$. Consider $\omega \neq \omega'$, then we have $\omega \leq \omega'^{\perp}$. As a consequence $a \wedge \omega \leq \omega'^{\perp} \leq a^{\perp} \vee \omega'^{\perp} = (a \wedge \omega')^{\perp}$, which proves that $a \wedge \omega \perp a \wedge \omega'$. From this it follows that $\kappa(a \wedge \omega) \cap \kappa(a \wedge \omega') = \emptyset$. □

Corollary 1. *We have*

$$\Sigma = \bigcup_{\omega \in \Omega} \kappa(\omega) \tag{106}$$

with

$$\kappa(\omega) \cap \kappa(\omega') = \emptyset \quad \text{for} \quad \omega \neq \omega' \tag{107}$$

Proposition 31. *Consider a_ω such that $a_\omega \leq \omega \; \forall \; \omega \in \Omega$. We have*

$$\kappa(\bigvee_{\omega \in \Omega} a_\omega) = \bigcup_{\omega \in \Omega} \kappa(a_\omega) \tag{108}$$

with

$$\kappa(a_\omega) \cap \kappa(a_{\omega'}) = \emptyset \quad \text{for} \quad \omega \neq \omega' \tag{109}$$

Proof: We have $\kappa(\vee_{\omega \in \Omega} a_\omega) = \cup_{\omega' \in \Omega} \kappa((\vee_{\omega \in \Omega} a_\omega) \wedge \omega')$. From (100) it follows that $(\vee_{\omega \in \Omega} a_\omega) \wedge \omega' = a_{\omega'}$. Hence $\kappa(\vee_{\omega \in \Omega} a_\omega) = \cup_{\omega' \in \Omega} \kappa(a_{\omega'})$. This proves (108). □

Let us now investigate the nonclassical parts of the state property system $(\Sigma, \mathcal{L}, \kappa)$.

Definition 32 (Nonclassical components)**.** *Suppose that* $(\Sigma, \mathcal{L}, \xi)$ *is the state property space of an entity satisfying axioms 1 and 2. For* $\omega \in \Omega$ *we introduce*

$$\Sigma_\omega = \{p \mid \omega \in \xi(p), p \in \Sigma\} \tag{110}$$
$$\mathcal{L}_\omega = \{a \mid a \leq \omega, a \in \mathcal{L}\} \tag{111}$$
$$\xi_\omega(p) = \xi(p) \cap \mathcal{L}_\omega \tag{112}$$

and we call $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *the nonclassical components of* $(\Sigma, \mathcal{L}, \xi)$ *corresponding to* ω*. We also introduce Cartan map corresponding to* ω

$$\kappa_\omega(a) = \{p \mid p \in \Sigma_\omega, a \in \xi_\omega(p)\} \tag{113}$$

Theorem 11. *Consider* $(\Sigma, \mathcal{L}, \xi)$ *the state property space of an entity satisfying axioms 1, 2 and 3. For* $a, b \in \mathcal{L}_\omega$*,* $(a_i)_i \subseteq \mathcal{L}_\omega$ *and* $p, q \in \Sigma_\omega$ *we have*

$$a \in \xi_\omega(p) \Leftrightarrow a \in \xi(p) \tag{114}$$
$$p \leq q \Leftrightarrow \xi_\omega(q) \subseteq \xi_\omega(p) \tag{115}$$
$$p \in \kappa_\omega(a) \Leftrightarrow p \in \kappa(a) \tag{116}$$
$$a \leq b \Leftrightarrow \kappa_\omega(a) \subseteq \kappa_\omega(b) \tag{117}$$
$$\kappa_\omega(a) = \kappa_\omega(b) \Rightarrow a = b \tag{118}$$
$$\kappa_\omega(\wedge_i a_i) = \cap_i \kappa_\omega(a_i) \tag{119}$$
$$p \perp_\omega q \Leftrightarrow p \perp q \tag{120}$$
$$\kappa_\omega(a^{\perp_\omega}) = \kappa(a^\perp \wedge \omega) = \kappa_\omega(a)^{\perp_\omega} \tag{121}$$

and $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *is a state property space that satisfies axioms 1, 2 and 3.*

Proof: Suppose that $a \in \xi_\omega(p)$. This means that $a \in \xi(p) \cap \mathcal{L}_\omega$, and hence $a \in \xi(p)$. Suppose now that $a \in \xi(p)$. Since $a \in \mathcal{L}_\omega$ we have $a \in \xi(p) \cap \mathcal{L}_\omega = \xi_\omega(p)$. This proves (114). Suppose that $p \leq q$, then we have $\xi(q) \subseteq \xi(p)$, and hence $\xi_\omega(q) = \xi(q) \cap \mathcal{L}_\omega \subseteq \xi(p) \cap \mathcal{L}_\omega = \xi_\omega(p)$. Suppose next that $\xi_\omega(q) \subseteq \xi_\omega(p)$, and consider $a \in \xi(q)$. Applying (114) this gives that $a \in \xi_\omega(q)$ and hence $a \in \xi_\omega(p)$. Applying again (114) this gives that $a \in \xi(p)$. Hence we have proven that $\xi(q) \subseteq \xi(p)$ and hence $p \leq q$. This proves (115). We have $p \in \kappa_\omega(a) \Leftrightarrow a \in \xi_\omega(a) \Leftrightarrow a \in \xi(p) \Leftrightarrow p \in \kappa(a)$. This proves (116). Suppose that $a \leq b$ and consider $p \in \kappa_\omega(a)$. Applying (116) this gives $p \in \kappa(a)$ and since $\kappa(a) \subseteq \kappa(b)$ this gives $p \in \kappa(b)$. Applying again (116) this gives $p \in \kappa_\omega(b)$. Hence we have proven that $\kappa_\omega(a) \subseteq \kappa_\omega(b)$. Suppose now that $\kappa_\omega(a) \subseteq \kappa_\omega(b)$, and consider $p \in \kappa(a)$. Applying (116) this gives $p \in \kappa_\omega(a)$ and hence $p \in \kappa_\omega(b)$. Applying again (116) this gives $p \in \kappa(b)$.

So we have proven that $\kappa(a) \subseteq \kappa(b)$ and from this follows that $a \leq b$. This proves (117). Suppose that $\kappa_\omega(a) = \kappa_\omega(b)$ and consider $p \in \kappa(a)$. Then we have $p \in \kappa_\omega(a)$ and hence $p \in \kappa_\omega(b)$. From this follows that $p \in \kappa(b)$. This means that we have proven that $\kappa(a) \subseteq \kappa(b)$. Since axiom 1 is satisfied for the state property space $(\Sigma, \mathcal{L}, \xi)$ we have $a = b$. This proves (118). We have $p \in \kappa_\omega(\wedge_i a_i) \Leftrightarrow p \in \kappa(\wedge_i a_i) \Leftrightarrow p \in \cap_i \kappa(a_i) \Leftrightarrow p \in \kappa(a_i)\ \forall\ i \Leftrightarrow p \in \kappa_\omega(a_i)\ \forall\ i \Leftrightarrow p \in \cap_i \kappa_\omega(a_i)$. This proves (119). Suppose that $p \perp_\omega q$. This means that there exists a test α such that $\alpha \cdot \alpha_\omega$ is true if the entity is in state p and $\widetilde{\alpha \cdot \alpha_\omega}^\omega$ is true if the entity is in state q. We have $\widetilde{\alpha \cdot \alpha_\omega}^\omega = \widetilde{\alpha} \cdot \alpha_\omega$. This means that α is true if the entity is in state p and $\widetilde{\alpha}$ is true if the entity is in state q. Hence $p \perp q$. Suppose now that $p \perp q$. This means that there exists a test α such that α is true if the entity is in state p and $\widetilde{\alpha}$ is true if the entity is in state q. Since $p, q \in \Sigma_\omega$ we have that ω is actual and hence α_ω is true. Hence $\alpha \cdot \alpha_\omega$ is true if the entity is in state p and $\widetilde{\alpha} \cdot \alpha_\omega = \widetilde{\alpha \cdot \alpha_\omega}^\omega$ is true if the entity is in state q. This means that $p \perp_\omega q$. This proves (120). Suppose that $a \in \mathcal{L}_\omega$. In this case $a \in \mathcal{L}$ and hence there exists an ortho test α testing a. Consider the test $\alpha \cdot \alpha_\omega$ and a state q such that $q \perp_\omega a$ and $q \in \Sigma_\omega$. This means that $q \perp_\omega r\ \forall\ r$ such that $r \in \kappa_\omega(a)$. Hence $q \perp r\ \forall\ r$ such that $r \in \kappa(a)$. Hence $q \in \kappa(a)^\perp = \kappa(a^\perp)$. This means that $\widetilde{\alpha}$ is true if the entity is in state q. Since $q \in \Sigma_\omega$ we also have that α_ω is true if the entity is in state q. Hence $\widetilde{\alpha} \cdot \alpha_\omega = \widetilde{\alpha \cdot \alpha_\omega}^\omega$ is true if the entity is in state q. Note that property tested by $\widetilde{\alpha \cdot \alpha_\omega}^\omega$ is $a^\perp \wedge \omega$. Hence we have proven that $q \perp_\omega a \Rightarrow q \in \kappa_\omega(a^\perp \wedge \omega)$, or $\kappa(a)^{\perp_\omega} \subseteq \kappa_\omega(a^\perp \wedge \omega)$. Consider $r \in \Sigma_\omega$ such that $r \perp_\omega a^\perp \wedge \omega$. This means that $r \perp_\omega s\ \forall\ s$ such that $a^\perp \wedge \omega \in \xi(s)$. From this follows that $r \perp s\ \forall\ s$ such that $a^\perp \wedge \omega \in \xi(s)$. Hence $r \in \kappa(a^\perp \wedge \omega)^\perp$. Since $r \in \Sigma_\omega$ we have $r \perp \omega^\perp$, and hence $r \perp a^\perp \wedge \omega^\perp$, such that $r \in \kappa(a^\perp \wedge \omega^\perp)^\perp$. From (97) and (98) we know that $\kappa(a^\perp) = \kappa(a^\perp \wedge \omega) \cup \kappa(a^\perp \wedge \omega^\perp)$, and hence $\kappa(a) = \kappa(a^\perp)^\perp = \kappa(a^\perp \wedge \omega)^\perp \cap \kappa(a^\perp \wedge \omega^\perp)^\perp$. Hence we have $r \in \kappa(a)$. This proves that for the entity being in state r we have α is true, and hence $\alpha \cdot \alpha_\omega$ is true. This proves that $\alpha \cdot \alpha_\omega$ is an ortho test for the orthogonality relation $\perp_\omega$. We can now denote $a^\perp \wedge \omega = a^{\perp_\omega}$. And it follows that we have proven (121). Remark that from (118) follows that axiom 1 is satisfied for $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$, and from (119) follows that axiom 2 is satisfied. From (121) follows that axiom 3 is satisfied for $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. □

5.3 A decomposition theorem

To see in more detail in which way the classical and nonclassical parts are structured within the lattice $\mathcal{L}$, we need to introduce some additional structures.

Definition 33 (Direct union of state property spaces)**.** *Consider a set of state property spaces* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *that all satisfy axioms 1, 2 and 3. The direct union* $\textcircled{\vee}_\omega(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *of these state property spaces is the state property space* $(\cup_\omega\Sigma_\omega, \textcircled{\vee}_\omega\mathcal{L}_\omega, \textcircled{\vee}_\omega\xi_\omega)$*, where*
(i) $\cup_\omega\Sigma_\omega$ *is the disjoint union of the sets* Σ_ω
(ii) $\textcircled{\vee}_\omega\mathcal{L}_\omega$ *is the direct union of the lattices* $\mathcal{L}_\omega$*, which means the set of sequences* $a = (a_\omega)_\omega$*, such that*

$$(a_\omega)_\omega \leq (b_\omega)_\omega \Leftrightarrow a_\omega \leq b_\omega \ \forall \omega \in \Omega \tag{122}$$

$$(a_\omega)_\omega \wedge (b_\omega)_\omega = (a_\omega \wedge b_\omega)_\omega \tag{123}$$

$$(a_\omega)_\omega \vee (b_\omega)_\omega = (a_\omega \vee b_\omega)_\omega \tag{124}$$

$$(a_\omega)_\omega^\perp = (a_\omega^{\perp_\omega})_\omega \tag{125}$$

(iii) $\textcircled{\vee}_\omega\xi_\omega$ *is defined as follows:*

$$\textcircled{\vee}_\omega\xi_\omega : \cup_\omega\Sigma_\omega \rightarrow \mathcal{P}(\textcircled{\vee}_\omega\mathcal{L}_\omega) \tag{126}$$

$$p_{\omega'} \mapsto \{(a_\omega)_\omega \mid a_{\omega'} \in \xi_{\omega'}(p_{\omega'}), a_\omega \in \mathcal{L}_\omega \ \forall \ \omega \neq \omega'\} \tag{127}$$

and hence the corresponding Cartan map is the following

$$\textcircled{\vee}_\omega\kappa_\omega : \textcircled{\vee}_\omega\mathcal{L}_\omega \rightarrow \cup_\omega\Sigma_\omega \tag{128}$$

$$(a_\omega)_\omega \mapsto \cup_{\omega\in\Omega}\kappa_\omega(a_\omega) \tag{129}$$

We remark that if $\mathcal{L}_\omega$ are complete orthocomplemented lattices, then also $\textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega$ is a complete orthocomplemented lattice. A fundamental decomposition theorem can now be proven.

Theorem 12 (Decomposition theorem)**.** *Consider the state property space* $(\Sigma, \mathcal{L}, \xi)$*, and suppose that axioms 1, 2 and 3 are satisfied. Then*

$$(\Sigma, \mathcal{L}, \xi) \cong \textcircled{\vee}_{\omega\in\Omega}(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega) \tag{130}$$

where Ω *is the set of classical states of* $(\Sigma, \mathcal{L}, \xi)$*,* Σ_ω *is the set of states and* $\mathcal{L}_\omega$ *the lattice of properties of the nonclassical component state property space* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$*.*

Proof: We use the notion of orthomorphism of state property systems, and need to prove that there exists an isomorphism of ortho state property systems between $(\Sigma, \mathcal{L}, \xi)$ and $\textcircled{\vee}_{\omega\in\Omega}(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. From (106) it follows that m can be defined in the following way:

$$m : \Sigma \rightarrow \cup_{\omega\in\Omega}\Sigma_\omega \tag{131}$$

$$p \mapsto p \tag{132}$$

The function n is defined in the following way:

$$n : \textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega \rightarrow \mathcal{L} \quad (133)$$

$$(a_\omega)_\omega \mapsto \vee_{\omega\in\Omega} a_\omega \quad (134)$$

The function m is a bijection by definition. Consider $(a_\omega)_\omega, (b_\omega)_\omega \in \textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega$ and suppose that $n((a_\omega)_\omega) = n((b_\omega)_\omega)$, hence $\vee_{\omega\in\Omega} a_\omega = \vee_{\omega\in\Omega} b_\omega$. Then $(\vee_{\omega\in\Omega} a_\omega) \wedge \omega' = (\vee_{\omega\in\Omega} b_\omega) \wedge \omega' \ \forall\ \omega' \in \Omega$. From (100) it follows that $(\vee_{\omega\in\Omega} a_\omega) \wedge \omega' = a_{\omega'}$ and $(\vee_{\omega\in\Omega} b_\omega) \wedge \omega' = b_{\omega'}$. Hence $a_{\omega'} = b_{\omega'}\ \forall\ \omega' \in \Omega$. As a consequence we have $(a_\omega)_\omega = (b_\omega)_\omega$. This proves that n is injective. Let us prove that n is surjective. Consider an arbitrary element $a \in \mathcal{L}$. From (103) it follows that $a = \vee_{\omega\in\Omega}(a \wedge \omega)$. Consider the element $(a \wedge \omega)_\omega \in \textcircled{\vee}_{\omega\in\Omega}\mathcal{L}_\omega$. Then $n((a \wedge \omega)_\omega) = a$ which proves that n is surjective. Hence we have proven that m as well as n are bijections. Let us show that we have an orthomorphism. We need to prove (67) and (72) hence:

$$m(p) \in \textcircled{\vee}_\omega \kappa_\omega((a_\omega)_\omega) \Leftrightarrow p \in \kappa(\vee_{\omega\in\Omega} a_\omega) \quad (135)$$

$$m(p) \in \textcircled{\vee}_\omega \kappa_\omega((a_\omega)^\perp_\omega) \Leftrightarrow p \in \kappa((\vee_{\omega\in\Omega} a_\omega)^\perp) \quad (136)$$

Let us calculate $\textcircled{\vee}_\omega \kappa_\omega((a_\omega)_\omega) = \cup_{\omega\in\Omega}\kappa_\omega(a_\omega) = \cup_{\omega\in\Omega}\kappa(a_\omega)$. On the other hand we have $\kappa(n((a_\omega)_\omega)) = \kappa(\vee_{\omega\in\Omega} a_\omega)$, and following (108), we have $\kappa(\vee_{\omega\in\Omega} a_\omega) = \cup_{\omega\in\Omega}\kappa(a_\omega)$. This means that (135) is satisfied. We have $\textcircled{\vee}_\omega \kappa_\omega((a_\omega)^\perp_\omega) = \textcircled{\vee}_\omega \kappa_\omega((a^{\perp_\omega}_\omega)_\omega) = \cup_\omega \kappa_\omega(a^{\perp_\omega}_\omega) = \cup_\omega \kappa_\omega(a^\perp \wedge \omega) = \kappa(\vee_{\omega\in\Omega}(a^\perp \wedge \omega)) = \kappa(a^\perp) = \kappa((\vee_{\omega\in\Omega} a_\omega)^\perp)$. This proves (136). Hence we have proven that (m, n) is an isomorphism of ortho state property spaces. □

6 Additional axioms

In the foregoing we have introduced three axioms. If these three axioms are satisfied we can decompose the state property space of an entity into its classical state property space such that for each classical state there is an underlying non classical property space describing the entity being in this classical state. We have proven that the classical state property space is isomorphic to the property space of classical physics. The underlying non classical state property spaces are however not necessarily isomorphic to the state property space of quantum mechanics. To make these underlying non classical state property spaces isomorphic to the state property space of quantum mechanics we need to introduce additional axioms.

6.1 The axiom of atomisticity

The axiom of property determination makes the pre-order relation on the set of properties $\mathcal{L}$ of a state property space into a partial order relation. The pre-order relation existing on the set of states is not necessarily a partial order relation. This means that it is possible for two states $p, q \in \Sigma$ to be different states even if $\xi(p) = \xi(q)$, which means that the properties which are actual if the entity is in state p are the same as the properties which are actual if the entity is in state q. The next axiom we introduce makes the pre-order relation on Σ into a trivial order, i.e. $p \leq q$ iff $p = q$.

Axiom 4 (Atomisticity)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1, 2 and 3 are satisfied. We say that the axiom of atomisticity is satisfied if for $p, q \in \Sigma$ we have*

$$\xi(q) \subseteq \xi(p) \Rightarrow p = q \tag{137}$$

Definition 34. *Consider a partially ordered set $Z, \leq$. We say that $s \in Z$ is an atom, if whenever $0 \leq a \leq s$ we have $a = 0$ ot $a = s$. A lattice $Z, \leq$, is atomistic, if there exists a set of atoms $\mathcal{A}$ which is ordering. This means that for $x, y \in Z$ we have $x \leq y \Leftrightarrow \{s \mid s \in \mathcal{A}, s \leq x\} \subseteq \{s \mid s \in \mathcal{A}, s \leq y\}$.*

Theorem 13. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1, 2, 3 and 4 are satisfied. For $p, q \in \Sigma$ and $a, b \in \mathcal{L}$ we have*

$$p \leq q \quad \Rightarrow \quad p = q \tag{138}$$

$$0 \leq a \leq s(p) \quad \Rightarrow \quad a = 0 \text{ or } a = s(p) \tag{139}$$

$$a \leq b \quad \Leftrightarrow \quad \{s(p) \mid s(p) \leq a\} \subseteq \{s(p) \mid s(p) \leq b\} \tag{140}$$

and $\mathcal{L}$ is an atomistic lattice with set of atoms the set of state properties $\mathcal{A} = \{s(p) \mid p \in \Sigma\}$.

Proof: Suppose $p \leq q$, then we have $\xi(q) \subseteq \xi(p)$, and hence $p = q$. This proves (138). Consider a such that $0 \leq a \leq s(p) = \wedge_{b \in \xi(p)} b$. If $a \neq 0$ there exists $q \in \Sigma$ such that $a \in \xi(q)$. Hence we have $s(p) \in \xi(q)$, and as a consequence we have $b \in \xi(q)\ \forall\ b \in \xi(p)$. Hence $\xi(p) \subseteq \xi(q)$, and hence $p = q$. This implies that $s(p) = s(q)$. Since $s(q) \leq a$ we have $a = s(p)$. So we have proven (139), which means that $s(p)$ is an atom of $\mathcal{L}$. If $a \leq b$ we obviously have that $\{s(p) \mid s(p) \leq a\} \subseteq \{s(p) \mid s(p) \leq b\}$. Suppose that $\{s(p) \mid s(p) \leq a\} \subseteq \{s(p) \mid s(p) \leq b\}$. From (39) follows that $a = \vee_{s(p) \leq a} s(p) \leq \vee_{s(p) \leq b} s(p) = b$, Hence we have proven (140), which means that $\mathcal{A}$ is an ordering set for $\mathcal{L}$, and hence $\mathcal{L}$ is a complete orthocomplemented atomistic lattice. □

Proposition 32. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2 and 3 are satisfied. The classical state property space* $(\Omega, \mathcal{C}, \xi_c)$ *corresponding to* $(\Sigma, \mathcal{L}, \xi)$ *satisfies axiom 4. If* $(\Sigma, \mathcal{L}, \xi)$ *satisfies also axiom 4, then each non classical component* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *of the decomposition of* $(\Sigma, \mathcal{L}, \xi)$ *satisfies axiom 4.*

Proof: Consider $\omega(p), \omega(q) \in \Omega$ such that $\xi_c(\omega(q)) \subset \xi_c(\omega(p))$. From (84) follows that then $\omega(p) \leq \omega(q)$, and hence $\omega(p) \wedge \omega(q) = \omega(p)$. Suppose now that $\omega(p) \neq \omega(q)$, then from (82) follows that $\omega(p) \perp \omega(q)$. But then $\omega(p) \wedge \omega(q) = 0$, which would lead to $\omega(p) = 0$. This is not possible, and hence this proves that $\omega(p) = \omega(q)$. Hence we have proven that $(\Omega, \mathcal{C}, \kappa_c)$ satisfies axiom 4. Suppose now that axiom 4 is satisfied for $(\Sigma, \mathcal{L}, \xi)$, and consider $p, q \in \Sigma_\omega$ such that $\xi_\omega(q) \subseteq \xi_\omega(p)$. From (115) follows then that $p \leq q$ and hence $\xi(q) \subseteq \xi(p)$. Since axiom 4 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ we have that $p = q$. Hence we have proven that axiom 4 is satisfied for $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. $\square$

Proposition 33. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2, 3 and 4 are satisfied. A property* $a \in \mathcal{L}$ *is classical, hence* $a \in \mathcal{C}$*, iff* a *is a central element of the lattice* $\mathcal{L}$*, i.e.* $x = (x \wedge a) \vee (x \wedge a^\perp)\ \forall\ x \in \mathcal{L}$*. The lattice of properties* $\mathcal{L}_\omega$ *of a non classical component property space* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *is an irreducible lattice.*

Proof: From (97) follows that a classical property a is a central element of the lattice $\mathcal{L}$. Consider a central element a of the lattice $\mathcal{L}$, and an arbitrary state $p \in \Sigma$. Because axiom 4 is satisfied we have that $s(p)$ is an atom of $\mathcal{L}$. We have $s(p) \wedge a \leq s(p)$ and $s(p) \wedge a^\perp \leq s(p)$ and hence $s(p) \wedge a = s(p)$ or $s(p) \wedge a = 0$, and $s(p) \wedge a^\perp = s(p)$ or $s(p) \wedge a^\perp = 0$. Since a is a central element of $\mathcal{L}$ we have $s(p) = (s(p) \wedge a) \vee (s(p) \wedge a^\perp)$, and hence we cannot have $s(p) \wedge a = 0$ and $s(p) \wedge a^\perp = 0$, which means that at least one of $s(p) \wedge a = s(p)$ or $s(p) \wedge a^\perp = s(p)$ is true. From this follows that $s(p) \leq a$ or $s(p) \leq a^\perp$. Hence $a \in \xi(p)$ or $a^\perp \in \xi(p)$. Since axiom 3 is satisfied there exists an ortho test α testing a and hence $\widetilde{\alpha}$ testing $a^\perp$. From the foregoing follows that this ortho test is a classical test, and hence a is a classical property. Consider a central element $a \in \mathcal{L}_\omega$ of the lattice of properties of a non classical component $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. Since $a \in \mathcal{C}$, we have $a = 0$ or $a = \omega$ which proves that $\mathcal{L}_\omega$ is irreducible. $\square$

6.2 The axiom of weak modularity

If we consider a closed subspace $A \in \mathcal{L}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$. The closed subspace A in itself is a Hilbert space. This means that we can consider the set $\mathcal{L}(A)$ of closed subspaces contained in A. One can prove that $\mathcal{L}(A)$ is a complete orthocomplemented lattice. The relative orthocomplementation $B^{\perp_A}$ for $B \in \mathcal{L}(A)$ is defined as follows: $B^{\perp_A} = B^\perp \cap A$. An important equality for $\perp_A$ to be an orthocomplementaion is the following: $(B^{\perp_A})^{\perp_A} = B$. This gives $B = (B^\perp \cap A)^\perp \cap A$ or $B = (B \vee A^\perp) \wedge A$. This is the way the requirement of 'weak modularity' is usually introduced, hence more specifically: for $B, A \in \mathcal{L}$ and $B \leq A$ we have $B = (B \vee A^\perp) \wedge A$. We however want to introduce 'weak modularity' in an operational way. To formulate the following axiom we first introduce the idea of relative state property space.

Proposition 34 (Relative state property space)**.** *Suppose we have a state property space $(\Sigma, \mathcal{L}, \xi)$ and for $a \in \mathcal{L}$ we consider $(\Sigma, \mathcal{L}, \xi)_a = (\kappa(a), \mathcal{L}_a, \xi_a)$ where*

$$\mathcal{L}_a = \{b \mid b \leq a\} \tag{141}$$

$$\xi_a : \kappa(a) \rightarrow \mathcal{P}(\mathcal{L}_a) \quad p \mapsto \xi(p) \cap \mathcal{L}_a \tag{142}$$

then $(\Sigma, \mathcal{L}, \xi)_a$ is a state property space and for $b \in \mathcal{L}_a$ we have

$$\kappa_a(b) = \kappa(b) \tag{143}$$

We call $(\Sigma, \mathcal{L}, \xi)_a$ the state property space relative to property $a \in \mathcal{L}$.

Proof: Suppose the entity is in state $p \in \kappa(a)$ and the property $b \leq a$ is actual. This means that $b \in \xi(p)$. Since $b \leq a$ we have $b \in \xi(p) \cap \mathcal{L}_a = \xi_a(p)$. On the other hand, suppose we have $p \in \kappa(a)$ and $b \leq a$, and $b \in \xi_a(p)$. This means that $b \in \xi(p)$, and hence, if the entity is in state p the property b is actual. This proves that $(\Sigma, \mathcal{L}, \xi)_a$ is a state property space describing the same entity as the one described by the state property space $(\Sigma, \mathcal{L}, \xi)$. Suppose that $p \in \kappa_a(b)$, and hence $b \in \xi_a(p)$. This means that $b \in \xi(p)$, and hence $p \in \kappa(b)$. Hence we have $\kappa_a(b) \subseteq \kappa(b)$. Suppose now that $p \in \kappa(b)$ and hence $b \in \xi(p)$. Since we have $b \leq a$ we also have $b \in \mathcal{L}_a$, and hence $b \in \xi(p) \cap \mathcal{L}_a = \xi_a(p)$. From this follows that $p \in \kappa_a(b)$. Hence we have proven that $\kappa_a(b) \subseteq \kappa(b)$. As a consequence we have $\kappa_a(b) = \kappa(b)$. $\square$

The operational meaning of the relative state property space is the following. We study the entity S in the special circumstance when we manage to keep the property a actual during the study. This means concretely that we can consider a test α testing a, and hence we consider only the states $\kappa(a)$ of the entity which make this test true.

Proposition 35. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1 and 2 are satisfied, and $a \in \mathcal{L}$. Then the relative state property space $(\Sigma, \mathcal{L}, \xi)_a$ satisfies axioms 1 and 2.*

Proof: Consider $b, c \in \mathcal{L}_a$ such that $\kappa_a(b) = \kappa_a(c)$. From (143) follows that $\kappa(b) = \kappa(c)$, and since axiom 1 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ we have $b = c$. This proves that axiom 1 is satisfied for $(\Sigma, \mathcal{L}, \xi)_a$. Consider $b \in \mathcal{L}_a$. Since axiom 2 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ there exists $(b_i)_i \subseteq \mathcal{T}$ such that $\kappa(b) = \cap_i \kappa(b_i)$. Consider $\mathcal{T}_a = \{b \wedge a \mid b \in \mathcal{T}\}$. Since $b \leq a$ we have $b = b \wedge a$. Hence $\kappa(b) = \kappa(b) \cap \kappa(a)$ and as a consequence we have $\kappa(b) = \cap_i \kappa(b_i) \cap \kappa(a) = \cap_i(\kappa(b_i) \cap \kappa(a)) = \cap_i \kappa(b_i \wedge a)$. Consider now an arbitrary property $b \in \mathcal{L}_a$. Since $b \in \mathcal{L}$ there exists $(b_i)_i \subseteq \mathcal{T}$ such that $\kappa(b) = \cap_i \kappa(b_i)$. This gives that $\kappa(b) = \cap_i \kappa(b_i \wedge a)$ for $(b_i \wedge a)_i \subseteq \mathcal{T}_a$. Hence we have proven that $(\Sigma, \mathcal{L}, \xi)_a$ satisfies axiom 2. $\square$

The axiom 3 of orthocomplementation is however not necessarily satisfied by a relative state property space. The next axiom, the axiom of weak modularity, is meant to make sure that also the axiom 3 of orthocomplementation is satisfied for an arbitrary relative state property space. Before we formulate the axiom of weak modularity, let us analyze why the axiom 3 of orthocomplementation is not necessarily satisfied for a relative state property space. We start by introducing the notion of relative inverse.

Definition 35 (Relative inverse)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ and for $a \in \mathcal{L}$ the relative state property space $(\Sigma, \mathcal{L}, \xi)_a$. For a test β we consider the test $\beta \cdot \alpha$ where α is test testing the property a, and we introduce $\widetilde{\beta \cdot \alpha}^{\alpha} = \widetilde{\beta} \cdot \alpha$.*

Proposition 36. $\widetilde{\beta \cdot \alpha}^{\alpha}$ *is an inverse for tests testing properties of the relative state property space $(\Sigma, \mathcal{L}, \xi)_a$.*

Proof: Consider an arbitrary property $b \in \mathcal{L}_a$ and a test β testing b. Since $b \leq a$ we have that also $\beta \cdot \alpha$ tests property b. Obviously also $\widetilde{\beta \cdot \alpha}^{\alpha} = \widetilde{\beta} \cdot \alpha$ is a test testing a property of $\mathcal{L}_a$, and we have $\widetilde{\widetilde{\beta \cdot \alpha}^{\alpha}}^{\alpha} = \widetilde{\widetilde{\beta} \cdot \alpha}^{\alpha} = \widetilde{\widetilde{\beta}} \cdot \alpha = \beta \cdot \alpha$. This proves that $\widetilde{\beta \cdot \alpha}^{\alpha}$ defines an inverse on tests of properties of $\mathcal{L}_a$. $\square$

Definition 36 (Relative orthogonality)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$, and for $a \in \mathcal{L}$ the relative state property space $(\Sigma, \mathcal{L}, \xi)_a$. Consider a test α testing a. For $p, q \in \kappa(a)$ we say that p is relatively orthogonal to q with respect to α, and denote $p \perp_\alpha q$, if there exists a test $\beta \cdot \alpha$ such that $\beta \cdot \alpha$ is true if the entity is in state p and $\widetilde{\beta \cdot \alpha}^{\alpha}$ is true if the entity is in state q.*

Proposition 37. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$, and for $a \in \mathcal{L}$ the relative state property space $(\Sigma, \mathcal{L}, \xi)_a$, and test α testing a. For $p, q \in \kappa(a)$ we have*

$$p \perp_\alpha q \Leftrightarrow p \perp q \tag{144}$$

Proof: Suppose that $p \perp_\alpha q$. This means that there exists $\beta \cdot \alpha$ such that $\beta \cdot \alpha$ is true if the entity is in state p and $\widetilde{\beta} \cdot \alpha$ is true if the entity is in state q. This means that β is true if the entity is in state p and $\widetilde{\beta}$ is true if the entity is in state q. Hence $p \perp q$. Suppose that $p \perp q$. This means that there exists a test γ such that γ is true if the entity is in state p and $\widetilde{\gamma}$ is true if the entity is in state q. If we consider $\gamma \cdot \alpha$ and $\widetilde{\gamma} \cdot \alpha$, then, since $p, q \in \kappa(a)$ and α tests a, we have that $\gamma \cdot \alpha$ is true if the entity is in state p and $\widetilde{\gamma} \cdot \alpha$ is true if the entity is in state q. This proves that $p \perp_\alpha q$. $\square$

Proposition 38. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which 1, 2 and 3 are satisfied, and for $a \in \mathcal{L}$ the relative state property space $(\Sigma, \mathcal{L}, \xi)_a$. Axiom 3 is satisfied for the state property space $(\Sigma, \mathcal{L}, \xi)_a$ if and only if for $b \leq a$ we have $b = (b \vee a^\perp) \wedge a$.*

Proof: Consider an ortho test α testing property a and an ortho test β testing property b and $p \in \kappa(a)$ such that $p \perp_\alpha b$. From proposition 37 follows that $p \perp b$ and hence, since axiom 3 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ we have $\widetilde{\beta}$ is true if the entity is in state p. Since $p \in \kappa(a)$ we have that $\widetilde{\beta} \cdot \alpha$ is true if the entity is in state p. Hence $p \in \kappa(b^\perp \wedge a)$. On the other hand, consider $q \in \kappa(a)$ and $q \perp b^\perp \wedge a$. Since $b^\perp \wedge a \in \mathcal{L}$ there exists an ortho test γ, such that $\widetilde{\gamma}$ is true if the entity is in state q. This means that $q \in \kappa((b^\perp \wedge a)^\perp \wedge a))$. The test $\widetilde{\gamma} \cdot \alpha$ will test b, and hence be an ortho test for the relative inverse with respect to a, if and only if $\kappa(b) = \kappa((b^\perp \wedge a)^\perp \wedge a))$. This is equivalent to $b = (b^\perp \wedge a)^\perp \wedge a) = (b \vee a^\perp) \wedge a$. $\square$

Axiom 5 (Weak modularity)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1, 2 and 3 are satisfied. We say that the axiom of weak modularity is satisfied if for $a, b \in \mathcal{L}$ we have*

$$b \leq a \quad \Rightarrow \quad b = (b \vee a^\perp) \wedge a \tag{145}$$

Operationally this axiom means the following. Consider two properties $b \leq a$, an ortho test β testing b, and a test α testing a. Then $\widetilde{\beta} \cdot \alpha$ tests $b^\perp \wedge a$. Consider an ortho test $\gamma \approx \widetilde{\beta} \cdot \alpha$, then also γ tests $b^\perp \wedge a$. The test $\widetilde{\gamma} \cdot \alpha$ tests the property $(b^\perp \wedge a)^\perp \wedge a$. The axiom of weak modularity means that we want $\widetilde{\gamma} \cdot \alpha$ also to test the property b. Hence we want $\widetilde{\gamma} \cdot \alpha \approx \beta$.

Proposition 39. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2 and 3 are satisfied. The classical state property space* $(\Omega, \mathcal{C}, \xi_c)$ *corresponding to* $(\Sigma, \mathcal{L}, \xi)$ *satisfies axiom 5. If* $(\Sigma, \mathcal{L}, \xi)$ *satisfies also axiom 5, then each non classical component* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *of the decomposition of* $(\Sigma, \mathcal{L}, \xi)$ *satisfies axiom 5.*

Proof: Consider $a, b \in \mathcal{C}$ such that $b \leq a$. From (101) follows that $(b \vee a^\perp) \wedge a = (b \wedge a) \vee (a^\perp \wedge a) = b \vee 0 = b$. This proves that $(\Omega, \mathcal{C}, \xi_c)$ satisfies axiom 5. Suppose now that axiom 5 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ and consider $a, b \in \mathcal{L}_\omega$ such that $b \leq a$. Using (121), and hence $b^{\perp_\omega} = b^\perp \wedge \omega$, we have $(b \vee a^{\perp_\omega}) \wedge a = (b^{\perp_\omega} \wedge a)^{\perp_\omega} \wedge a = (b^\perp \wedge \omega \wedge a)^{\perp_\omega} \wedge a = (b^\perp \wedge a)^{\perp_\omega} \wedge a = (b^\perp \wedge a)^\perp \wedge \omega \wedge a = (b^\perp \wedge a)^\perp \wedge a = (b \vee a^\perp) \wedge a = b$. This proves that axiom 5 is satisfied for $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. □

6.3 The axiom 'the covering law'

The covering law is the root of the linear structure of quantum mechanics, which means that it is a very important axiom. In some sense it demands something similar to the axiom of atomisticity, but then for all parts of the lattice of properties.

Axiom 6 (The covering law)**.** *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2, 3 and 4 are satisfied. For* $a, b \in \mathcal{L}$ *and* $p \in \Sigma$ *we have*

$$s(p) \wedge a = 0 \text{ and } a \leq b \leq a \vee s(p) \Rightarrow b = a \text{ or } b = a \vee s(p) \tag{146}$$

Proposition 40. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2 and 3 are satisfied. The classical state property space* $(\Omega, \mathcal{C}, \kappa_c)$ *corresponding to* $(\Sigma, \mathcal{L}, \xi)$ *satisfies axiom 6. If* $(\Sigma, \mathcal{L}, \xi)$ *satisfies also axiom 6, then each non classical component* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *of the decomposition of* $(\Sigma, \mathcal{L}, \xi)$ *satisfies axiom 6.*

Proof: Consider $a, b \in \mathcal{C}$ and $\omega(p) \in \Omega$, such that $a \wedge \omega(p) = 0$ and $a \leq b \leq a \vee \omega(p)$. Making use of theorem 10 we have $\kappa_c(a) \subseteq \kappa_c(b) = \kappa_c(a \vee \omega(p)) = \kappa_c(a) \cup \{\omega(p)\}$. Hence $\kappa_c(b) = \kappa_c(a)$ or $\kappa_c(b) = \kappa_c(a) \cup \{\omega(p)\}$. From (86) follows then that $b = a$ or $b = a \vee \omega(p)$. This proves that $(\Omega, \mathcal{C}, \kappa_c)$ satisfies axiom 6. Suppose now that axiom 6 is satisfied for $(\Sigma, \mathcal{L}, \xi)$ and consider $a, b \in \mathcal{L}_\omega$ and $p \in \Sigma_\omega$ such that $a \wedge s(p) = 0$ and $a \leq b \leq a \vee s(p)$. From axiom 6 being satisfied for $(\Sigma, \mathcal{L}, \xi)$ follows that $b = a$ or $b = a \vee s(p)$, which proves that axiom 6 is satisfied for $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. □

6.4 The axiom of plane transitivity

The seventh axiom that brings us directly to the structure of one of the three standard Hilbert spaces is much more recent [18].

Axiom 7 (Plane transitivity)**.** *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *for which axioms 1, 2 and 3 are satisfied. The state property space is plane transitive if for an arbitrary classical state* $\omega \in \Omega$ *and states* $p, q \in \Sigma_\omega$ *there exist two distinct atoms* $s_1, s_2 \in \mathcal{L}_\omega$ *and an automorphism* (m, n) *of* $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ *such that* $n|_{[0,s_1 \vee s_2]}$ *is the identity and* $m(p) = q$*.*

Both classical entities and quantum entities can be described by a state property space where the set of properties is a complete atomistic orthocomplemented lattice that satisfies the covering law, is weakly modular and plane transitive. In section 8 we consider the converse, namely how this structure leads us to classical physics and to quantum physics. But first we want to look into one of the basic notions of quantum mechanics, namely the notion of superposition state.

7 Superposition

One of the aspects which is often put forward as the most characteristic feature of quantum mechanics is the existence of 'superposition states'. In principle, the notion of superposition of states is intrinsically linked to the linearity of the Hilbert space. It is however possible to introduce it on a more fundamental level, which is what we will do in this section.

Definition 37 (Superposition)**.** *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *and states* $p, q, r \in \Sigma$*. We say that* r *is a superposition of* p *and* q *if* $\xi(p) \cap \xi(q) \subseteq \xi(r)$*. More generally, for a set of states* $\Gamma \subseteq \Sigma$ *we say that* r *is a superposition of* Γ *if* $\cap_{p \in \Gamma} \xi(p) \subseteq \xi(r)$*. We call*

$$\bar{\Gamma} = \{r \mid r \in \Sigma, \cap_{p \in \Gamma} \xi(p) \subseteq \xi(r)\} \tag{147}$$

the superposition set corresponding to $\Gamma \subseteq \Sigma$*.*

Proposition 41. *Consider a state property space* $(\Sigma, \mathcal{L}, \xi)$ *and* $\Gamma \subseteq \Sigma$*. For* $\Gamma, \Delta \subseteq \Sigma$ *we have*

$$\Gamma \subseteq \bar{\Gamma} \tag{148}$$

$$\Gamma \subseteq \Delta \Rightarrow \bar{\Gamma} \subseteq \bar{\Delta} \tag{149}$$

$$\bar{\bar{\Gamma}} = \bar{\Gamma} \tag{150}$$

which proves that $^-$ *is a closure operator.*

Proof: We have $\cap_{p\in\Gamma}\xi(p) \subseteq \xi(r) \ \forall \ r \in \Gamma$, and hence that $\Gamma \subseteq \bar{\Gamma}$, which proves (148). Suppose we have $\Gamma \subseteq \Delta \subseteq \Sigma$ and consider $r \in \bar{\Gamma}$. We have $\cap_{p\in\Delta}\xi(p) \subseteq \cap_{p\in\Gamma}\xi(p) \subseteq \xi(r)$ and hence $r \in \bar{\Delta}$. As a consequence we have $\bar{\Gamma} \subseteq \bar{\Delta}$, which proves (149). From (148) follows that $\bar{\Gamma} \subseteq \bar{\bar{\Gamma}}$. Consider $r \in \bar{\bar{\Gamma}}$, which means that $\cap_{p\in\bar{\Gamma}}\xi(p) \subseteq \xi(r)$. We have $\cap_{p\in\Gamma}\xi(p) \subseteq \cap_{p\in\bar{\Gamma}}\xi(p) \subseteq \xi(r)$, and hence $r \in \bar{\Gamma}$. Hence we have that $\bar{\bar{\Gamma}} \subseteq \bar{\Gamma}$. This means that we have proven (150). □

Proposition 42. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1 and 2 are satisfied, and $\Gamma \subseteq \Sigma$. We have*

$$r \in \bar{\Gamma} \Leftrightarrow s(r) \leq \vee_{p\in\Gamma} s(p) \tag{151}$$

Proof: Consider $r \in \bar{\Gamma}$, which means that $\cap_{p\in\Gamma}\xi(p) \subseteq \xi(r)$. We have $s(r) = \wedge_{a\in\xi(r)} a \leq \wedge_{a\in\cap_{p\in\Gamma}\xi(p)} a = \wedge_{a\in\xi(p)\forall p\in\Gamma} a = \wedge_{s(p)\leq a\forall p\in\Gamma} a = \vee_{p\in\Gamma} s(p)$. Suppose now that $s(r) \leq \vee_{p\in\Gamma} s(p)$. This means that $\wedge_{a\in\cap_{p\in\Gamma}\xi(p)} a \leq \wedge_{a\in\xi(r)} a$ and hence $\cap_{p\in\Gamma}\xi(p) \subseteq \xi(r)$, which proves that $r \in \bar{\Gamma}$. □

Definition 38 (Superselection)**.** *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$. We say that $p, q \in \Sigma$ are separated by a superselection rule, and we denote p ssr q, if the only superpositions of p and q are contained in p or in q. Hence, if for $r \in \Sigma$ such that $\xi(p) \cap \xi(q) \subseteq \xi(r)$ we have $\xi(p) \subseteq \xi(r)$ or $\xi(q) \subseteq \xi(r)$.*

Proposition 43. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1 and 2 are satisfied. For $p, q \in \Sigma$ we have*

$$p \text{ ssr } q \Leftrightarrow \kappa(s(p) \vee s(q)) = \{r \mid r \in \Sigma, r \leq p \text{ or } r \leq q\} \tag{152}$$

Proof: Suppose that p ssr q and consider $r \in \kappa(s(p) \vee s(q))$ and hence $s(r) \leq (s(p) \vee s(q)$. From (151) follows that $r \in \{p, q\}^- = \{r \mid r \in \Sigma, r \leq p \text{ or } r \leq q\}$. Hence $r \leq p$ or $r \leq q$. Suppose that $\kappa(s(p) \vee s(q)) = \{r \mid r \in \Sigma, r \leq p \text{ or } r \leq q\}$ and consider $r \in \{p, q\}^-$. From (151) follows that $s(r) \leq s(p) \vee s(q)$ and hence $r \in \kappa(s(p) \vee s(q))$. As a consequence we have $r \leq p$ or $r \leq q$. This proves that p ssr q. □

Theorem 14. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1, 2, 3, 4 and 5 are satisfied. For $p, q \in \Sigma$ we have*

$$p \text{ ssr } q \Rightarrow p = q \text{ or } p \perp q \tag{153}$$

Proof: Suppose that p ssr q and $p \not\leq q$, and consider r such that $r \in \kappa((s(p) \vee s(q)) \wedge s(q)^\perp)$. This means that $r \in \kappa((s(p) \vee s(q))$ and $r \in \kappa(s(q)^\perp)$. From (152) we have that $r \leq p$ or $r \leq q$, but since axiom 4 is satisfied, this gives $r = p$ or $r = q$. However, since $r \in \kappa(s(q)^\perp)$ we cannot have $r = q$. Hence $r = p$. Hence we have proven that $\kappa((s(p) \vee s(q)) \wedge s(q)^\perp) = \kappa(s(p) \wedge s(q)^\perp)$. We have $s(p) \wedge s(q)^\perp \leq s(p)$ and since $s(p)$ is an atom of $\mathcal{L}$ we have $s(p) \wedge s(q)^\perp = s(p)$ or $s(p) \wedge s(q)^\perp = 0$. If $s(p) \wedge s(q)^\perp = s(p)$ we have $s(p) \leq s(q)^\perp$ and hence $p \perp q$, which is not true. This means that $s(p) \wedge s(q)^\perp = 0$, and hence $(s(p) \vee s(q)) \wedge s(q)^\perp = 0$. From this follows that $(s(p)^\perp \wedge s(q)^\perp) \vee s(q) = I$, and as a consequence we have $((s(p)^\perp \wedge s(q)^\perp) \vee s(q)) \wedge s(q)^\perp = s(q)^\perp$. Since axiom 5 is satisfied we have $((s(p)^\perp \wedge s(q)^\perp) \vee s(q)) \wedge s(q)^\perp = s(p)^\perp \wedge s(q)^\perp$. Hence we have $s(q)^\perp = s(p)^\perp \wedge s(q)^\perp$, and as a consequence $s(q)^\perp \leq s(p)^\perp$. From this willows that $s(p) \leq s(q)$, and since $s(q)$ is an atom of $\mathcal{L}$ we have $s(p) = s(q)$, and hence $p = q$. $\square$

Theorem 15. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$ for which axioms 1, 2, 3, 4 and 6 are satisfied. For $p, q \in \Sigma$ we have*

$$p \text{ ssr } q \Rightarrow p = q \text{ or } p \perp q \tag{154}$$

Proof: In the same way as in the proof of theorem 14 we prove that $(s(p)^\perp \wedge s(q)^\perp) \vee s(q) = I$ if p ssr q and $p \not\leq q$. This means that I covers $s(p)^\perp \wedge s(q)^\perp$. Since $s(p)^\perp \wedge s(q)^\perp \leq s(q)^\perp \leq I$, and axiom 6 is satisfied, we have $s(p)^\perp \wedge s(q)^\perp = s(q)^\perp$ and hence $s(q)^\perp \leq s(p)^\perp$. From this follows that $s(p) \leq s(q)$, and since $s(q)$ is an atom, we have $s(p) = s(q)$ and hence $p = q$. $\square$

From these theorems follows that if axioms 1, 2, 3, 4 and 5 are satisfied or if axioms 1, 2, 3, 4 and 6 are satisfied, and two different states p and q are separated by a superselection rule then they are orthogonal. It also means that for two different states p and q that are not orthogonal there always exists a third state r which is a superposition of p and q.

8 Hilbert Space Representations

In this section we make further steps to get closer to standard quantum mechanics in a complex Hilbert space. A first step is based on Piron's representation theorem for an irreducible complete orthocomplemented weakly modular lattice satisfying the covering law [6]. Piron proved that such a lattice can be represented as the set of closed subspaces of a generalized Hilbert space.

8.1 Representation in Generalized Hilbert spaces

Starting from the general decomposition theorem 12 we have proven in section 5.3, and using the extra axioms 4, 5, 6 introduced in section 6, we can prove the following theorem for each one of the non classical components of the decomposition.

Theorem 16. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$, and suppose that axioms 1, 2, 3, 4, 5 and 6 are satisfied. Consider $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ the decomposition of $(\Sigma, \mathcal{L}, \xi)$ in its non classical components. For each non-classical component $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$, which has at least four orthogonal states, there exists a vector space V_ω, over a division ring K_ω, with an involution of K_ω, which means a function*

$$^* : K_\omega \to K_\omega \tag{155}$$

such that for $k, l \in K_\omega$ we have:

$$(k^*)^* = k \tag{156}$$

$$(k \cdot l)^* = l^* \cdot k^* \tag{157}$$

and an Hermitian product on V_ω, which means a function

$$\langle\ ,\ \rangle : V_\omega \times V_\omega \to K_\omega \tag{158}$$

such that for $x, y, z \in V_\omega$ and $k \in K_\omega$ we have:

$$\langle x + ky, z\rangle = \langle x, z\rangle + k\langle x, y\rangle \tag{159}$$

$$\langle x, y\rangle^* = \langle y, x\rangle \tag{160}$$

$$\langle x, x\rangle = 0 \Leftrightarrow x = 0 \tag{161}$$

and such that for $M \subset V_\omega$ we have:

$$M^\perp + (M^\perp)^\perp = V_\omega \tag{162}$$

where $M^\perp = \{y \mid y \in V_\omega, \langle y, x\rangle = 0, \forall x \in M\}$. Such a vector space is called a generalized Hilbert space or an orthomodular vector space. And we have that:

$$(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega) \cong (\mathcal{R}(V_\omega), \mathcal{L}(V_\omega), \nu_\omega) \tag{163}$$

where $\mathcal{R}(V_\omega)$ is the set of rays of V, $\mathcal{L}(V_\omega)$ is the set of biorthogonally closed subspaces (subspaces that are equal to their biorthogonal) of V_ω, and ν_ω makes correspond with each ray the set of biorthogonally closed subspaces that contain this ray.

Proof: If axioms 1, 2, 3, 4, 5 and 6 are satisfied for $(\Sigma, \mathcal{L}, \xi)$, then from theorem 11 and propositions 32, 39 and 40 follows that $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ satisfies axioms 1, 2, 3, 4, 5 and 6. Hence the lattice $\mathcal{L}_\omega$ is a complete orthocomplemented atomistic weakly modular lattice satisfying the covering law. Furthermore from proposition 33 follows that $\mathcal{L}_\omega$ is irreducible, and since $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ has at least four orthogonal states, it follows that $\mathcal{L}_\omega$ has at least four orthogonal atoms. This means that for $\mathcal{L}_\omega$ we can employ Piron's representation theorem [6, 8, 27, 28], and hence infer that there exists a vector space V_ω, over a division ring K_ω, with an involution * of K_ω and an Hermitian product $\langle\ ,\ \rangle$ on V_ω and such that for $M \subset V_\omega$ we have:

$$M^\perp + (M^\perp)^\perp = V_\omega \tag{164}$$

where $M^\perp = \{y \mid y \in V_\omega, \langle y, x\rangle = 0, \forall x \in M\}$, and such that

$$\mathcal{L}_\omega \cong \mathcal{L}(V_\omega) \tag{165}$$

where $\mathcal{L}(V_\omega)$ is the set of all biorthogonal subspaces of V_ω, i.e.

$$\mathcal{L}(V_\omega) = \{M \mid M \subseteq V_\omega, (M^\perp)^\perp = M\} \tag{166}$$

Each atom $s(p)$ of the lattice $\mathcal{L}_\omega$ is represented by a ray, i.e. a one dimensional subspace of V_ω. This means that Σ_ω can be put equal to $\mathcal{R}(V_\omega)$ the set of rays of the vector space V_ω. If we define ν_ω as the function from $\mathcal{R}(V_\omega)$ to $\mathcal{P}(\mathcal{L}(V_\omega))$, that makes correspond with each ray the set of biorthogonally closed subspaces that contain this ray, we have proven that $(\mathcal{R}(V_\omega), \mathcal{L}(V_\omega), \nu_\omega)$ is isomorphic to $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$. □

8.2 Representation in classical Hilbert spaces

Maria Pia Solèr has proven that if V_ω contains an infinite orthonormal sequence, then $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and V_ω is the corresponding Hilbert space [17]. Holland has shown that it is enough to demand the existence of a nonzero $\lambda \in K$ and an infinite orthogonal sequence $(e_n)_n \in V_\omega$ such that $< e_n, e_n >= \lambda$ for every n. To be precise, either $(V_\omega, K, < \cdot, \cdot >)$ or $(V_\omega, K, - < \cdot, \cdot >)$ is then a classical Hilbert space [29]. In [18] we proved some alternatives to Solèr's result, by means of automorphisms of $\mathcal{L}(V_\omega)$.

Proposition 44. *Let $(V, K, < \cdot, \cdot >)$ be an orthomodular space and let $\mathcal{L}(V)$ be the lattice of its closed subspaces. The following are equivalent:*
(1) $(V, K, < \cdot, \cdot >)$ is an infinite dimensional Hilbert space over $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

(2) V is infinite–dimensional and given two orthogonal atoms $\bar{x}, \bar{y}$ in $\mathcal{L}(V)$, there is a unitary operator U such that $U(\bar{x}) = \bar{y}$.
(3) There exist $a, b \in \mathcal{L}(V)$, where b is of dimension at least 2, and an ortholattice automorphism f of $\mathcal{L}(V)$ such that $f(a) \lneq a$ and $f|_{[0,b]}$ is the identical map.
(4) V is infinite dimensional and given two orthogonal atoms $\bar{x}, \bar{y}$ in $\mathcal{L}(V)$ there exist distinct atoms $\bar{x}_1, \bar{y}_2$ and an ortholattice automorphism f of $\mathcal{L}(V)$ such that $f|_{[0,\bar{x}_1 \vee \bar{y}_2}$ is the identity and $f(\bar{x}) = \bar{y}$.

Condition (2) is Holland's Ample Unitary Group axiom [29] and (3) is due to Mayet [30]. Using the properties listed in section 2 of [30], one can easily prove that (4) implies (2).

Theorem 17. *Consider a state property space $(\Sigma, \mathcal{L}, \xi)$, and suppose that axioms 1, 2, 3, 4, 5, 6 and 7 are satisfied. Consider $\circledvee_{\omega \in \Omega}(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$ the decomposition of $(\Sigma, \mathcal{L}, \xi)$ in its non classical components. Each nonclassical component $(\Sigma_\omega, \mathcal{L}_\omega, \xi_\omega)$, which has at least four orthogonal states, is isomorphic to the canonical state property space $(\Sigma(\mathcal{H}_\omega), \mathcal{L}(\mathcal{H}_\omega), \xi_{\mathcal{H}_\omega})$ where $\mathcal{H}_\omega$ are real, complex or quaternionic Hilbert spaces.*

Proof: an immediate consequence of proposition 44. $\square$

Theorem 17 proves that if axioms 1, 2, 3, 4, 5, 6 and 7 are satisfied, our theory reduces to standard quantum mechanics with super selection variables, and eventually on a quaternionic Hilbert space.

References

[1] Birkhoff, G. and von Neumann, J. (1936), The logic of quantum mechanics, *Annals of Mathematics*, **37**, 823-843.

[2] von Neumann, J. V. (1932), *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin.

[3] Wigner, E.P. (1959), *Group Theory and its Applications to Quantum Mechanics of Atomic Spectra*, Academic Press, New York.

[4] Mittelstaedt, P. (1963), *Philosophische Probleme der Modernen Physik*, Bibliographisches Institut, Manheim.

[5] Mackey, G. (1963), *Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.

[6] Piron, C. (1964), Axiomatique quantique, *Helvetica Physica Acta*, **37**, 439-468.

[7] Aerts, D. and Aerts, S. (2004). Towards a general operational and realistic framework for quantum mechanics and relativity theory. In A. C. Elitzur, S. Dolev and N. Kolenda (Eds.), *Quo Vadis Quantum Mechanics? Possible Developments in Quantum Theory in the 21st Century* (pp. 153-208). New York: Springer.

[8] Piron, C. (1976), *Foundations of Quantum Physics*, Benjamin, Massachusetts.

[9] Aerts, D. (1981), *The One and the Many: Towards a Unification of the Quantum and Classical Description of One and Many Physical Entities*, Doctoral dissertation, Brussels Free University.

[10] Aerts, D. (1982), Description of many physical entities without the paradoxes encountered in quantum mechanics, *Foundations of Physics*, **12**, 1131-1170.

[11] Aerts, D. (1983), Classical theories and non classical theories as a special case of a more general theory, *Journal of Mathematical Physics*, **24**, 2441-2453.

[12] Aerts, D. (1999), Foundations of quantum physics: a general realistic and operational approach, *International Journal of Theoretical Physics*, **38**, 289-358, archive ref and link: quant-ph/0105109.

[13] Aerts, D. (1999), Quantum mechanics: structures, axioms and paradoxes, in *Quantum Mechanics and the Nature of Reality*, eds. Aerts, D. and Pykacz, J., Kluwer Academic, Dordrecht, archive ref and link: quant-ph/0106132.

[14] Aerts, D., Colebunders, E., Van der Voorde, A, Van Steirteghem, B. (1999), State property systems and closure spaces: a study of categorical equivalence, *International Journal of Theoretical Physics*, **38**, 359-385, archive ref and link: quant-ph/0105108.

[15] Aerts, D. (2002), Being and change: foundations of a realistic operational formalism, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Non-locality, Probability and Axiomatics*, eds. Aerts, D, Czachor, M. and Durt, T, World Scientific, Singapore.

[16] Keller, H. (1980), Ein nicht-klassischer Hilbertscher Raum, *Mathematische Zeitschrift*, **172**, 41-49.

[17] Solèr, M. P. (1995), Characterization of Hilbert spaces by orthomodular spaces, *Communications in Algebra*, **23**, 219-243.

[18] Aerts, D. and Van Steirteghem, B. (2000), Quantum axiomatics and a theorem of M. P. Solèr, *International Journal of Theoretical Physics*, **39**, 497-502, archive ref and link: quant-ph/0105107.

[19] Randall, C. and Foulis, D. (1983), Properties and operational propositions in quantum mechanics, *Foundations of Physics*, **13**, 835.

[20] Foulis, D., Piron, C. and Randall, C. (1983). Realism, operationalism, and quantum mechanics, *Foundations of Physics*, **13**, 813.

[21] Aerts, D., Colebunders, E., Van der Voorde, A. and Van Steirteghem, B. (2002). On the amnestic modification of the category of state property systems. *Applied Categorical Structures*, **10**, 469-480.

[22] Aerts, D. and Deses, D. (2002). State property systems and closure spaces: extracting the classical and nonclassical parts. In D. Aerts, M. Czachor and T. Durt (Eds.), *Probing the Structure of Quantum Mechanics: Nonlinearity, Non-locality, Probability and Axiomatics* (pp. 130-148). Singapore: World Scientific.

[23] Aerts, D. and Valckenborgh, F. (2002). The linearity of quantum mechanics at stake: the description of separated quantum entities. In D. Aerts, M. Czachor and T. Durt (Eds.), *Probing the Structure of Quantum Mechanics: Nonlinearity, Non-locality, Probability and Axiomatics* (pp. 20-46). Singapore: World Scientific.

[24] Aerts, D. and Deses, D. (2005). State property systems and orthogonality. *International Journal of Theoretical Physics*, **44**, 919-929.

[25] Aerts, D., Deses, D. and Van der Voorde, A. (in press). Classicality and connectedness for state property systems and closure spaces. *International Journal of Theoretical Physics.*

[26] Aerts, D. and Pulmannova, S. (2006). Representation of state property systems. *Journal of Mathematical Physics*, **47**, 1-18

[27] Maeda, F., and Maeda, S. (1970). *Theory of symmetric lattices*, Springer–Verlag, Berlin.

[28] Faure, Cl.-A., and Frölicher, A. (1995). Dualities for infinite–dimensional projective geometries, *Geometriae Dedicata*, **56**, 225-236.

[29] Holland, S.S. Jr. (1995). Orthomodularity in infinite dimensions; a theorem of M. Solèr, *Bulletin of the American Mathematical Society*, **32**, 205-234.

[30] Mayet, R. (1998). Some characterizations of the underlying division ring of a Hilbert lattice by automorphisms, *International Journal of Theoretical Physics*, **37**, 109-114.