MacNeille Completions of Commutative Basic Posets

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Common generalizations of MV-algebras [1] and orthomodular lattices are lattice effect algebras [2]. An effect algebra $(E; \oplus, 0, 1)$ is a set $E$ with two special elements $0, 1$ and a partial binary operation $\oplus$ which is commutative and associative at which these equalities hold if one of their sides exists. In every effect algebra we can define a partial order by which is commutative and associative at which these equalities hold if one of their sides exists. In every effect algebra we can define a partial order by

$$b \leq a \iff \exists c \in E \text{ such that } a \oplus c = b,$$

and we set $a^\perp = (a^\perp)$. A basic poset $P$ is called commutative if, for all $a \leq b$, we have $b^\perp = (a^\perp)^{(b^\perp)}$.

Similarly as for effect algebras and D-posets, commutative basic posets are equivalent to so called WD-posets.

**Definition 2** A weak difference on a bounded poset $(P, \leq, 0, 1)$ is a partial binary operation $\ominus$ on $P$ such that $b \ominus a$ is defined if and only if $a \leq b$ subject to conditions

(WD1) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

(WD2) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(1 \ominus a) \ominus (1 \ominus b) = b \ominus a$.

A WD-poset is a bounded poset with a weak difference.

For subsets $U, Q$ of a WD-poset $(P, \ominus, \leq, 0, 1)$, we will write $U \subseteq Q$ iff $u \leq q$ for all $u \in U$, $q \in Q$.

In such case we will write $Q \ominus U = \{q \ominus u : q \in Q, u \in U\}$. A WD-poset $(P, \ominus, \leq, 0, 1)$ is called strongly D-continuous[3] iff for all $U, Q \subseteq P$ with $U \subseteq Q$ the following condition is satisfied:

(SDC) $\bigwedge(Q \ominus U) = 0$ iff every lower bound of $Q$ is under every upper bound of $U$.

If we define on the MacNeille completion $MC(P)$ the partial operation $\ominus$ by the same formula as in [3] for D-posets then we can prove (similarly as in [3]):

**Theorem 3** An WD-poset $(P, \ominus, \leq, 0, 1)$ has a MacNeille completion a complete WD-poset $(MC(P), \ominus, \leq, 0, 1)$ iff $P$ is strongly D-continuous and for any $b, c \in MC(P)$ such that $b \leq c$ there is $a \in MC(P)$ satisfying $b = c \ominus a$.

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**References**


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