Orthonormal Bases and Quasi-splitting Subspaces in Pre-Hilbert Spaces

David Buhagiar∗, Emmanuel Chetcuti† and Hans Weber‡

Let $S$ be a real or complex pre-Hilbert space and $H$ its completion. In the Hilbert space model for quantum mechanics, the events of a quantum system can be identified with projections on a Hilbert space or, equivalently, a collection of closed subspaces of a Hilbert space. Two classes of closed subspaces of $S$ that can naturally replace the lattice of projections in a Hilbert space are those of orthogonally closed subspaces $E(S)$, and splitting subspaces $F(S)$. It is known that $E(S) \subseteq F(S)$ and that equality between these two classes holds if and only if $S$ is complete.

The class $E_q(S)$ of quasi-splitting subspaces of $S$ was introduced in [1] as an intermediate between $E(S)$ and $F(S)$. A subspace $M$ of $S$ is quasi-splitting if it is closed in $S$ and $M \oplus M^\perp_S$ is a dense subspace of $S$. Equivalently, a closed subspace $M$ of $S$ is quasi-splitting if $\overline{M + M^\perp} = M + M^\perp$. If $S$ is complete, then $E(S) = E_q(S) = F(S)$. The inclusions $E(S) \subseteq E_q(S) \subseteq F(S)$ hold though, in general, they are proper. Motivated by the Amemiya-Araki-Piron Theorem, the authors of [1] conjectured that: $E_q(S) = E(S)$ if and only if $S$ is a Hilbert space and also settled this in the affirmative for the case when the Hamel dimension of $H/S$ is finite. This question is closely related to the problem of characterizing those pre-Hilbert spaces that admit an orthonormal basis (ONB), i.e. an orthonormal system (ONS) that is total.

We show that if $S$ has an ONB, then $E_q(S) = E(S)$ if and only if $S$ is complete. In particular, this means that $E(S) \neq E_q(S)$ when $S$ is an incomplete separable pre-Hilbert space. We also investigate when a pre-Hilbert space has an ONB. It is shown that when every linear complement of $S$ in $H$ is separable, then $S$ has an ONB. This means, for example, that $S$ admits an ONB when the Hamel dimension of $H/S \leq \aleph_0$. In particular, all hyperplanes have an ONB. The relation between $\dim S$, $\dim H$ and the Hamel dimension of $S$ is also considered.

The talk is based on results obtained in [2].

References
