Unfortunately, Johan van Benthem (who was supposed to co-teach with us) could not come.
Course Material: Lecture Slides

The slides will be available (on a day-to-day basis) at

http://sonja.tiddlyspot.com/#NASSLLI-2012

Today’s slides are already there!!!
Relevant Textbooks and Surveys


- J. van Benthem, 2010, ”Modal Logic for Open Minds”, CSLI Publications, Stanford, 2011, especially the chapters on agency


Prerequisites

basic first-order logic, basic modal logic
Syllabus

1. “Hard” (classical) $DEL$: epistemic logic and information update
2. “Soft” $DEL$: dynamic logics for belief revision
3. Non-standard $DEL$: inference actions, questions, probabilistic dynamics and preference change (?)
4. Long-term behavior of Information: iterated revision, convergence and cycles, truth-tracking, Learning theory, epistemic temporal logic (??)
5. Social Interaction and Games: from pragmatics of communication to rational agency. (???)

This plan is tentative: depending on time, we will update it.
Epistemic Puzzle no. 1: To learn is to falsify

Our starting example concerns a “love triangle”: suppose that Alice and Bob are a couple, but Alice has just started an affair with Charles.

At some point, Alice sends to Charles an email, saying:

“Don’t worry, Bob doesn’t know about us”.

But suppose now that Bob accidentally reads the message (by, say, secretly breaking into Alice’s email account).

Then, paradoxically enough, after seeing (and believing) the message which says he doesn’t know..., he will know!
So, in this case, learning the message is a way to falsify it.

As we’ll see, this example shows that standard belief-revision postulates may fail to hold in such complex learning actions, in which the message to be learned refers to the knowledge of the hearer.
Epistemic Puzzle no. 2: Self-fulfilling falsehoods

Suppose Alice becomes somehow convinced that Bob knows everything (about the affair).

This is false (Bob doesn’t have a clue), but nevertheless she’s so convinced that she makes an attempt to warn Charles by sending him a message:

”Bob knows everything about the affair!”.

As before, Bob secretly reads (and believes) the message. While false at the moment of its sending, the message becomes true: now he knows.
So, communicating a false belief (i.e. Alice’s action) might be a self-fulfilling prophecy: Alice’s false belief, once communicated, becomes true.

In the same time, the action of (reading and) believing a falsehood (i.e. Bob’s action) can be self-fulfilling: the false message, once believed, becomes true.
Epistemic Puzzle no. 3: Self-enabling falsehoods

Suppose that in fact Alice was faithful, despite all the attempts made by Charles to seduce her.

Out of despair, Charles comes up with a “cool” plan of how to break up the marriage:

he sends an email which is identical to the one in the second puzzle (bearing Alice’s signature and warning Charles that Bob knows about their affair.) Moreover, he makes sure somehow that Bob will have the opportunity to read the message.

Knowing Bob’s quick temper, Charles expects him to sue for a divorce; knowing Alice’s fragile, volatile sensitivity, he also expects that, while on the rebound, she’d be open for a possible relationship with himself (Charles).
The plan works: as a result, *Bob is mislead into “knowing” that he has been cheated.*

He promptly sends Alice a message saying: ”I’ll see you in court”.

After divorce, Charles makes his seductive move, playing the friend-in-need. Again, *the original message becomes true:* now, Alice does have an affair with Charles, and Bob knows it.

*Sending a false message has enabled its validation.*
About the U.S. Army’s exploration of psi research and military applications of the paranormal.

General Brown: *When did the Soviets begin this type of research?*

Brigadier General Dean Hopgood: *Well, Sir, it looks like they found out about our attempt to telepathically communicate with one of our nuclear subs. The Nautilus, while it was under the Polar cap.*

General Brown: *What attempt?*

Dean: *There was no attempt. It seems the story was a French hoax.*
Dean: *But the Russians think the story about the story being a French hoax is just a story, Sir.*

General Brown: *So they started doing psi research because they thought we were doing psi research, when in fact we weren’t doing psi research?*

Dean: *Yes sir. But now that they *are* doing psi research, we’re gonna have to do psi research, sir.*

Dean: *We can’t afford to have the Russian’s leading the field in the paranormal.*
Epistemic Puzzle no. 4: Muddy Children

Suppose there are 4 children, all of them being good logicians, exactly 3 of them having dirty faces. Each can see the faces of the others, but doesn’t see his/her own face.

The father publicly announces:

“At least one of you is dirty”.

Then the father does another paradoxical thing: starts repeating over and over the same question “Do you know if you are dirty or not, and if so, which of the two?”
After each question, the children have to answer publicly, sincerely and simultaneously, based only on their knowledge, without taking any guesses. No other communication is allowed and nobody can lie.

One can show that, after 2 rounds of questions and answers, all the dirty children will come to know they are dirty! So they give this answer in the 3rd round, after which the clean child also comes to knows she’s clean, giving the correct answer at the 4th round.
First Question: What’s the point of the father’s first announcement (”At least one of you is dirty”)?

Apparently, this message is not informative to any of the children: the statement was already known to everybody! But the puzzle wouldn’t work without it: in fact this announcement adds information to the system! The children implicitly learn some new fact, namely the fact that what each of them used to know in private is now public knowledge.
Second Question: What’s the point of the father’s repeated questions?

If the father knows that his children are good logicians, then at each step the father knows already the answer to his question, before even asking it! However, the puzzle wouldn’t work without these questions. In a way, it seems the father’s questions are “abnormal”, in that they don’t actually aim at filling a gap in father’s knowledge; but instead they are part of a Socratic strategy of teaching-through-questions.

Third Question: How can the children’s statements of ignorance lead them to knowledge?
This is another story encoding the same puzzle:

On the island of Amazonia, women are dominant and the law says that, if at any point a woman knows her husband is cheating on her, she must shoot him the same day at noon in the main square.

Now the queen (truthfully) tells the women: “At least one of your husbands is a cheater. Whenever somebody’s husband is cheating, all the other women know it”.

For 16 days, nothing happens. Then, in the 17th day, shootings are heard.

**Question:** How many husbands died?
Puzzle no 5: Sneaky Children

Let us modify the last example a bit.

Suppose the children are somehow rewarded for answering as quickly as possible, but they are punished for incorrect answers; thus they are interested in getting to the correct conclusion as fast as possible.

Suppose also that, after the first round of questions and answers, two of the dirty children “cheat” on the others by secretly announcing each other that they’re dirty, while none of the others suspects this can happen.
As a result, they both will answer truthfully “I know I am dirty” in the second round.

One can easily see that the third dirty child will be totally deceived, coming to the “logical” conclusion that... she is clean!

So, after giving this wrong answer in the third round, she ends up by being punished for her credulity, despite her impeccable logic.
Clean Children Always Go Crazy

What happens to the clean child?

Well, assuming she doesn’t suspect any cheating, she is facing a contradiction: two of the dirty children answered too quickly, coming to know they’re dirty before they were supposed to know!

If the third child simply updates her knowledge monotonically with this new information (and uses classical logic), then she ends up believing everything: she goes crazy!
The Dangers of Mercy

In the Amazonia version of the story, assume that again there are exactly 17 cheating husbands (out of 1 million husbands on the island), while the rest of 999983 husbands are faithful.

Consider what happens now if all the wives of the 17 cheating husbands secretly decide to break the Queen’s rules, by quietly sparing the lives of their husbands, even when they get to know that they are cheating. We also assume that all the other wives do not suspect this: not only that they strictly obey by the Queen’s rules, but they believe that it is common knowledge that everybody else obeys by those same rules.

It’s easy to see that, in this case, 17 days will pass without any shooting. But it’s also easy to show that in the 18th day, shots will be heard. How many husbands will die in this scenario? How many of these are innocent?
The students in a high-school class know for sure that the date of the exam has been fixed in one of the five (working) days of next week: it’ll be the last week of the term, and it’s got to be an exam, and only one exam.

But they don’t know in which day.

Now the Teacher announces her students that the exam’s date will be a surprise: even in the evening before the exam, the students will still not be sure that the exam is tomorrow.
Intuitively, one can prove (by backward induction, starting with Friday) that, IF this announcement is true, then the exam cannot take place in any day of the week.

So, using this argument, the students come to “know” that the announcement is false: the exam CANNOT be a surprise.

GIVEN THIS, they feel entitle to dismiss the announcement, and... THEN, surprise: whenever the exam will come (say, on Tuesday), it WILL indeed be a complete surprise!
Consider the following game $G$, where Alice (a) is the \textit{first and third} player, and Bob (b) the \textit{second}:

\begin{itemize}

\item $v_0 : a$ -> $v_1 : b$ -> $v_2 : a$ -> $o_4 : 4, 5$
\item $o_1 : 3, 0$
\item $o_2 : 2, 3$
\item $o_3 : 5, 2$

\end{itemize}

In the leaves ("outcomes", denoted by $o$’s), the \textit{first number} is Alice’s \textit{payoff}, while \textit{the second} is Bob’s \textit{payoff}.
Backward Induction Method

We iteratively eliminate the obviously “bad” moves (that lead to “bad” payoffs for the player making the move) in stages, proceeding backwards from the leaves. The first elimination stage gives us:

\begin{align*}
v_0 : a & \rightarrow v_1 : b & \rightarrow v_2 : a \\
o_1 : 3,0 & \rightarrow o_2 : 2,3 & \rightarrow o_3 : 5,2
\end{align*}
Next Stage:

\[ v_0 : a \rightarrow v_1 : b \]

\[ o_1 : 3, 0 \rightarrow o_2 : 2, 3 \]
Backward Induction: The Outcome

*Final Stage:*

\[ v_0 : a \]

\[ o_1 : 3, 0 \]

So, according to this method, the **outcome of the game should by** \( o_1 \): Alice gets 3 dollars, while Bob gets nothing!

So the game stops at the first step, and the players have to be “rationally” satisfied with \((3, 0)\), when they could have got \((4, 5)\) or at least \((5, 2)\) if they continued to play!

This conclusion strucks most people as pretty “irrational”.
But... it seems to be an inescapable conclusion of (commonly known) “rationality”!

Indeed, suppose that it is common “knowledge” that everybody is “rational”: always plays to maximize his/her profit. Then, in particular, Alice is rational, so when choosing between outcomes $o_3$ and $o_4$ (at node $v_2$), she will choose $o_3$ (giving her 5 dollars rather than 4). This justifies the first elimination stage.

Now, since “rationality” is common “knowledge”, Bob knows that Alice is “rational”, so he can “simulate” the above elimination argument in his mind: so now Bob “knows” that, if the node $v_2$ is reached during the game, then Alice will choose outcome $o_3$. 
Given this information, he knows that, if arriving at node $v_1$, the only possible outcomes would be $o_2$ and $o_3$. From these two, $o_2$ gives Bob higher payoff (3 instead of 2). Since Bob is rational, it follows that, if node $b_1$ is reached during, Bob would choose $o_2$. This justifies the second elimination stage.

Again, all this is known to Alice: she knows that Bob is rational and that he knows that she is rational, so she can “simulate” all the above argument, concluding that at the initial node $v_0$, the possible outcomes are only $o_1$ and $o_2$. Being rational herself, she has to choose $o_1$ (giving her a higher payoff $3 > 2$).
In the view of the above argument, let’s re-examine Bob’s reasoning when he plans his move for node $v_1$ in the Centipede game:

Based on the above argument, Bob knows that, IF “rationality” is “common knowledge”, then Alice should choose outcome $o_1$, thus finishing the game.
So Bob reasons like this: IF node \( v_1 \) would be reached AT ALL, then this could only happen if the above assumption ("common knowledge of rationality") was wrong! So, in this eventuality, he will have to give up his "knowledge of Alice’s rationality": she would have already made what appeared as an “irrational” choice (of \( v_1 \) over \( o_1 \)). Even if he started the game believing that Alice was rational, he may now reassess this assumption.

This undermines the justification for the first elimination step, at least in Bob’s mind: once he’s not sure of Alice’s rationality, he cannot be sure anymore that she will choose \( o_3 \) over \( o_4 \), if given this opportunity.
Rational Pessimism about Other’s Rationality

So, the question is: what will Bob believe about Alice if he’d seen her “irrationally” choosing $v_1$ over $o_1$?

One possible option is the “pessimistic” one: in this case, Bob would conclude that Alice is “irrational”, and that moreover she will continue to be “irrational” from then on.

This seems pretty reasonable: after all, if Alice has once behaved weird, what’s to stop her from doing it again?!

Let’s assume that Bob does indeed adopt this attitude of “rational pessimism” towards Alice’s rationality.
The Consequences of Pessimism

So Bob thinks that, if node $v_1$ would be reached then (Alice is “irrational”, so that), if she’d be given the opportunity to choose between $o_3$ and $o_4$, Alice would “stupidly” choose $o_4$. So, as far as Bob’s beliefs go, the “first” elimination stage goes now as follows:

```
  $v_0 : a$  $v_1 : b$  $v_2 : a$  $o_4 : 4,5$
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  $o_1 : 3,0$  $o_2 : 2,3$
```
Next Stage

Given that, and Bob’s rationality, the next stage according to Bob is:

\[ v_0 : a \rightarrow v_1 : b \rightarrow v_2 : a \rightarrow o_4 : 4, 5 \]

\[ o_1 : 3, 0 \]
We assumed common knowledge of rationality, so in fact Alice IS rational: Bob is wrong to revise his belief, since she would never choose $o_4$ over $o_3$. Even her choice of $v_1$ over $o_1$ is perfectly justified, if we assume that she knows Bob’s belief revision policy. Then the best move for Alice is to first choose $v_1$ and later choose $o_3$:
So, assuming common knowledge of “rationality”, we “proved” **both** that the backward induction outcome **is reached** and that **it is not reached**!

Moreover, this reasoning can be **generalized** (as pointed out by Binmore, Bonanno, Bicchieri, Reny, Brandenburger and others): the argument underlying the backward induction method seems to give rise to a **fundamental paradox** (the so-called “**BI paradox**”).
Epistemic Logic was first formalized by Hintikka (1962), who also sketched the first steps in formalizing doxastic logic.

They were further developed and studied by philosophers and logicians (Parikh, Stalnaker, van Benthem etc.), computer-scientists (Halpern, Vardi, Fagin etc.) and economists (Aumann, Brandeberger, Samet etc.).
For a set $\Phi$ of facts, a $\Phi$-Kripke model is a triple

$$\mathcal{S} = (S, \{R_i\}_{i \in I}, \parallel \cdot \parallel, s_*)_{i \in I}$$

consisting of

1. a set $S$ of "possible worlds"
2. a family of binary accessibility relations $R_i \subseteq S \times S$, indexed by labels $i \in I$,
3. and a valuation $\parallel \cdot \parallel : \Phi \rightarrow \mathcal{P}(S)$, assigning to each $p \in \Phi$ a set $\parallel p \parallel_S$ of states
4. a designated world $s_*$: the "actual" one.
Kripke Semantics: Modalities

For atomic sentences and for Boolean connectives, we use the same semantics (and notations) as on epistemic-doxastic models.

For every sentence \( \varphi \), we can define a new sentence \([R_i]\varphi\) by (universally) quantifying over \( R_i \)-accessible worlds:

\[
s \models [R_i] \varphi \text{ iff } t \models \varphi \text{ for all } t \text{ such that } s R_i t.
\]

The operator \([iR] \varphi\) is called a “(universal) Kripke modality”. When the relation \( R = R_1 \) is unique (i.e. \( \text{card}(I) = 1 \)), we can leave it implicit and abbreviate \([R] \varphi\) as \( \square \varphi \).

The dual existential modality is given by

\[
< R_i > \varphi := \neg [R_i] \neg \varphi.
\]

Again, when \( R \) is unique, we can abbreviate \(< R > \varphi\) as \( \Diamond \varphi \).
In a context when we interpret a modality $\Box \varphi$ as knowledge, we use the notation $K\varphi$ instead, and we denote by $\sim$ the underlying binary relation $R$.

When we interpret the modality $\Box \varphi$ as belief, we use the notation $B\varphi$ instead, and we denote by $\rightarrow$ the underlying binary relation $R$. 
A multi-agent Kripke model is simply a Kripke model

\[ S = (S, \{ a \rightarrow \}_{a \in A}, \| \|) \]

in which the set of labels (for relations) is denoted by \( A \), and in which
of think of these labels as being “names” of epistemic agents.
Scenario 1: The concealed coin

Two players $a$, $b$ and a referee $c$ play a game. In front of everybody, the referee throws a fair coin, catching it in his palm and fully covering it, before anybody (including himself) can see on which side the coin has landed.
Modalities

The operator

\[ \Box_a \varphi := \left[ \overset{a}{\rightarrow} \right] \varphi \]

may be interpreted as knowledge (in which case we use the notation \( K_a \varphi \) instead) or as belief (in which case we use \( B_a \varphi \) instead), depending on the context.

Its existential dual

\[ \Diamond_a \varphi := \neg \Box_a \neg \varphi \]

denotes a sense of “epistemic/doxastic possibility”.
An epistemic model (or S5-model) is a Kripke model in which all the accessibility relations are equivalence relations, i.e. reflexive, transitive and symmetric (or equivalently: reflexive, transitive and Euclidean).
**Knowledge properties**

S5 epistemic models validate the following:

- The **Veracity** (known as axiom T in modal logic) $K \varphi \Rightarrow \varphi$ corresponds to the **reflexivity** of the relation $\sim$.

- **Positive Introspection** (known as axiom 4 in modal logic) $K \varphi \Rightarrow KK \varphi$ corresponds to the **transitivity** of the relation $\sim$.

- **Negative Introspection** (known as axiom 5 in modal logic) $\neg K \varphi \Rightarrow K \neg K \varphi$ corresponds to **Euclideaness** of the relation $\sim$:

  if $s \sim t$ and $s \sim w$ then $t \sim w$.

In the context of the other two, Euclideaness is equivalent to **symmetry**:

if $s \sim t$ then $t \sim s$. 
A doxastic model (or KD45-model) is a Φ-Kripke model satisfying the following properties:

- **(D) Seriality**: for every $s$ there exists some $t$ such that $s \rightarrow t$;
- **(4) Transitivity**: If $s \rightarrow t$ and $t \rightarrow w$ then $s \rightarrow w$
- **(5) Euclideaness**: If $s \rightarrow t$ and $s \rightarrow w$ then $t \rightarrow w$
Putting together in the same structure the belief arrows $\rightarrow$ from the previous example with the knowledge arrows from before, now denoted by $\sim$, we obtain a Kripke model for both knowledge AND belief.
Belief properties

Doxastic models validate the following:

- **Consistency** of beliefs (known as axiom D in modal logic)
  \[ \neg (B\varphi \land \neg B\varphi) \]
  corresponds to the **seriality** of the relation \( \rightarrow \).

- **Positive Introspection for Beliefs** (axiom 4)
  \[ B\varphi \Rightarrow BB\varphi \]
  corresponds to the **transitivity** of the relation \( \rightarrow \).

- **Negative Introspection for Beliefs** (axiom 5)
  \[ \neg B\varphi \Rightarrow B\neg B\varphi \]
  corresponds to **Euclideaness** of the relation \( \rightarrow \).
But, we can see that, in the setting of Kripke models, the properties specific to “epistemic-doxastic models” are NOT automatically satisfied.

So Kripke semantics is more general than the “sphere semantics”.

In fact, one can use Kripke semantics to interpret various weaker notions of “knowledge”, e.g. a type of knowledge that is truthful (factive) and positively introspective, but NOT necessarily negative introspective.

An S4-model for knowledge is a Kripke model satisfying only reflexivity and transitivity (but not necessarily symmetry or Euclideaness).
Similarly, by dropping the corresponding semantic conditions, one can use Kripke models to represent non-introspective beliefs, or even inconsistent beliefs.
The Problem of Logical Omniscience

However, it is easy to see that any Kripke modality $\square = [R]$ still validates Kripke’s axiom

$$(K) \quad \square(\varphi \Rightarrow \psi) \Rightarrow (\square \varphi \Rightarrow \square \psi),$$

and still satisfies the **Necessitation Rule**:

if $\varphi$ is valid, then $\square \varphi$ is valid.

So, if we interpret the modality as “knowledge” or “belief”, then every logical validity is known/believed, and similarly every logical entailment between two propositions is known/believed.

This means that Kripke semantics can only model “ideal” reasoners, who may have limited access to external truths, but have unlimited inference powers.
“Common” Modalities

The sentence \( C \square \varphi \) is obtained by quantifying over all worlds that are accessible by any concatenations of arrows:

\[
\text{if } s \models_s C \square \varphi \text{ iff } t \models_s \varphi \text{ for every } t \text{ and every a finite chain (of length } n \geq 0) \text{ of the form } s = s_0 \rightarrow a_1 \rightarrow s_1 \rightarrow a_2 \rightarrow s_2 \cdots \rightarrow a_n \rightarrow s_n = t.
\]

\( C \square \varphi \) may be interpreted as common knowledge (in which case we use the notation \( C_k \varphi \) instead) or common true belief (in which case we use \( C_b \varphi \) instead), depending on the context.

In epistemic-doxastic models, we have both \( C_k \) and \( C_b \).
Common Knowledge Within a Group

Common knowledge (or common true belief) can also be considered in a restricted form, as common knowledge within a given (sub)group $G \subseteq \mathcal{A}$. The definition is the same, except that we restrict the concatenated arrows to arrows within the group $G$:

$$s \models s C\Box_{G}\varphi \text{ iff } t \models s \varphi \text{ for every } t \text{ and every a finite chain of the form } s = s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_n} s_n = t, \text{ with } a_1, \ldots, a_n \in G.$$ 

Obviously, full common knowledge/belief $C\Box$ (as previously defined) corresponds to the case that $G$ is the set $\mathcal{A}$ of all agents:

$$C\Box \varphi = C\Box_{\mathcal{A}} \varphi$$
Common Knowledge as an Infinite Conjunction

If we make the abbreviation

\[ E_G \varphi := \bigwedge_{a \in G} \Box_a \varphi \]

(“everybody knows \( \varphi \)”), then we can easily check that:

\[ s \models_s C \Box_G \iff s \text{ satisfies all the (infinitely many) sentences} \]

\[ \varphi, E_G \varphi, E_G E_G \varphi, E_G E_G E_G \varphi, \ldots \]
In *this sense*, we can say that $E_G \varphi$ is equivalent to the “infinite conjunction”

$$\varphi \land E_G \varphi \land E_G E_G \varphi \land \ldots$$

However, the most used modal-epistemic languages are *finitary*, so that $C \Box$ cannot be defined as the (impossible to form) infinite conjunction. Instead, $C \Box$ is usually taken as a **primitive** operator (introduced via the above semantic clause).
Validities involving common modalities

“Fixed-Point Axiom” (also known as “mix”)

\[ C\Box_G \varphi \Rightarrow (\varphi \land E_G C\Box_G \varphi) \]

where the operator \( E_G \) (“everybody in \( G \) knows...”) was defined as abbreviation in previous slide.

“Induction Axiom”

\[ C\Box_G (\varphi \Rightarrow E_G \varphi) \Rightarrow (\varphi \Rightarrow C\Box_G \varphi) \]
A **complete proof system** for modal (or epistemic, or doxastic) logic with common modality $C\Box$ consists of:

- the axioms and rules of the multi-modal system $K$ (or $S5$, or $KD45$),
- the Fixed Point Axiom,
- the Induction Axiom, and
- the Normality conditions (axiom $K$ and the Necessitation Rule) for the modality $C\Box$. 
"Distributed" Modalities

The sentence $D\square \varphi$ is obtained by quantifying over all worlds that are \textbf{simultaneously accessible} by all arrows (from a given world):

$$s \models_S D\square \varphi \text{ iff } t \models_S \varphi \text{ for every } t \text{ such that } s \xrightarrow{a} t \text{ holds for all } a \in A.$$ 

In other words, $D\square$ is the Kripke modality corresponding to the \textbf{intersection of all epistemic relations $\bigcap_{a \in A} \xrightarrow{a}$}.

When the relations $\xrightarrow{a}$ are \textit{reflexive} (corresponding to some form of "knowledge"), $D\square \varphi$ may be interpreted as \textbf{distributed knowledge} (in which case we use the notation $Dk\varphi$ instead).

When the relations $\xrightarrow{a}$ represent beliefs, one can also interpret $D\square$ as \textbf{"distributed belief" $DB\varphi$}, but in this case it might actually be \textit{inconsistent}. 


Distributed Knowledge Within a Group

As for common knowledge, distributed knowledge can also be considered in a restricted form: distributed knowledge within a given (sub)group $G \subseteq A$. The definition is the same, except that we restrict the intersection of the arrows within the group $G$:

$$s \models D\Box_G \varphi \text{ iff } t \models \varphi \text{ for every } t \text{ such that } s \xrightarrow{a} t \text{ holds for all } a \in G.$$

Distributed knowledge precisely captures the implicit (or “virtual”) knowledge of the group $G$: what the agents in $G$ could come to know if they would pool together all their private knowledge.
Distributed Knowledge: Complete Axiomatization

It is easy to see that each of the semantic properties (reflexivity, transitivity, Euclideaness) corresponding to logical postulates usually attributed to “knowledge” (Veracity, Positive Introspection, Negative Introspection) holds for the intersection relation $\bigcap_{a \in G} a \rightarrow$ whenever it holds for each of the arrows $\rightarrow$ (for each $a \in G'$).

Thus, a complete axiomatization of modal/epistemic/doxastic logic with distributed knowledge is given by:

- the axioms and rules for modal/epistemic/doxastic logic;
- the corresponding axioms and rules for Distributed Knowledge;
- the axiom

$$\Box_a \varphi \Rightarrow D \Box_G \varphi$$

, for every $a \in G'$.
Scenario 2: The coin revealed

The referee $c$ opens his palm and shows the face of the coin to everybody (to the public, composed of $a$ and $b$, but also to himself): they all see it’s Heads up, and they all see that the others see it etc.

So this is a “public announcement” that the coin lies Heads up. We denote this event by $!H$. Intuitively, after the announcement, we have common knowledge of $H$, so the model of the new situation is:

$H$
Scenario 3: 'Legal’ Private Viewing

Instead of Scenario 2: in front of everybody, the referee (c) uncovers the coin, so that (they all see that) he, and only he, can see the upper face. This changes the initial model to

Now, c knows the real state. E.g. if it’s Heads, he knows it, and disregards the possibility of Tails. a and b don’t know the real state, but they know that c knows it. c’s viewing of coin is a ”legal”, non-deceitful action, although a private one.
Fair-Game Announcements

Equivalently: in front of everybody, an announcement of the upper face of the coin is made, but in such a way that (it is common knowledge that) only $c$ hears it.

Such announcements (first modeled by H. van Ditmarsch) are called fair-game announcements, they can be thought of as “legal moves” in a fair game: nobody is cheating, all players are aware of the possibility of this move, but only some of the players (usually the one who makes the move) can see the actual move. The others know the range of possible moves at that moment, and they know that the “insider” knows his move, but they don’t necessarily know the move.
Scenario 4: Cheating

Suppose that, after Scenario 1, the referee $c$ has taken a peek at the coin, before covering it. Nobody has noticed this. Indeed, let’s assume that $c$ knows that $a$ and $b$ did not suspect anything.

This is an instance of cheating: a private viewing which is ”illegal”, in the sense that it is deceitful for $a$ and $b$. Now, $a$ and $b$ think that nobody knows on which side the coin is lying. But they are wrong!
We indicated the *real world* here. In the actual world (above), $a$ and $b$ think that the only possibilities are the worlds below. That is, they *do not even consider the ”real” world as a possibility.*
Scenario 5: Secret Communication

After cheating (Scenario 4), $c$ engages in another "illegal" action: he secretly sends an email to his friend $a$, informing her that the coin is Heads up. Suppose the delivery and the secrecy of the message are guaranteed: so $a$ and $c$ have common knowledge that $H$, and that $b$ doesn’t know they know this.

Indeed, $b$ is completely fooled: he doesn’t suspect that $c$ could have taken a peek, nor that he could have been engaged in secret communication.
The model is
Both of the above actions were examples of completely private announcements

\[
!_G \varphi
\]

of a sentence \( \varphi \) to a group \( G \) of agents: in the first case \( G = \{c\} \), in the second case \( G = \{a, c\} \).

The “insiders” (in \( G \)) know what’s going on, the “outsiders” don’t suspect anything.
Scenario 5’: Wiretapping?

In Scenario 5’, everything goes on as in Scenario 5, except that in the meantime b is secretly breaking into c’s email account (or wiretapping his phone) and reading c’s secret message. Nobody suspects this illegal attack on c’s privacy. So both c and a think their secret communication is really secret and unsuspected by b: the deceivers are deceived.

What is the model of the situation after this action?! Things are getting rather complicated!
Scenario 6

This starts right after Scenario 3, when it was common knowledge that $c$ knew the face. $c$ attempts to send a secret message to $a$ announcing that $H$ is the case. $c$ is convinced the communication channel is fully secure and reliable; moreover, he thinks that $b$ doesn’t even suspect this secret communication is going on. But, in fact, unknown and unsuspected by $c$, the message is intercepted, stopped and read by $b$. As a result, it never makes it to $a$, and in fact $a$ never knows or suspects any of this. As for $b$, he knows all of the above: not only now he knows the message, but he knows that he “fooled” everybody, in the way described above.
The Update Problem

We need to find a general method to solve all the above problems, i.e. to compute all these different kinds of updates.
Dynamic Modalities

To express informational changes, DEL borrows an idea from PDL (Propositional Dynamic Logic), namely the use of dynamic modalities

\[ [\alpha] \varphi \]

The intended meaning is that: **if the action** \( \alpha \) **is performed (in the current state) then the sentence** \( \varphi \) **will become true after this.**

The “if” part is interpreted as a material conditional: \( [\alpha] \varphi \) is be definition true in states at which \( \alpha \) cannot be performed.

The “then” part contains an implicit **universal quantifier**: when \( \alpha \) is a non-deterministic action (i.e. when executed in a given input-state \( s \), may result in either one of several possible output-states), \( [\alpha] \varphi \) is taken to mean that \( \varphi \) will be true **at all possible output-states** (that can result by executing \( \alpha \) in state \( s \)).
Traditional (PDL) Semantics

In PDL, dynamic modalities $[\alpha]$ are treated exactly like any other Kripke modalities: the models $S$ are equipped with a set of states $S$ and with binary relations $R_\alpha \subseteq S \times S$, one for each dynamic modalities, and the usual Kripke clause for $[R_\alpha]$ gives the semantics for $[\alpha]$:

$$s \models S [\alpha] \varphi \text{ iff } t \models S \varphi \text{ for all } t \text{ such that } s R_\alpha t.$$ 

The only difference is in the interpretation: the relations $R_\alpha$ are interpreted as transition relations for “actions”, relating an input-state $s$ to the possible output-states $t$ that might result by executing action $\alpha$ on input $s$.

For this reason, many times the “transition” notation

$$s \rightarrow^\alpha t$$

is used instead of $s R_\alpha t$. 
Epistemic Actions as Model Transformers

However, in DEL we will consider \textit{transition relations between states living in DIFFERENT models}.

An action $\alpha$ will be thought of now as a \textbf{model transformer}, consisting of

1. a map

$$\mathbf{S} \mapsto \mathbf{S}^{\alpha}$$

that takes any initial Kripke model $\mathbf{S}$ into an “updated” model $\mathbf{S}^{\alpha}$, and

2. a binary transition relation

$$\xRightarrow{\alpha} \subseteq \mathbf{S} \times \mathbf{S}^{\alpha}$$

between the input-states of the initial model $\mathbf{S}$ and the output-states (living in the updated model $\mathbf{S}^{\alpha}$).
The semantics of dynamic modalities will thus be given, for any state $s \in S$:

$$s \models_s [\alpha] \varphi \text{ iff } t \models_{S^\alpha} \varphi \text{ for all } t \in S^\alpha \text{ such that } s \rightarrow_S^\alpha t.$$ 

Note that the truth-value of $\varphi$ (at some state $s$) in the model $S$ depends on the truth-value of $\varphi$ (at some other states $t$) in the updated model $S^\alpha$. 
Dual (Existential) Dynamic Modalities

\( \langle \alpha \rangle \varphi ::= \neg[\alpha] \neg \varphi \)
Public Announcement Logic (*PAL*)

The syntax of **basic** *PAL* is obtained by adding to basic multi-modal logic dynamic modalities for public announcements:

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \Box_a \varphi \mid [! \varphi] \varphi \]

These formulas are interpreted on multi-agent Kripke models.

If we use $K$ instead of $\Box$ in the above syntax, and interpret formulas on epistemic (S5) models, we obtain **epistemic logic with public announcements**.

Similarly, if we use $B$ instead of $\Box$, we obtaining the **doxastic logic with public announcements**, with the proviso that ‘doxastic models” will now mean $K45$ (*rather than $KD45$*): as we’ll see, axiom $D$ (consistency of beliefs) is inconsistent with public announcements!
Formal Semantics

As a model transformer, a public announcement $!\varphi$ maps any model $S = (S, \xrightarrow{a}, \parallel \bullet \parallel)$ to a new model $S^{!\varphi} = (S_{\varphi}, \xrightarrow{a_{\varphi}}, \parallel \bullet \parallel_{\varphi})$, given by:

$$S_{\varphi} := \parallel \varphi \parallel_{S},$$

$$s \xrightarrow{a_{\varphi}} t \text{ iff } s \xrightarrow{a} t, \text{ for } s, t \in S_{\varphi},$$

$$\parallel p \parallel_{\varphi} := \parallel p \parallel \cap S_{\varphi}$$

The transition relation $\xrightarrow{!\varphi}$ relates any state $s \in S$ satisfying $\varphi$ to the same state $s$ in the model $S^{\varphi}$. 

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A complete axiomatization of basic $PAL$ is given by:
the the axioms and rules of multi-agent modal logic $K$;
and the following **Reduction Axioms** (or “Recursion Axioms”):

**Atomic Permanence**: $[\![\varphi]\!]p \iff \varphi \Rightarrow p$ (for atomic facts $p$),

**Announcement-Negation**: $[\![\varphi]\!][\neg\psi] \iff \varphi \Rightarrow \neg[\![\varphi]\!]\psi$

**Announcement-Conjunction**: $[\![\varphi]\!](\psi \land \theta) \iff [\![\varphi]\!]\psi \land [\![\varphi]\!]\theta$

**Announcement-Knowledge**: $[\![\varphi]\!]\Box_a \psi \iff \varphi \Rightarrow \Box_a[\![\varphi]\!]\psi$.

**NOTE 1**: These are axiom schemata, rather than single axioms.

**NOTE 2**: The logic is **NOT closed under substitution**!
A complete axiomatization of basic epistemic (or doxastic) logic with public announcements is given by taking:

the multi-agent version of the axioms and rules of the logic $S5$ (i.e. $KT45$) for epistemic logic, or the axioms $K45$ for doxastic logic;

together with the above Reduction Axioms, stated for $K$ (respectively, for $B$).

NOTE: If we allow (as we should) models in which some beliefs are false, then the axiom $D$ of consistency of beliefs cannot be maintained in the presence of public announcement operators!

Find a counterexample!

This IS a problem! It has to do with the fact that (false) beliefs cannot be revised in $PAL$. Tomorrow we’ll fix this problem!
By recursively using the Reduction Axioms, we can “rewrite” every formula of PAL to an equivalent formula of basic modal (epistemic, doxastic) logic:

**Proposition:** *PAL has the same expressivity as basic modal logic.*

So, in principle, all dynamic modalities are eliminable from *PAL*.

However, there is a difference in *succinctness* between *PAL* and basic *ML*:

*PAL is exponentially more succinct.*
Announcements about Announcements

It is important that in PAL we can iterate all the constructions: we can announce, not only facts

\[ !p \]

or combinations of facts (Boolean formulas)

\[ !(p \lor \neg q) , \]

but also epistemic formulas

\[ !(\neg K_a p) \]

and can even make announcements about other announcements

\[ !(\neg K_a [!q] \neg K_a p) . \]

This last fact is essential for having the following nice closure property.
Closure of public announcements under composition

Here is an interesting valid schema (all of whose instance can be derived in the above system):

$$[!\varphi][!\psi]\theta \iff [!(\varphi \land [!\varphi]\psi)]\theta$$

Semantically, this captures the closure of the class of public announcements under sequential compositions:

performing successively two public announcements

$$!\varphi; !\psi$$

is equivalent to performing only one more complex public announcement

$$!(\varphi \land [!\varphi]\psi)$$
Adding back the common knowledge operator $C\Box$ to the syntax, we obtain the public announcement logic with common knowledge $PAC$.

However, in this law there is no reduction law for common knowledge:
the formula $[!p]C\Box q$ is not equivalent to any formula in the logic without dynamic modalities.

So $PAC$ is more expressive then modal-epistemic logic with common knowledge.

Axiomatization??!!
Towards an Axiomatization

First, let us observe that the Induction Axiom for common knowledge

\[ C \square_G (\varphi \Rightarrow E_G \varphi) \Rightarrow (\varphi \Rightarrow C \square_G \varphi) \]

is equivalent to the following inference rule:

**Induction Rule:**

*From* \( \chi \Rightarrow \psi \) *and* \( \chi \Rightarrow E_G \chi \), *infer* \( \chi \Rightarrow C \square_G \psi \).

This is the rule that we will generalize to prove expressions of the form \([! \varphi]C \square_G \psi\).
A complete axiomatization of PAC is given by:

- the axioms and rules of epistemic/doxastic logic with common knowledge modalities;
- the Reduction Axioms of PAL;
- the Normality conditions for dynamic modalities:
  - Necessitation: from $\psi$ infer $[!\varphi]\psi$
  - Kripke’s axiom: $[!\varphi](\psi \Rightarrow \theta) \Rightarrow ([!\varphi]\psi \Rightarrow [!\varphi]\theta)$

the Composition Axiom:

$$[!\varphi][!\psi]\theta \iff [!(\varphi \land [!\varphi]\psi)]\theta$$

the Announcement Rule:

From $\chi \Rightarrow [!\varphi]\psi$ and $\chi \land \varphi \Rightarrow E_G\chi$, infer $\chi \Rightarrow [!\varphi]C\Box_G\psi$. 
Another Axiomatization of \( P\text{AC} \)

Another solution (due to van Benthem) is to **extend the language of classical epistemic logic** with **some appropriate static modality that “pre-encodes” the dynamics of common knowledge**: “**conditional common knowledge**” \( C\Box \varphi \psi \).

The usual common knowledge operator will be then recovered as just a special case:

\[
C\Box_G \varphi = C\Box^T \varphi ,
\]

where \( T \) is any tautology.

As a result, in this extended language we WILL have a Reduction Law for common knowledge:

\[
[\varphi]C\Box_G \psi \iff \varphi \Rightarrow C\Box \varphi [\varphi] \psi .
\]
Of course, to have a full elimination of all dynamic modalities, we will also need to have now a **Reduction Law for “conditional common knowledge”**!

\[ [\!\! \varphi ] C \square_G^\theta \psi \iff \text{??????????} \]

But, luckily, we don’t have to extend the language further: we get such a Reduction Law for free!

The resulting complete logic is called **PACC**.
Semantics for Conditional Common Knowledge

In a model $S = (S, a, \| \bullet \|)$, put

$$s \models_s C^{\theta}_G \varphi \iff t \models_s \varphi$$

for every $t \in S$ and every chain of the form $s = s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_n} s_n = t$,

with all $a_1, \ldots, a_n \in G$ and all $s_1, \ldots s_n \models_s \theta$.

Essentially, this makes $C^{\theta}_G \varphi$ equivalent to the infinite conjunction

$$\varphi \land E_G (\theta \Rightarrow \varphi) \land E_G (\theta \Rightarrow E_G (\theta \Rightarrow \varphi)) \land \ldots$$

where $E_G \varphi = \bigwedge_{a \in G} \Box_a \varphi$ is the “everybody knows/believes” operator.
A complete axiomatization of the “static” epistemic logic ECC of conditional common knowledge/belief consists of the axioms and rules of any doxastic/epistemic logic, plus the following:

the Kripke axioms for conditional common knowledge/belief

\[ C\Box_G^\theta (\varphi \Rightarrow \psi) \Rightarrow C\Box_G^\theta \varphi \Rightarrow C\Box_G^\theta \psi, \]

a Fixed-Point Axiom

\[ C\Box_G^\theta \varphi \Rightarrow (\varphi \land E_G (\theta \Rightarrow C\Box_G^\theta \varphi)), \]

an “Induction Axiom”

\[ C\Box_G^\theta (\varphi \Rightarrow E_G (\theta \Rightarrow \varphi)) \Rightarrow (\varphi \Rightarrow C\Box_G^\theta \varphi), \]

and a Necessitation Rule: From \( \varphi \) infer \( C\Box_G^\theta \varphi. \)
Reduction Laws for (Conditional) Common Knowledge

A complete axiomatization of the logic $PACC$ of public announcements with conditional common knowledge is given by putting to the axioms and rules of $PAL$, together with the above axioms of $ECCE$ and the following Reduction Law:

Announcement-Conditional-Common-Knowledge:

$$[!\varphi]C^G_\theta \psi \iff \varphi \Rightarrow C_G^{<!\varphi>\theta} [!\varphi] \psi$$

**NOTE:** As a special case of this, we get:

$$[!\varphi]C_G \psi \iff \varphi \Rightarrow C_G^{!\varphi} [!\varphi] \psi$$

Again, by recursively using the Reduction Axioms, we can “rewrite” every formula of $PACC$ to eliminate dynamic modalities:

**Proposition:** $PACC$ has the same expressivity as $ECC$. 

Let $d_i$ be an atomic sentence saying that "child $i$ is dirty".

Then the following abbreviations can be made in $PAC$:

\[
\text{vision} := \bigwedge_{a \in A} \bigwedge_{b \neq a} \left( (d_b \leftrightarrow \square_a d_b) \land (\neg d_b \leftrightarrow \square_a \neg d_b) \right)
\]

\[
\text{at least one} := \bigvee_{a \in A} d_a
\]

\[
\text{exactly } k := \bigvee_{G \subseteq A, |G|=k} \left( \bigwedge_{a \in G} d_a \land \bigwedge_{a \notin G} \neg d_a \right)
\]

\[
\text{nobody knows} := \bigwedge_{a \in A} (\neg \square_a d_a \land \neg \square_a \neg d_a)
\]

\[
\text{dirties know} := \bigwedge_{a \in A} (d_a \Rightarrow \square_a d_a)
\]
Then the Muddy children scenario can be encoded in the formula:

\[
(\text{exactly } k \land C \Box \text{vision}) \Rightarrow [!(\text{at least one})][!(\text{nobody knows})]^{k-1}\text{dirties know}
\]

**EXERCISE:** Assuming that our basic epistemic/doxastic modalities satisfy the axioms of \( K45 \), **prove the validity of this sentence by PAC** by first reducing it to a purely epistemic formula (using the Reduction Laws), and then showing that the resulting formula is valid either by semantic methods or axiomatically (proving it in epistemic logic with common knowledge).
Answering Our Questions

Answer to First Question: The point of Father’s first announcement (“At least one of you is dirty”) is that it converts something that was only known to everybody into common knowledge. This public announcement changes the epistemic situation (i.e. the model)!

Answer to Second Question: The point of Father’s repeated questions is that answering the question changes the model. The truth value of the answer may be different in the new model, so answering the same question the second (or third...) time is non-redundant.

Answer to Third Question: Since the model keeps changing (by repeated public announcements of “ignorance”), more and more information is gained. At some point, the truth-value of the sentence expressing “ignorance” changes from true to false: repeated learning of “ignorance” has lead to “knowledge”!
EXTENSIONS OF \textit{PAL}: Adding Distributed Knowledge

Adding to \textit{PAL} the corresponding \textit{distributed knowledge} operator $D\square_G \varphi$, we obtain the logic \textit{PAD}.

If we add both common knowledge and distributed knowledge, we obtain the logic \textit{PACD}.
A complete axiomatization of $PAD$ (or $PACD$) is given by:

- the axioms and rules of $PAL$ (or $PAC$);
- the usual axioms for distributed knowledge;
- the following Reduction Law:

**Announcement-Distributed-Knowledge:**

$$[!\varphi]D \square_G \psi \iff \varphi \Rightarrow D \square_G [!\varphi] \psi.$$
Decidability

Proposition: The logics \(PAL\), \(PAC\), \(PACC\), \(PAD\) and \(PACD\) are \textit{decidable} and have \textit{the (strong) finite model property}. 
Suppose we introduce a dynamic modality $[!a] \psi$, corresponding to the action by which agent $a$ **publicly announces** “all (s)he knows”.

We interpret this in a *language-independent manner*: $a$ announces which states (s)he considers possible (or equivalently, which states she knows to be impossible).
At a state $s$ in a model $\mathbf{S} = (S, \xrightarrow{a}, \parallel \bullet \parallel)$, this acts as the public announcement $!s(a)$ of the set

$$s(a) := \{ t \in S : s \xrightarrow{a} t \},$$

representing agent $a$’s current information cell (in the partition induced by $a$’s equivalence relation).

So the semantics of $!a$ is given by deleting all states outside $s(a)$ and keep everything else the same; i.e. by relativizing (i.e. restricting all the components of) $\mathbf{S}$ to the set $s(a)$. 

**Semantics of $!a$**


Proof System for \(!a\)

The Reduction Law for “knowledge” will use distributed knowledge.

So we need to start with a static epistemic logic with distributed knowledge, and add the following Reduction Laws:

\[
[!a]p \iff p
\]

\[
[!a]\neg \varphi \iff \neg[!a]\varphi
\]

\[
[!a](\varphi \land \psi) \iff [!a]\varphi \land [!a]\psi
\]

\[
[!a]\Box_b \varphi \iff D\Box_{\{a,b\}}[!a]\varphi
\]

\[
[!a]D\Box_G \varphi \iff D\Box_{G \cup \{a\}}[!a]\varphi
\]

But what about common knowledge?
Again, we have to introduce a new conditional modality, formalizing yet another ("static") epistemic attitude.

For $A, B \subseteq A$, we read $C \square^B_A \psi$ as saying that:

**group $A$ has common knowledge of $\psi$ conditional on the knowledge of (all agents of) group $B$.**

Formally, if we put for any group $G \subseteq A$

$$
\dfrac{G}{\rightarrow} = \bigcap_{a \in G} \dfrac{a}{\rightarrow},
$$

then $C \square^B_A$ is defined as the Kripke modal logic for

$$
\left( \bigcup_{a \in A} \dfrac{B \cup \{a\}}{\rightarrow} \right)^*.
$$
In other words:

\[ s \models_S C \square^B_A \varphi \iff t \models_S \varphi \text{ for all } t \in S, \text{ all } a_1, \ldots, a_n \in A \text{ and all chains } s = s_0 \xrightarrow{B \cup \{a_1\}} s_1 \xrightarrow{B \cup \{a_2\}} s_2 \cdots \xrightarrow{B \cup \{a_n\}} s_n = t. \]

If we put

\[ E \square^B_A \varphi := \bigwedge_{a \in A} D \square_{B \cup \{a\}} \varphi, \]

then we can easily check that

\[ C \square^B_A \varphi \text{ is equivalent to } \varphi \land E \square^B_A \varphi \land E \square^B_A (E \square^B_A \varphi) \land \ldots \]
The static logic $C\Box^B_A$ has $C\Box^B_A$ as the only modalities (one for each pair of groups $A, B \subseteq A$).

All the standard epistemic operators are definable as abbreviations:

$$\Box_a \psi := C\Box^\emptyset_{\{a\}} = C\Box^\{a\}_{\{a\}} \psi$$

$$C\Box_A \psi := C\Box^\emptyset_A \psi$$

$$D\Box_A \psi := C\Box^A_\emptyset \psi = C\Box^A_{\{a\}} \psi = C\Box^A_{\{a\}} \psi = C\Box^A_{\{a\}} \psi,$$ for any $a \in A$

$$E\Box^B_A \psi := \bigwedge_{a \in A} \Box^B_{\{a\}} \psi.$$
Axiomatization of the Static Logic $C\Box_A^B$

The static logic $C\Box_A^B$ is completely axiomatized by:

The standard epistemic or doxastic axioms and rules (of the systems $S5$ or $KD4$) for $C\Box_A^B$;

the **Monotonicity Axiom**:

$$C\Box_A^B \Rightarrow C\Box_{A'}^B$$, for $A \supseteq A'$, $B \subseteq B'$,

the **Fixed Point Axiom**:

$$C\Box_A^B \varphi \Rightarrow (\varphi \land E_A^B(C\Box_A^B \varphi))$$,

the "**Induction Axiom**"

$$C\Box_A^B (\varphi \Rightarrow E_A^B \varphi) \Rightarrow (\varphi \Rightarrow C\Box_A^B \varphi)$$.
Axioms for the dynamic logic of “tell us all you know”

To get a complete axiomatization, take:

the axioms and rules of the above static logic $C \square^B_A$, and

the following Reduction Laws:

\[
egin{align*}
&[!a]p \iff p \\
&[!a] \neg \varphi \iff \neg [!a] \varphi \\
&[!a](\varphi \land \psi) \iff [!a] \varphi \land [!a] \psi \\
&[!a]C \square^B_A \varphi \iff C \square^B_{A \cup \{a\}} [!a] \varphi
\end{align*}
\]
As a model transformer, the fully private announcement $!_G \varphi$ to a subgroup $G$ maps any model $S = (S, \xrightarrow{a}, \| \cdot \|)$ to a new model $S^{!_G \varphi} = (S', \xrightarrow{a}', \| \cdot \|')$, given by:

$$S' := S + \| \varphi \|_S = S \times \{0\} \cup \| \varphi \|_S \times \{1\},$$

where $+$ is the disjoint union operator; all the arrows are

$$(s, 1) \xrightarrow{a'} (t, 1) \text{ iff } s \xrightarrow{a} t, a \in G, s, t \in \| \varphi \|_S,$$

$$(s, 1) \xrightarrow{a'} (t, 0) \text{ iff } s \xrightarrow{a} t, a \not\in G, s \in \| \varphi \|_S,$$

$$(s, 0) \xrightarrow{a'} (t, 0) \text{ iff } s \xrightarrow{a} t :$$

$$\| p \|' := \| p \| \times \{0\} \cup (\| p \| \cap S') \times \{1\}$$

The transition relation $\xrightarrow{!_G \varphi}$ relates any state $s \in S$ satisfying $\varphi$ to the state $(s, 1)$ in the model $S^{!_G \varphi}$. 
The Logic of Private Announcements

A complete axiomatization for the basic logic of private announcements (without common knowledge) is obtained by replacing $!$ by $!_G$ in all the axioms of $PAL$, but in the same time replacing the Reduction Axiom for $\Box_a$ with the following two axioms:

$$[!_G \phi] \Box_a \psi \iff \phi \Rightarrow \Box_a [!_G \phi] \psi$$, for $a \in G$;

$$[!_G \phi] \Box_b \psi \iff \phi \Rightarrow \Box_b \psi$$, for $b \notin G$.

Now, we cannot read $\Box$ as knowledge, but rather as belief, since it won’t satisfy Veracity: outsiders may acquire false beliefs. Moreover, these beliefs might even become inconsistent!

As before, we put

$$E_G \psi := \bigwedge_{a \in G} \Box_a \psi$$
Private announcements are NOT closed under composition

Unfortunately, **private announcements are not closed under sequential composition:**
given sentences \( \varphi, \psi \) and distinct groups \( A, B \subseteq A \), in general there do not exist a sentence \( \theta \) and a group \( C \subseteq A \), such that performing successively the announcements

\[
!_A \varphi; !_B \psi
\]

would be equivalent to performing only one announcement

\[
!_C \theta.
\]

(This works only for announcements \( !_A \varphi, !_A \psi \) TO THE SAME GROUP \( A \).)
No “Composition Axiom” For Private Announcements

This means there CANNOT be any Composition Axiom for private announcements, asserting that for all sentences $\varphi, \psi$ and groups $A, B \subseteq A$, there would exist a sentence $\theta$ and a group $C \subseteq A$, such that

$$[^{A}\varphi][^{B}\psi] \chi \iff [^{C}\theta] \chi$$

for all sentences $\chi$. 
To achieve completeness, we need a **Generalized Announcement Rule**.

In fact, **DUE TO THE LACK OF A COMPOSITION AXIOM**, we need a very wild generalization of this rule, one that would enable us to prove theorems of the form

\[ \chi_1 \Rightarrow [!G_1 \varphi_1][!G_2 \varphi_2] \cdots [!G_n \varphi_n]C \Box G \psi. \]

Such a rule can be given, but it’s **extremely complicated**.

It is much easier to move to full DEL, by **introducing a general notion of “epistemic action”** into our logics.
“Standard DEL”: the dynamic logic of hard information

• studies the **multi-agent information flow of “hard information”** (absolutely certain, fully introspective “knowledge”) as well as “soft”, but essentially un-revisable, information (“beliefs” that change monotonically, but are never overturned);

• gives an answer to the Update Problem, based on the BMS (Baltag, Moss and Solecki) setting: **logics of epistemic actions**;

• it arose from generalizing previous work on logics for public/private announcements.

• this dynamics is **essentially monotonic** (no belief revision!), though *it can model very complex forms of communication.*
Models for ‘Events’

Until now, our Kripke models capture only epistemic situations, i.e. they only contain static information: they all are state models. We can thus represent the result of each of our Scenarios, but not what is actually going on. Our scenarios involve various types of changes that may affect agents’ beliefs or state of knowledge: a public announcement, a ’legal’ (non-deceitful) act of private learning, ’illegal’ (unsuspected) private learning etc.

We want to use now Kripke models to represent such types of epistemic events, in a way that is similar to the representations we have for epistemic states.
An event model (or “action model”)

\[ \Sigma = (\Sigma, \rightarrow, \text{pre}) \]

is just like an Kripke model, except that its elements are now called actions (or “simple events”) and instead of the valuation we have a precondition map \( \text{pre} \), associating a sentence \( \text{pre}_\sigma \) to each action \( \sigma \).
An event model is **epistemic**, or respectively a **doxastic**, event model if it satisfies the S5, or respectively the KD45, conditions.
Interpretation

We call of the simple events $\sigma \in \Sigma$ as *deterministic* actions of a particularly simple kind: they do not change the ”facts” of the world, but the agents’ beliefs. In other words, they are “purely epistemic” actions.

For $\sigma \in \Sigma$, we interpret $pre_\sigma$ as giving the *precondition* of the action $\sigma$: this is a sentence that is true in a world iff $\sigma$ can be performed. In a sense, $pre_\sigma$ gives the implicit information carried by $\sigma$.

Finally, the accessibility relations express the agents’ knowledge/beliefs about the current action taking place.
The Product Update

Given a state model $S = (S, \rightarrow, \parallel)$ and an action model $\Sigma = (\Sigma, \rightarrow, pre)$, we define their update product

$$S \otimes \Sigma = (S \otimes \Sigma, \rightarrow, \parallel)$$

to be a new state model, given by:

1. $S \otimes \Sigma$ is

$$\{(s, \sigma) \in S \times \Sigma : s \models_{S} pre_{\sigma}\}.$$

2. $(s, \sigma) \rightarrow (s', \sigma')$ iff $s \rightarrow s'$ and $\sigma \rightarrow \sigma'$.

3. $\parallel_{p}^{S \otimes \Sigma} = \{(s, \sigma) \in S \otimes \Sigma : s \in \parallel_{p}^{S}\}$. 
As before, we can consider pointed event models, if we want to specify the actual event taking place.

Naturally, if initially the actual state was $s$ and then the actual event is $\sigma$, then the actual output-state is $(s, \sigma)$; i.e. the transition relations between the two models are given by

$$s \rightarrow^{\sigma} (s, \sigma) \quad \text{whenever } s \models s \text{ pre}_\sigma.$$
The product arrows encode the idea that: two output-states are indistinguishable iff they are the result of indistinguishable actions performed on indistinguishable input-states.

This comprises two intuitions:

1. “No Miracles”: knowledge can only gained from (the epistemic appearance of) actions;

2. “Perfect Recall”: once gained, knowledge is never lost.

The fact that the valuation is the same as on the input-state tells us that these actions are purely epistemic.
Examples: Public Announcement

The event model $\Sigma !\varphi$ for public announcement $!\varphi$ consists of a single action, with precondition $\varphi$ and reflexive arrows:

$$a, b, c...$$

EXERCISE: Check that, for every state model $S$, $S \otimes \Sigma !\varphi$ is indeed the result of deleting all non-$\varphi$ worlds from $S$. 

More Examples: Taking a Peek

The action in Scenario 4: C takes a peek at the coin and sees the Head is up, without anybody noticing.

There are two actions in this model: the real event (on the left) is the cheating action of c ”taking a peek”. The action on the right is the apparent action skip, having any tautological sentence true as its precondition: this is the action in which nothing happens. This is what the outsiders (a and b) think it is going on: nothing, really.
The Product Update

We can now check that the product of

\[
\begin{array}{c}
H \\
\end{array}
\begin{array}{c}
\rightarrow \\
a,b,c
\end{array}
\begin{array}{c}
T \\
\rightarrow \\
a,b,c
\end{array}
\]

and

\[
\begin{array}{c}
H \\
\end{array}
\begin{array}{c}
\rightarrow \\
a,b,c
\end{array}
\begin{array}{c}
true \\
\rightarrow \\
a,b,c
\end{array}
\]

is indeed what intuitively should be:
More generally, a fully **private announcement** \(!_G \varphi\) of \(\varphi\) to a subgroup \(G\) is described by the action on the left in the event model

\[
G \xrightarrow{\varphi} A/G \rightarrow \{true\} A
\]

This subsumes both taking a peak (Example 4) and the secret communication in Example 5.
Fair-Game Announcements

The following event model represents the situation in which it is common knowledge that an agent $c$ privately learns whether $\varphi$ or $\neg \varphi$ is the case:

$$
\begin{array}{c}
\mathcal{A}(\varphi) \\
\mathcal{A}\{c\} \\
\neg \varphi
\end{array}
\rightleftharpoons
\begin{array}{c}
\mathcal{A}
\end{array}
$$

This is a “fair-game announcement” $Fair_\varphi$.

The case $\varphi := H$ represents the action in Example 3 ("legal viewing" of the card by $c$).
Recall Scenario 5: the supposedly secret message from \( c \) to \( a \) is secretly intercepted by \( b \). This is an instance of a private announcements with (secret) interception by a group of outsiders.
For any action $\sigma \in \Sigma$, we can consider the corresponding dynamic modality $[\sigma] \varphi$. This is a property of the original model, expressing the fact that, if action $\sigma$ happens, then $\varphi$ will come to be true after that.

We can easily define the epistemic proposition $[\sigma] \varphi$ by:

$$s \models S [\sigma] \varphi \text{ iff } (s, \sigma) \in S \otimes \Sigma \text{ implies } (s, \sigma) \models S \otimes \Sigma \varphi$$
Reduction Laws

If $\sigma \in \Sigma$ is a simple epistemic action, then we have the following properties (or “axioms”):

- **Preservation of “Facts”**. For all atomic $p \in \Phi$ :
  \[
  [\sigma]p \iff \text{pre}_\sigma \Rightarrow p
  \]

- **Partial Functionality**:
  \[
  [\sigma]\lnot \varphi \iff \text{pre}_\sigma \Rightarrow \lnot [\sigma]\varphi
  \]

- **Normality**:
  \[
  [\sigma](\varphi \land \psi) \iff [\sigma]\varphi \land [\sigma]\psi
  \]

- **“Action-Knowledge Axiom”**:
  \[
  [\sigma][\sigma]a\varphi \iff \text{pre}_\sigma \Rightarrow \bigwedge_{\sigma \xrightarrow{a} \sigma'} \Box a[\sigma']\varphi
  \]
Here, □ can be either knowledge $K$ or belief $B$, depending on whether the model is doxastic or epistemic.

The **Action-Knowledge Axiom** helps us to *compute the state of knowledge/belief* of an agent *after* an event, in terms of the agent’s *initial state of knowledge or belief* and of the event’s *appearance* to the agent.

In plain words, the Action-Knowledge axiom says that:

*a proposition $\varphi$ will be known (to an agent $a$) AFTER a (deterministic) epistemic event $\sigma$ iff, whenever the event $\sigma$ can take place, it is ALREADY known (to agent $a$, BEFORE the event) that $\varphi$ will be true after ANY event that is indistinguishable (to agent $a$) from $\sigma$.***
Instances of Action-Knowledge Axiom

If $a \in G$, $b \not\in G$, $c \neq a$, then:

\[
[!\theta] \Box_a \varphi \iff \theta \Rightarrow \Box_a [!\theta] \varphi
\]

\[
[!G \theta] \Box_a \varphi \iff \theta \Rightarrow \Box_a [!G \theta] \varphi
\]

\[
[!G \theta] \Box_b \varphi \iff \theta \Rightarrow \Box_b \varphi
\]

\[
[\text{Fair}_a \theta] \Box_a \varphi \iff \theta \Rightarrow \Box_a [\text{Fair}_a \theta] \varphi
\]

\[
[\text{Fair}_a \theta] \Box_c \varphi \iff \theta \Rightarrow \Box_c ([[\text{Fair}_a \theta] \varphi \land [\text{Fair}_a \neg \theta] \varphi)
\]
(Sequential) Composition of Epistemic Events

Given two event models $\Sigma = (\Sigma, \rightarrow, pre)$ and $\Sigma' = (\Sigma', \rightarrow, pre)$, their sequential composition is a new event model

$$\Sigma; \Sigma' = (\Sigma \times \Sigma', \rightarrow, pre),$$

where

1. $S \otimes \Sigma$ is the Cartesian product of the two models.

2. $(\sigma, \sigma') \rightarrow (\rho, \rho')$ iff $\sigma \rightarrow \rho$ and $\sigma' \rightarrow \rho'$.

3. $pre_{(\sigma, \sigma')} := pre_\sigma \land [\sigma]pre_{\sigma'},$
   or equivalently $pre_{(\sigma, \sigma')} := \langle \sigma \rangle pre_{\sigma'},$
   where $\langle \alpha \rangle \varphi := \neg [\alpha] \neg \varphi$ is the existential dynamic modality.

NOTATION: We denote by $\sigma; \sigma'$ the pair $(\sigma, \sigma')$ as an event in $\Sigma; \Sigma'$. 
As before, we can consider **pointed event models**, if we want to specify the **actual event** taking place.

Naturally, when the actual first event is $\sigma \in \Sigma$ and then the actual second event is $\sigma' \in \Sigma'$, then the actual composed event in $\Sigma; \Sigma'$ is $\sigma; \sigma' = (\sigma, \sigma')$. 
“Composition” is indeed Sequential Composition

It is easy to see that taking the Product Update with a composed event model is equivalent to taking successive Product Updates with the two event models:

\[ S \otimes (\Sigma; \Sigma') = (S \otimes \Sigma) \otimes \Sigma'. \]

Moreover, the transition relation for the event \( \sigma; \sigma' \) is the composition of the two transition relations:

\[ s \xrightarrow{\sigma; \sigma'} t \text{ iff } \exists w (s \xrightarrow{\sigma} w \xrightarrow{\sigma'} t). \]

This confirms our intended interpretation: intuitively, \( \sigma; \sigma' \) is the sequential composition of the two events: the action obtained by performing first action \( \sigma \) then action \( \sigma' \).
The composition of the private announcement $!_a \varphi$

$\xymatrix@C=15pt@R=10pt{a \ar[r]^{b,c} & true_{a,b,c}}$

and the private announcement $!_b \psi$

$\xymatrix@C=15pt@R=10pt{b \ar[d]^{a,c} \ar[r] & true_{a,b,c}}$
is the event model
Complete Axiomatization of \( DEL \) with Common Knowledge

A complete axiomatization of standard dynamic epistemic logic with common knowledge is given by:

the axioms and rules of epistemic/doxastic logic with common knowledge;

the Reduction Axioms of \( DEL \);

the Normality conditions for dynamic modalities:

• **Necessitation**: From \( \psi \) infer \([\sigma]\psi\).

• **Kripke’s axiom**: \( [\sigma](\psi \Rightarrow \theta) \Rightarrow ([\sigma]\psi \Rightarrow [\sigma]\theta) \)

the \( DEL \) Composition Axiom: \([\alpha][\beta]\theta \iff [\alpha; \beta]\theta\).

the **Action Rule**: From \( \chi_\beta \Rightarrow [\beta]\psi \) and \( \chi_\beta \land \text{pre}_\beta \Rightarrow E_{\{a \in G : \gamma\rightarrow a\} \chi_{\gamma}} \)

for all \( \beta, \gamma \) such that \( \alpha \rightarrow^* G \beta, \gamma \), infer \( \chi_\alpha \Rightarrow [\alpha]C\Box_G \psi \).
In the above, we used

\[ G^* = (\bigcup_{a \in G} a)^* \]

for the reflexive-transitive closure of the union of all the \( a \to \) relations with \( a \in G \). In other words:

\[ \alpha \xrightarrow{G^*} \beta \text{ iff } (\exists) \text{ a chain } \alpha = \alpha_0 \xrightarrow{a_1} \alpha_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} \alpha_n = \beta \text{ with all } a_i \in G. \]
Application of Action Rule

Here is an instantiation of the Action Rule for the case of common knowledge $C\Box_G \psi$ among a group $G \subseteq A$ achieved after a private announcement $!_A \varphi$ to another group $A \subseteq A$:

**Private Announcement Rule:**

*From the theorems*

\[
\chi_1 \Rightarrow [!_A \varphi] \psi , \quad \chi_1 \land \varphi \Rightarrow E_{G \cap A} \chi_1 , \quad \chi_1 \land \varphi \Rightarrow E_{G \setminus A} \chi_2 , \quad \chi_2 \Rightarrow \psi \quad \text{and} \quad \chi_2 \Rightarrow E_G \chi_2 ,
\]

*infer*

\[
\chi_1 \Rightarrow [!_A \varphi] C\Box_G \psi .
\]
One may want to close DEL modalities under iteration (Kleene star), in the manner of PDL:

Add a new operator \([\alpha]^*\psi\), with the semantics given by

\[
s \models s [\alpha]^*\psi \iff s \models s [\alpha]^n\psi \text{ for every } n \in \mathbb{N},
\]

where \([\alpha]^n\) is given recursively by

\[
[\alpha]^0\psi := \psi,
\]

\[
[\alpha]^{n+1}\psi := [\alpha]^n[\alpha]\psi.
\]

The resulting logic is called DEL*, while the result of adding iteration \([!\varphi]^*\psi\) to PAL is called PAL*.
Iteration would be very good to have, e.g. to capture the Muddy Children story without having to specify the exact number of muddy children.

But unfortunately, $PAL^*$, and hence also $DEL^*$, are undecidable, and moreover they are not axiomatizable:

**Theorem** (Miller and Moss 2005): $PAL^*$ is not axiomatizable (hence undecidable).
Another extension is obtained by adding an operator that quantifies over arbitrary public announcements:

\[ s \models s [!]\psi \iff s \models s [!]\varphi\psi \text{ for every epistemic formula } \varphi. \]

The resulting logic is undecidable, but still axiomatizable (Balbiani et alia 2007).

This logic has been used by van Benthem to give a new solution to Fitch’s Knowability Paradox, which was proposed as an attack to Verificationism.