Logical and Algebraic Structures from Quantum Computation

Roberto Giuntini, Hector Freytes, Antonio Ledda, Francesco Paoli, Giuseppe Sergioli

Quantum computation has suggested new forms of quantum logic, called quantum computational logics [1]. The basic semantic idea is that the meaning of a sentence can be identified with a vector in \( \otimes^n \mathbb{C}^2 \), called q-register, that physically represents a pure state of a compound physical system. The quantum logical counterpart of the set of classical gates is the set of unitary (and therefore reversible) operators that, if applied to a q-register, give another q-register as output (e.g. negation Not\(^{(n)}\); square root of the identity \( \sqrt{T^{(n)}} \); square root of the negation \( \sqrt{\text{Not}^{(n)}} \); Toffoli \( T^{(n,m,1)} \)). Actually, however, a prepared state is seldom a pure state, but rather a mixture of pure states, called q-mixes and mathematically represented by density operators of \( \otimes^n \mathbb{C}^2 \). The framework of density operators encompasses pure states, which are in 1 : 1 correspondence with projection operators onto unidimensional subspaces.

By focusing on the set of density operators of \( \otimes^n \mathbb{C}^2 \), endowed with operators corresponding to the above-mentioned quantum logical gates, one can define an array of quantum computational logics [1], differing from one another along two degrees of freedom: (i) the language, i.e. the number and type of quantum gates under consideration; (ii) the consequence relation, which can be either weak or strong. Actually, it turns out that confining ourselves to \( \mathbb{C}^2 \) instead of arbitrary \( n \)-fold tensor products \( \otimes^n \) entails no loss of generality in that the consequence relations are the same in both cases. We therefore restrict ourselves to the structure whose universe is the set of density operators of \( \mathbb{C}^2 \), endowed with operations corresponding to an irreversible (non-linear) operation (the Lukasiewicz truncated sum \( \oplus \)) and the negation gate \( \neg \). By resorting to the representation of density operators of \( \mathbb{C}^2 \) via the Pauli matrices, we are allowed to replace such operators by inner or surface points of the Bloch-Poincaré sphere, and we can even shift down by one dimension by disregarding the component which plays no role in probability assignments. We therefore end up with a structure whose universe is the set of complex numbers in the closed disc with center in the origin and unitary radius, which is more conveniently transposed to the first quadrant and scaled down by one half. This structure is an algebra \( \langle C, \oplus, \neg, 0, 1 \rangle \) of type \( \langle 2, 1, 0, 0 \rangle \), and thus it makes sense to investigate the variety of this type it generates. Such a variety is the variety of quasi-MV algebras [2], which happens to be a generalization of Chang’s MV algebras. The above-mentioned algebras can be expanded by an operation of square root of negation, thus obtaining the class of \( \sqrt{\text{ quasi-MV algebras} \} \) [3]. In the present talk we shall report on some recent developments concerning the algebraic theory of these structures. In particular, we shall dwell on the following results: (i) both varieties have the finite model property, and quasi-MV algebras have the strong finite model property; (ii) both varieties have the congruence extension property; (iii) it is possible to describe free algebras and Gumm-Ursini ideals in both varieties; (iv) it is possible to provide a complete description of the lattices of subvarieties of both varieties; (v) it is possible to characterize the qMV term reducts and term subreducts of \( \sqrt{\text{ quasi-MV algebras} \) algebras; (vi) quasi-MV algebras have the amalgamation property; (vii) quasi-MV algebras can be given two alternative representations as labelled MV algebras; (viii) Cartesian \( \sqrt{\text{ quasi-MV algebras} \) algebras can be represented as subalgebras of intervals of Abelian lattice-ordered groups endowed with projection and rotation operators.

References


*Dipartimento di Scienze Pedagogiche e Filosofiche, Università di Cagliari, Italy; e-mail: giuntini@unica.it, hfreytes@dm.uba.ar, antonio.ledda@inwind.it, paoli@unica.it, curvenellamemoria@virgilio.it