Orthogonalization Time Revisited

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The question of how fast a quantum state can evolve into a state orthogonal to it has attracted a considerable attention in connection with quantum measurements and quantum information processing. Indeed, since only orthogonal states can be distinguished without ambiguity by a measurement, the transition to an orthogonal state can be viewed as a “unit step” in a process of computation [1]. Thus, the minimum orthogonalization time determines the maximum number of distinct states that the system can pass through per unit time—which, for a computer, would correspond to the maximum number of operations per second. Starting with Mandelstam and Tamm’s [2] classical result, it was later shown by Anandan and Aharonov [3] and Vaidman [4] that the minimum time \( \tau \) required for arriving to an orthogonal state is bounded by

\[
\tau \geq \frac{\hbar}{4\Delta E},
\]

(1)

where \( (\Delta E)^2 = \langle \psi | H^2 | \psi \rangle - (\langle \psi | H | \psi \rangle)^2 \), and \( | \psi \rangle \) is the wavefunction of the system. A different bound was obtained in [5], namely,

\[
\tau \geq \frac{\hbar}{4E}.
\]

(2)

Here \( \langle \psi | H | \psi \rangle \) is the quantum-mechanical average energy of the system (the energy of the ground state is taken to be zero). Both bounds (1) and (2) are tight, and achieved for a quantum state such that \( \Delta E = E \). Bound (2) was first established for an autonomous physical system. Later [6, 7] the bound was generalized for the case of a system driven by an external (in general, time-dependent) Hamiltonian. Since then, a vast literature has been devoted to various aspects of this problem. However, what remained unnoticed is the paradoxical situation of the existence of two bounds based on two different characteristics of the quantum state, seemingly independent of one another. This problem is addressed by the present paper. We will show that there exists a better bound that takes into account both characteristics, \( \Delta E \) and \( E \). Using variational equations for the energy eigenvalues under the constraints imposed by the requirement that both the real and the imaginary parts of the inner product of the initial and final states of the system must vanish, it is possible to derive a system of parametric equations that determine the minimum time \( \tau \) in terms of the average energy \( E \) and the ratio \( \delta = \Delta E / E \). For \( \delta = 1 \), the new bound coincides with (1) and (2). However, the value of \( \tau \) increases up to two times for ratios deviating from 1.

References


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