On a Connection between Piron Lattices and Kripke Frames

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Whither Quantum Structures in the XXIth Century? Brussels, Belgium
Outline

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2. Quantum Kripke Frames
   - Definition and Main Result
   - Relations with Other Structures
   - Probabilistic Quantum Kripke Frames

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Consider an *isolated* quantum system described by some Hilbert space $\mathcal{H}$ over complex numbers.

Two non-zero vectors $|\psi\rangle$ and $|\phi\rangle$ are said to be *orthogonal*, denoted as $|\psi\rangle \perp |\phi\rangle$, if the inner product $\langle \psi|\phi \rangle$ is 0.

This binary relation on vectors induces a binary relation on one-dimensional subspaces of $\mathcal{H}$ and thus on states of the quantum system, which is also called *orthogonality relation*.

By studying this relation, we get many representation theorems for lattices emerging from quantum logic via Kripke frames.
Ortholattices and Orthogonality Spaces [Goldblatt, 1974]

- ‘That the \(\bot\)-closed subsets of an orthogonality space form an ortholattice under the partial ordering of set inclusion is a result of long standing (cf. Birkhoff,[1]§V.7).’
- ‘Every ortholattice is, within isomorphism, a subortholattice of the lattice of \(\bot\)-closed subsets of some orthogonality space.’
- **Ortholattice**: an orthocomplemented lattice
  **Orthogonality space**: a Kripke frame in which the binary relation is *irreflexive* and *symmetric*. 

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Property Lattices and State Spaces [Moore, 1995]

- **Property lattice**: a *complete atomistic orthocomplemented lattice*.
- **State space**: a Kripke frame \((\Sigma, \perp)\) in which the binary relation \(\perp\) is *irreflexive, symmetric* and *separated* in the following sense:
  
  \textit{there is} \(w \in \Sigma\) \textit{such that} \(w \perp s\) \textit{and} \(w \not\perp t\), \textit{for any} \(s, t \in \Sigma\) \textit{such that} \(s \neq t\).

- The main result in this paper is a **duality** between
  - a category with *property lattices* as objects, and
  - a category with *state spaces* as objects.
What about Piron Lattices?

- Compared to lattices emerging from quantum theory, i.e. lattices of closed linear subspaces of Hilbert spaces, both ortholattices and property lattices are too general.
- **Piron lattice**: an *irreducible, complete, atomistic, orthocomplemented lattices satisfying weak modularity and the Covering Law.*
- They are also called *irreducible propositional systems.*
Piron’s Theorem (1964)

The lattice of bi-orthogonally closed subspaces of a generalized Hilbert space is always a Piron lattice; and every Piron lattice of rank at least 4 is isomorphic to such a lattice.

A Corollary of the Amemiya-Araki-Piron Theorem

Generalized Hilbert spaces over the real numbers, the complex numbers and the quaternions are Hilbert spaces over these *-fields, in such a way that bi-orthogonally closed subspaces are exactly closed linear subspaces.
In [Baltag and Smets, 2005], the authors give a representation theorem for Piron lattices (satisfying Mayet’s condition) using quantum dynamic frames.

A quantum dynamic frame is a tuple $(\Sigma, \{P \rightarrow\}_{P \in \mathcal{L}})$, where $\Sigma$ is a non-empty set, $\mathcal{L}$ is a subset of the power set of $\Sigma$ and $P \rightarrow$ is a binary relation on $\Sigma$ for all $P \in \mathcal{L}$.

The orthogonality relation, denoted as $\perp$, is defined as follows:

\[ s \perp t \iff \text{there is no } P \in \mathcal{L} \text{ such that } s \xrightarrow{P} t. \]
The Work in this Talk

I will

- define a kind of Kripke frames, and
- use them to give a representation theorem for Piron lattices.

This work

- inspired by Baltag and Smets’ work, provides an alternative way of defining quantum dynamic frame;
- continues the logical study of the orthogonality relation extending Moore’s result and thus Goldblatt’s result.
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Some Terminologies of Kripke Frames

A Kripke frame $\mathcal{F}$ is a tuple $(\Sigma, \rightarrow)$, where $\Sigma$ is a non-empty set and $\rightarrow \subseteq \Sigma \times \Sigma$.

- Write $s \not\rightarrow t$ for $(s, t) \not\in \rightarrow$.
- Given $P \subseteq \Sigma$, the orthocomplement of $P$ (w.r.t. $\rightarrow$) is defined as follows:

$$\sim P \overset{\text{def}}{=} \{ s \in \Sigma \mid s \not\rightarrow t, \text{ for every } t \in P \}$$

- $P$ is bi-orthogonally closed, if $P = \sim \sim P$. 

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Quantum Kripke Frames

Definition (Quantum Kripke Frame)

A quantum Kripke frame (QKF) $\mathcal{F}$ is a Kripke frame $(\Sigma, \rightarrow)$ satisfying the following conditions:

1. $\rightarrow$ is reflexive and symmetric.
2. (Existence of Good Approximation) if $s \notin \sim P$ and $\sim \sim P = P$, then there is $t \in P$ such that $s \rightarrow u$ if and only if $t \rightarrow u$ for each $u \in P$;
3. (Separation) if $s \neq t$, then there is $w \in \Sigma$ such that $w \rightarrow s$ and $w \nrightarrow t$;
4. (Superposition) for any $s, t \in \Sigma$, there is $w \in \Sigma$ such that $w \rightarrow s$ and $w \rightarrow t$. 
Good Approximations are the Best

Consider $s \in \Sigma$ and $P \subseteq \Sigma$ such that good approximation of $s$ in $P$ exists according to condition (ii).

There is $t \in P$ such that $s \rightarrow u \iff t \rightarrow u$, for every $u \in P$.

Condition (iii), i.e. Separation, guarantees that the $t$ with this property is unique.

This $t$ will be called the best approximation of $s$ in $P$.

Given a bi-orthogonally closed $P \subseteq \Sigma$, define a partial function $P?(\cdot) : \Sigma \rightarrow \Sigma$ as follows:

$$P?(s) \overset{\text{def}}{=} \begin{cases} \text{the best approximation } t \text{ of } s \text{ in } P, & \text{if } s \not\in \sim P \\ \text{undefined}, & \text{otherwise} \end{cases}$$
Main Results

Theorem 1
For any quantum Kripke frame $\mathcal{F} = (\Sigma, \rightarrow)$, $(\mathcal{L}_\mathcal{F}, \subseteq, \sim(\cdot))$ is a Piron lattice, where $\mathcal{L}_\mathcal{F} = \{P \subseteq \Sigma | \sim\sim P = P\}$ and $\sim(\cdot)$ is the orthocomplement operation (w.r.t. $\rightarrow$).

Theorem 2
Every Piron lattice $\mathcal{L}$ is isomorphic to $(\mathcal{L}_\mathcal{F}, \subseteq, \sim(\cdot))$ for some quantum Kripke frame $\mathcal{F}$. 
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Quantum Kripke frames are a special kind of Moore’s state spaces, because conditions (i) and (iii) are equivalent to the conditions on state spaces, despite the fact that quantum Kripke frames take as primitive the non-orthogonality relation instead of the orthogonality relation.
Given the close relations
- between *quantum Kripke frames* and *Piron lattices*,
- between *Piron lattices* and *quantum dynamic frames*,
- between *Piron lattices* and *irreducible Hilbert geometries*,
we can conceive of using quantum Kripke frames:
- a representation theorem for quantum dynamic frames,
- a representation theorem for irreducible Hilbert geometries.
Quantum Kripke Frames and Quantum Dynamic Frames

Proposition

Given a quantum Kripke frame $\mathcal{F} = (\Sigma, \rightarrow)$, let $L_{\mathcal{F}}$ denote 
$\{ P \subseteq \Sigma \mid P = \sim P \}$, and for each $P \in L_{\mathcal{F}}$, define $P? \subseteq \Sigma \times \Sigma$ such that:

$$s \xrightarrow{P?} t \iff s \notin \sim P \text{ and } t = P?(s).$$

Then $(\Sigma, \{ P? \}_{P \in L_{\mathcal{F}}})$ is a quantum dynamic frame.

Proposition

Every quantum dynamic frame is isomorphic to $(\Sigma, \{ P? \}_{P \in L_{\mathcal{F}}})$ for some quantum Kripke frame $\mathcal{F} = (\Sigma, \rightarrow)$. 

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Hilbert Geometries

A Hilbert geometry is a tuple \((G, l, \perp)\), where \((G, l)\) is a projective geometry, i.e. \(G\) is a non-empty set and \(l \subseteq G \times G \times G\) such that:

(G1) \(l(a, b, a)\);
(G2) if \(l(a, p, q), l(b, p, q)\) and \(p \neq q\), then \(l(a, b, p)\);
(G3) if \(l(p, a, b)\) and \(l(p, c, d)\), then there exists \(q \in G\) such that \(l(q, a, c)\) and \(l(q, b, d)\);

moreover, \(\perp \subseteq G \times G\) satisfies the following:

(O1) if \(a \perp b\), then \(a \neq b\);
(O2) if \(a \perp b\), then \(b \perp a\);
(O3) if \(a \neq b\), \(a \perp p\), \(b \perp p\) and \(l(c, a, b)\), then \(c \perp p\);
(O4) if \(a \neq b\), then there is \(q \in G\) such that \(l(q, a, b)\) and \(q \perp a\);
(O5) if \(S \subseteq G\) is a subspace such that \(S \perp \perp = S\), then \(S \vee S \perp = G\).

This is Definition 51 on page 499 of [Stubbe and van Steirteghem, 2007].
Proposition

Let $\mathcal{F} = (\Sigma, \rightarrow)$ be a quantum Kripke frame. Define a relation $l_{\mathcal{F}} \subseteq \Sigma \times \Sigma \times \Sigma$ such that for any $u, v, w \in \Sigma$, $(u, v, w) \in l_{\mathcal{F}}$, if and only if one of the following holds:

- $v = w$;
- $s \rightarrow u$ implies that $s \rightarrow v$ or $s \rightarrow w$, for every $s \in \Sigma$.

Then $(\Sigma, l_{\mathcal{F}}, \not\rightarrow)$ is an irreducible Hilbert geometry.

Proposition

Every irreducible Hilbert geometry $(G, l, \perp)$ is isomorphic to $(\Sigma, l_{\mathcal{F}}, \not\rightarrow)$ for some quantum Kripke frame $\mathcal{F} = (\Sigma, \rightarrow)$. 

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Quantum Kripke Frames and Classical Frames

Definition (Classical Frame)

A classical frame $\mathcal{F}$ is a Kripke frame $(\Sigma, \rightarrow)$ in which $\rightarrow$ is the identity relation, i.e. $\rightarrow = \{(s, t) \in \Sigma \times \Sigma \mid s = t\}$.

Bi-orthogonally closed subsets of a classical frame form a Boolean lattice.
Quantum Kripke Frames and Classical Frames

**Definition (Classical Frame)**

A classical frame $F$ is a Kripke frame $(\Sigma, \rightarrow)$ in which $\rightarrow$ is the identity relation, i.e. $\rightarrow = \{(s, t) \in \Sigma \times \Sigma | s = t\}$.

Bi-orthogonally closed subsets of a classical frame form a Boolean lattice.

**Proposition**

Let $\mathcal{F} = (\Sigma, \rightarrow)$ be a Kripke frame satisfying conditions (i) to (iii) in the definition of quantum Kripke frames but not condition (iv), i.e. superposition. Then

- $\mathcal{F}$ is a quantum Kripke frame, iff superposition holds;
- $\mathcal{F}$ is a classical frame, iff $\rightarrow$ is transitive.

Moreover, if $\Sigma$ has at least 2 elements, then superposition and transitivity of $\rightarrow$ can not hold simultaneously.
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Minimal Requirement of Probabilistic QKFs

- A probabilistic quantum Kripke frame should consist of a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ and a function $\rho : \Sigma \times \Sigma \to [0, 1]$.
- Given such a pair and $s \in \Sigma$, define $\mu_s : \mathcal{L}_{\mathfrak{F}} \to [0, 1]$ such that
  \[\mu_s(P) = \begin{cases} 
    0 & \text{if } s \in \sim P, \\
    \rho(s, P?(s)) & \text{otherwise}.
  \end{cases}\]

Minimal Requirement of Probabilistic Quantum Kripke Frames

For every $s \in \Sigma$, $\mu_s$ defined in the above way is a quantum probability measure on the Piron lattice $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot))$. 
Quantum Probability Measure

A quantum probability measure is a function $p$ from a Piron lattice $\mathcal{L} = (L, \leq, (\cdot)')$ to $[0, 1]$ such that:

- $p(I) = 1$;
- $\sum_{i\in A} p(b_i)$ exists and is equal to $p(\bigvee_{i\in A} b_i)$, for every $\{b_i \mid i \in A\} \subseteq L$ with $A$ at most countable and $b_i \leq b'_j$ when $i \neq j$.
- $p(b) = p(c) = 0$ implies that $p(b \lor c) = 0$, for every $b, c \in L$.

This definition is adapted from Definition (4.38) on page 82 of [Piron, 1976].
A probabilistic quantum Kripke frame $\mathcal{F}_P$ is a tuple $(\mathcal{F}, \rho)$, where $\mathcal{F} = (\Sigma, \rightarrow)$ is a quantum Kripke frame and $\rho$ is a function from $\Sigma \times \Sigma$ to $[0, 1]$ satisfying the following:

1. $\rho(s, t) = \rho(t, s)$;
2. $\rho(s, t) = 0$, if and only if $(s, t) \notin \rightarrow$;
3. if $\{t_i \mid i \in I\} \subseteq \Sigma$ satisfies that $I$ is at most countable and $t_i \perp t_j$ whenever $i \neq j$, then $\sum_{i \in I} \rho(s, t_i) \leq 1$; and equality holds if and only if $s \in \sim \cup \{t_i \mid i \in I\}$;
4. if $P \in \mathcal{L}_{\mathcal{F}}, \ s \notin \sim P$ and $t \in P$, then $\rho(s, t) = \rho(s, P?(s)) \cdot \rho(P?(s), t)$.

Proposition

This definition satisfies the minimal requirement.
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Simplifying the Definition of Quantum Kripke Frames

- Condition (ii), i.e. existence of good approximation, looks complicated, because it involves quantification over subsets of $\Sigma$.

**Theorem in [Goldblatt, 1984]**

There is no first-order formula $\varphi$ in the language with one binary relation symbol such that, for any pre-Hilbert space $\mathcal{P}$, the following are equivalent:

- $(\mathcal{P}, \perp) \models \varphi$;
- orthomodularity holds in the lattice of orthoclosed subspaces of $\mathcal{P}$.

- It’s interesting to see whether condition (ii) can be simplified under some specific constraints, e.g. those on ‘dimension’.
Axiomatizing Quantum Kripke Frames

- Kripke frames with various properties are often described by the modal propositional language with one unary modality □.
- Try to find a proof system in this language, which is sound and complete w.r.t. the class of quantum Kripke frames.
- This logic will have classical negation ¬ and orthocomplement (quantum negation) ∼ can be defined as □¬.
- One of the challenges is that conditions (ii) and (iii) involve saying that a state can not access another state, which is a characteristic feature of undefinable properties of modal language.

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More Work on Probabilistic Quantum Kripke Frames

- Characterize quantum Kripke frames that are induced by Hilbert spaces with some conditions involving probability.
- Capture the notions of *quantum probability measure* and *mixed states* in this framework.
Complete Axiomatizations for Quantum Actions.

Semantics Analysis of Orthologic.
*Journal of Philosophical Logic, 3*:19–35.

Orthomodularity Is Not Elementary.

Some Characterizations of the Underlying Division Ring of a Hilbert Lattice by Automorphisms.

Categories of Representations of Physical Systems.

*Foundations of Quantum Physics.*
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Characterization of Hilbert Spaces with Orthomodularity Spaces.
*Communications in Algebra, 23*:219 – 243.

Propositional Systems, Hilbert Lattices and Generalized Hilbert Spaces.
Thank you very much!